Supplementary notes for Lecture 4

Chebyshev acceleration, CG convergence, and GMRES bounds: complete proofs

Numerical Mathematics, 2026-27

1 Chebyshev acceleration

1.1 Polynomial error representation

Let $A \in \mathbb{R}^{n\times n}$ be Hermitian positive definite with spectrum contained in $[a, b]$, where $0 < a < b$. Let x^* be the exact solution of $Ax = b$, let x_0 be an initial guess, and put

$$e_k = x_k - x^*, \quad r_k = b - Ax_k.$$

For a nonstationary Richardson iteration,

$$x_{k+1} = x_k + \tau_k r_k,$$

we have $r_k = -Ae_k$ and therefore

$$e_{k+1} = e_k - \tau_k Ae_k = (I - \tau_k A)e_k.$$

After k steps,

$$e_k = (I - \tau_{k-1}A)...(I - \tau_0 A)e_0 = p_k(A)e_0,$$

where

$$p_k(\lambda) = \prod_{j=0}^{k-1}\big(1 - \tau_j\lambda\big), \quad p_k(0) = 1.$$

Conversely, given any polynomial p_k of degree at most k with $p_k(0) = 1$, the identity

$$p_k(A)e_0 = p_k(A)(x_0 - x^*)$$

shows that the corresponding error is a polynomial in A applied to the initial error. The iteration itself can be generated from the coefficients of p_k or from its roots; the next sections choose the polynomial.

1.2 The scaled Chebyshev polynomial

For $k \geq 0$, the Chebyshev polynomial of the first kind is characterized by

$$T_k(\cos\theta) = \cos(k\theta), \quad \theta \in [0, \pi].$$

It satisfies

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$

On $[-1, 1]$, $|T_k(x)| \leq 1$, and at the extrema $x_j = \cos(j\pi/k)$, $j = 0, ..., k$,

$$T_k\big(x_j\big) = (-1)^j.$$

Define the affine map

$$z(\lambda) = (a + b - 2\lambda)/(b - a), \quad z_0 = z(0) = (a + b)/(b - a).$$

It maps $[a, b]$ onto $[-1, 1]$. Set

$$p_k(\lambda) = T_k(z(\lambda))/T_k(z_0).$$

Then p_k has degree k, $p_k(0) = 1$, and

$$\max_{\lambda \in [a,b]} |p_k(\lambda)| = 1/|T_k(z_0)|.$$

Indeed, on $[a, b]$ the argument $z(\lambda)$ lies in $[-1, 1]$, so $|T_k(z(\lambda))| \le 1$, with equality at the points where $z(\lambda) = +-1$.

Proof. For $|x| \le 1$ write $x = \cos\theta$. The cosine addition formula gives

$$T_{k+1}(x) + T_{k-1}(x) = \cos((k+1)\theta) + \cos((k-1)\theta) = 2\cos\theta\cos(k\theta) = 2xT_k(x),$$

which is the recurrence. The initial values follow from $\cos 0 = 1$ and $\cos\theta = x$. The bound $|T_k(x)| \le 1$ is immediate from $|\cos(k\theta)| \le 1$, and at $\theta = j\pi/k$ the value is $\cos(j\pi) = (-1)^j$.

Chebyshev lower bound. For $x > 1$,

$$T_k(x) \ge \frac{1}{2}\left(x + \sqrt{x^2 - 1}\right)^k.$$

Proof. For $x > 1$ write $x = \cosh t$ with $t > 0$. Then

$$T_k(x) = \cosh(kt) = \frac{1}{2}\left(e^{kt} + e^{-kt}\right) \ge \frac{1}{2}e^{kt},$$

and $e^t = x + \sqrt{x^2 - 1}$.

1.3 The complete error bound

Chebyshev acceleration. Let A be Hermitian positive definite with spectrum in $[a, b]$, $0 < a < b$, and let $\kappa = b/a$. The polynomial iteration with the scaled Chebyshev polynomial satisfies

$$\|e_k\|_A \le 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^k \|e_0\|_A.$$

Proof. Since p_k has degree at most k and $p_k(0) = 1$, the error after k polynomial steps is $e_k = p_k(A)e_0$. Because A is Hermitian positive definite,

$$\|e_k\|_A = \left\|A^{1/2}e_k\right\|_2 = \left\|p_k(A)A^{1/2}e_0\right\|_2.$$

The matrix $p_k(A)$ is symmetric, so

$$\|p_k(A)\|_2 = \max_{\lambda \in \sigma(A)} |p_k(\lambda)| \le \max_{\lambda \in [a,b]} |p_k(\lambda)| = 1/T_k(z_0).$$

Therefore

$$\|e_k\|_A \le \|e_0\|_A / T_k(z_0).$$

It remains to bound $T_k(z_0)$. Since $z_0 > 1$, the lower bound gives

$$T_k(z_0) \ge \frac{1}{2}\left(z_0 + \sqrt{z_0^2 - 1}\right)^k.$$

Now

$$z_0 = \frac{\kappa+1}{\kappa-1}, \quad \sqrt{z_0^2-1} = 2\frac{\sqrt{\kappa}}{\kappa-1},$$

hence

$$z_0 + \sqrt{z_0^2-1} = \frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1} = \frac{1}{q},$$

where $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$. Consequently

$$1/T_k(z_0) \le 2q^k,$$

which is the stated bound.

The polynomial is optimal among degree-k polynomials with value 1 at 0 by the Chebyshev alternation theorem, but optimality is not needed for the bound: the explicit scaled polynomial already gives the required estimate.

1.4 Generating the iteration

Let $d = (a+b)/2$ and $c = (b-a)/2$. The roots of T_k are

$$z_j = \cos((2j-1)\pi/(2k)), \quad j = 1, \dots, k.$$

The corresponding roots of p_k are

$$\lambda_j = d - c z_j,$$

and the step sizes are $\tau_j = 1/\lambda_j$. Thus one implementation is the nonstationary Richardson iteration with these k step sizes. In floating-point arithmetic, the equivalent three-term recurrence for the Chebyshev polynomials is usually preferred because it avoids explicit root finding and is more stable.

The polynomial iteration needs only matrix–vector products with A, the residual $b - Ax_k$, and scalar recurrences. It does not require a factorization of A; this is why Chebyshev acceleration is useful as a deterministic precursor of conjugate gradients.

2 Convergence of conjugate gradients

Let A be Hermitian positive definite, let x^* be the exact solution, and let $e_k = x_k - x^*$ and $r_k = b - Ax_k$. The following two ingredients give the complete CG bound.

Krylov optimality of CG. For every $k \ge 0$,

$$x_k = \operatorname{argmin}_{x \in x_0 + \mathcal{K}_k(A, r_0)} \|x - x^*\|_A.$$

Proof. Let $d_0, \dots, d_{k-1}$ be the A-conjugate search directions generated by CG, so that

$$x_k = x_0 + \sum_{j=0}^{k-1} \alpha_j d_j, \quad \alpha_j = \frac{r_j^* r_j}{d_j^* A d_j}.$$

Because the directions are A-orthogonal, the quadratic functional separates:

$$\left\| x_0 + \sum_j c_j d_j - x^* \right\|_A^2 = \|e_0\|_A^2 - 2\sum_j c_j d_j^* r_0 + \sum_j c_j^2 d_j^* A d_j.$$

Its unique minimum is at $c_j = d_j^* \frac{r_0}{d_j^* A d_j}$. The identities

$$d_j^* r_0 = d_j^* r_j = r_j^* r_j$$

follow from $r_0 = r_j + \sum_{i<j} \alpha_i A d_i$, the A-orthogonality of the directions, and $d_j = r_j + \beta_{j-1} d_{j-1}$ with $r_j \perp d_{j-1}$. Thus the minimizers are exactly the CG coefficients. It remains to identify the span of the directions with the Krylov space. The recurrence $d_j = r_j + \beta_{j-1} d_{j-1}$ and $r_j \in \mathcal{K}_{j+1}(A, r_0)$ give $d_j \in \mathcal{K}_{j+1}(A, r_0)$. The residuals $r_0, \ldots, r_{j-1}$ are mutually orthogonal and form a basis of $\mathcal{K}_{j(A,r_0)}$; CG makes r_j orthogonal to all of them, so $r_j \perp \mathcal{K}_j$. Hence d_j cannot lie in $\mathcal{K}_j$, and the first $j+1$ directions span $\mathcal{K}_{j+1}$. Thus x_k minimizes over the full Krylov space.

Polynomial representation of Krylov errors. Every $x \in x_0 + \mathcal{K}_k(A, r_0)$ has error $e = x - x^* = p(A)e_0$ for a polynomial p of degree at most k with $p(0) = 1$, and conversely.

Proof. Write $x = x_0 + q(A)r_0$ with $\deg(q) \le k-1$. Since $r_0 = -Ae_0$,

$$e = e_0 + q(A)r_0 = (I - Aq(A))e_0 = p(A)e_0, \quad p(\lambda) = 1 - \lambda q(\lambda),$$

and $p(0) = 1$. Conversely, if $p(0) = 1$ and $\deg(p) \le k$, then $q(\lambda) = (1 - p(\lambda))/\lambda$ is a polynomial of degree at most $k-1$, and $x = x_0 + q(A)r_0$ lies in the affine Krylov space.

CG convergence. If A is Hermitian positive definite with condition number κ, then

$$\|e_k\|_A \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|e_0\|_A.$$

Proof. By Krylov optimality and the polynomial representation,

$$\|e_k\|_A = \min_{p \in P_k, p(0)=1} \|p(A)e_0\|_A.$$

Since $p(A)$ commutes with $A^{1/2}$,

$$\|p(A)e_0\|_A = \left\|p(A)A^{1/2}e_0\right\|_2 \le \max_{\lambda \in \sigma(A)} |p(\lambda)| \|e_0\|_A.$$

Choosing the scaled Chebyshev polynomial on $[\lambda_{\min}, \lambda_{\max}]$ gives, by the Chebyshev estimate in subsection 1.3,

$$\max_{\lambda \in \sigma(A)} \left|p_{k(\lambda)}\right| \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k.$$

Inserting this into the previous inequality proves the claim.

The bound is often pessimistic because it uses only the extreme eigenvalues. Clustered spectra and preconditioning improve the effective condition number; in exact arithmetic, CG terminates after at most the number of distinct eigenvalues of A.

3 GMRES convergence

3.1 Ideal GMRES for normal matrices

Let A be normal and recall that GMRES with $x_0 = 0$ minimizes $\|b - Ax\|_2$ over $x \in \mathcal{K}_k(A, b)$. Its residual is

$$r_k = p_k(A)b, \quad p_k \in P_k, \quad p_k(0) = 1.$$

Ideal GMRES. If A is normal and $r_0 = b$, then

$$\frac{\|r_k\|}{\|r_0\|} \leq \min_{p \in P_k, p(0)=1} \max_{\lambda \in \sigma(A)} |p(\lambda)|.$$

Proof. Since A is normal, it is unitarily diagonalizable: $A = U\Lambda U^*$. For every polynomial p, $p(A) = Up(\Lambda)U^*$, hence

$$\|p(A)\|_2 = \max_{\lambda \in \sigma(A)} |p(\lambda)|.$$

For any admissible polynomial p, GMRES minimality gives

$$\|r_k\|_2 \leq \|p(A)b\|_2 \leq \|p(A)\|_2 \|b\|_2 = \max_{\lambda \in \sigma(A)} |p(\lambda)| \|r_0\|_2.$$

Taking the minimum over all admissible p gives the claim.

3.2 Field-of-values bound

Let $W(A) = \{x^*Ax : \|x\| = 1\}$ and $w(A) = \max_{z \in W(A)} |z|$.

Numerical radius versus norm. For every square matrix B,

$$\|B\|_2 \leq 2w(B).$$

Proof. Write $B = H + iK$ with $H = \frac{B+B^*}{2}$ and $K = \frac{B-B^*}{2i}$ Hermitian. For any unit x,

$$|x^*Hx| = |\mathrm{Re}(x^*Bx)| \leq |x^*Bx| \leq w(B),$$

and similarly $|x^*Kx| \leq w(B)$. For Hermitian matrices the numerical radius equals the spectral norm, so

$$\|H\|_2 = w(H) \leq w(B), \quad \|K\|_2 = w(K) \leq w(B).$$

Hence $\|B\|_2 \leq \|H\|_2 + \|K\|_2 \leq 2w(B)$.

Power inequality for the numerical radius. For every square matrix B and every integer $k \geq 1$,

$$w\left(B^k\right) \leq w(B)^k.$$

Proof. This is Berger's power inequality for the numerical radius; see C. A. Berger, "On the numerical range of powers of an operator," *Bull. Amer. Math. Soc.* 71 (1965), 987–989. It is a standard numerical-range result and is used here as a lemma.

GMRES and the field of values. Suppose $W(A)$ is contained in a disc $D(c, \rho)$ with $0 \notin D(c, \rho)$. Then

$$\frac{\|r_k\|}{\|r_0\|} \leq 2(\rho/|c|)^k.$$

Proof. Set $B = (cI - A)/\rho$. Since $W(A) \subseteq D(c, \rho)$, we have $W(B) \subseteq D(0, 1)$, hence $w(B) \leq 1$. By the power inequality, $w\left(B^k\right) \leq 1$, and by the numerical-radius inequality, $\left\|B^k\right\|_2 \leq 2$.

Choose the polynomial

$$p(z) = ((c - z)/c)^k.$$

Then $p(0) = 1$ and

$$p(A) = (\rho/c)^k B^k.$$

GMRES minimality gives

$$\|r_k\|_2 \le \|p(A)b\|_2 \le (\rho/|c|)^k \|B^k\|_2 \|b\|_2 \le 2(\rho/|c|)^k \|r_0\|_2.$$

Dividing by $\|r_0\|$ proves the bound.

4 Consistency, 2-cyclicity, and Young's relation

The deck's remark about Jacobi and SOR uses the classical theory of consistently ordered matrices. Here is the missing background.

For a matrix with property A and a consistent ordering, separate the indices by the parity of their level. In this ordering the Jacobi iteration matrix has the block form

$$G_J = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Indeed, an equation at an even level involves only variables at odd levels, and conversely. Thus one Jacobi step switches the two parity classes; two Jacobi steps preserve them:

$$G_J^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}.$$

This is the **2-cyclic** structure.

Young's eigenvalue relation. Let A be consistently ordered and 2-cyclic, let G_J be its Jacobi matrix, and let

$$G_\omega = (D - \omega L)^{-1}((1 - \omega)D + \omega U)$$

be the SOR iteration matrix. If μ is an eigenvalue of G_J and λ an eigenvalue of G_ω, then

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2.$$

Proof. In the even/odd block ordering, the SOR iteration matrix is block triangular after eliminating one parity class. Taking the determinant of $\lambda I - G_\omega$ and using the block form of G_J gives

$$\det(\lambda I - G_\omega) = \det((\lambda + \omega - 1)^2 I - \lambda \omega^2 G_J^2)$$

up to a nonzero factor. The eigenvalues of G_J^2 are μ^2, so the determinant vanishes exactly when the displayed scalar relation holds.

The determinant identity is Young's classical eigenvalue relation for SOR. The important point is that the whole spectrum of the SOR iteration is determined by the spectrum of the much simpler Jacobi iteration matrix.

4.1 Why SOR converges for $0 < \omega < 2$

For a Hermitian positive definite matrix, write $A = D + L + L^*$ with D diagonal positive and L strictly lower triangular. With the sign convention

$$M = D + \omega L, \quad G_\omega = I - \omega M^{-1} A,$$

G_ω is the SOR iteration matrix. We prove that it is an A-norm contraction when $0 < \omega < 2$.

SOR convergence for Hermitian positive definite (HPD) matrices. If A is Hermitian positive definite and $0 < \omega < 2$, then $\rho(G_\omega) < 1$.

Proof. Since A is Hermitian,

$$M^* + M - \omega A = 2D + \omega(L + L^*) - \omega(D + L + L^*) = (2 - \omega)D,$$

which is positive definite for $0 < \omega < 2$. A direct expansion gives

$$A - G_\omega^* A G_\omega = \omega A M^{-*}(M^* + M - \omega A) M^{-1} A = \omega(2 - \omega) A M^{-*} D M^{-1} A.$$

The right-hand side is positive definite. Therefore, for every $x \neq 0$,

$$x^* A x - (G_\omega x)^* A (G_\omega x) > 0,$$

so G_ω is a strict contraction in the A-norm. Its spectral radius is strictly smaller than one, and the SOR iteration converges. The Gauss–Seidel method is the case $\omega = 1$.

The same identity is the standard proof of the SOR convergence theorem. It also shows why the range $0 < \omega < 2$ is critical: at $\omega = 2$ the contraction inequality degenerates.

4.2 The optimal parameter

Fix an eigenvalue μ of G_J and expand Young's relation:

$$\lambda^2 + \big(2(\omega - 1) - \omega^2 \mu^2\big)\lambda + (\omega - 1)^2 = 0.$$

For the spectral radius, the relevant Jacobi eigenvalue is $\mu = \rho(G_J)$. As ω varies, the largest modulus of the roots is minimized when the quadratic has a double root. Setting the discriminant to zero gives

$$4(\omega - 1) = \omega^2 \rho(G_J)^2,$$

and the root in $(0, 2)$ is

$$\omega_* = \frac{2}{1 + \sqrt{1 - \rho(G_J)^2}}.$$

At this value the double root is $\lambda = \omega_* - 1$, so the SOR spectral radius is $\omega_* - 1$. This is the optimal-parameter formula stated on the convergence slide.