Towards bifurcations of complex dimensions

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Motivation from dynamical systems

Consider the standard Hopf–Takens bifurcations in polar coordinates:

$$\begin{aligned} \dot{r} &= r\left(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}\right) \\ \dot{\varphi} &= 1 \end{aligned} \tag{1}$$

Theorem (Weak focus; D. Zubrinic, V. Zupanovic, (2005))

Let Γ be a part of a trajectory of (1) near the origin.

(*a*) *Assume that $a_0 \neq 0$. Then the spiral Γ is comparable with $r = e^{a_0\varphi}$, and hence $\dim_B \Gamma = 1$.*

(*b*) *Let k be fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \ldots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable to the spiral $r = \varphi^{-1/2k}$ and $\dim_B \Gamma = \frac{4k}{2k+1}$. Moreover, the spiral Γ is Minkowski measurable.*

What is a fractal?

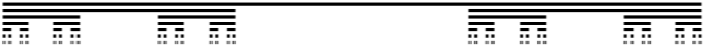


Figure: The middle-third Cantor set C.

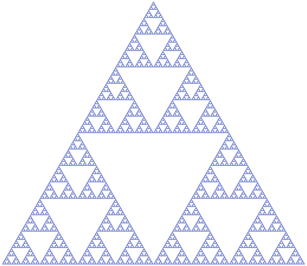


Figure: The Sierpiński gasket S.

Fractal dimensions

- There are several definitions of fractal dimension.
- e.g., similarity dimension, Hausdorff dimension, box counting dimension, Minkowski dimension, etc.

Figure: $\dim_H C = \dim_B C = \log_3 2$

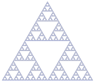


Figure: $\dim_H S = \dim_B S = \log_2 3 > 1$

- Mandelbrot: A set is fractal if its fractal dimension exceeds its topological dimension.
- None of the above dimensions give a completely satisfactory definition of a fractal.

Relative fractal drum (A, Ω)

- $\emptyset \neq A \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, Lebesgue measurable, i.e., $|\Omega| < \infty$
- **upper r-dimensional Minkowski content of** (A, Ω):

$$\overline{\mathcal{M}}^r(A,\Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

- **upper Minkowski dimension of** (A, Ω):

$$\overline{\dim}_B(A,\Omega) = \inf\{r \in \mathbb{R} \ : \ \overline{\mathcal{M}}^r(A,\Omega) = 0\}$$

- **lower Minkowski content** and **dimension** defined via lim inf

Minkowski measurability

- $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega) \Rightarrow \exists \dim_B(A,\Omega)$
- if $\exists D \in \mathbb{R}$ such that

$$0 < \underline{\mathcal{M}}^D(A,\Omega) = \overline{\mathcal{M}}^D(A,\Omega) < \infty,$$

  we say (A,Ω) is **Minkowski measurable**; in that case $D = \dim_B(A,\Omega)$
- if the above inequalities are not satisfied for D, we call (A,Ω) **Minkowski degenerated**

The relative distance zeta function

- (A, Ω) RFD in $\mathbb{R}^N$, $s \in \mathbb{C}$ and **fix** $\delta > 0$
- the **distance zeta function of** (A, Ω):

$$\zeta_{A,\Omega}(s;\delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N}\, dx$$

- dependence on δ is not essential
- the **complex dimensions of** (A, Ω) are defined as the poles of $\zeta_{A,\Omega}$
- take Ω to be an open neighborhood of A in order to recover the classical ζ_A

Holomorphicity theorem

Theorem

(*a*) $\zeta_{A,\Omega}(s)$ is **holomorphic on** $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$, and

(*b*) $\mathbb{R} \ni s < \overline{\dim}_B(A,\Omega) \Rightarrow$ the integral defining $\zeta_{A,\Omega}(s)$ diverges

(*c*) If $\exists D = \dim_B(A,\Omega) < N$ and $\underline{\mathcal{M}}^D(A,\Omega) > 0$, then

$\zeta_{A,\Omega}(x) \to +\infty$ when $\mathbb{R} \ni x \to D^+$

Definition (Complex dimensions)

Assume $\zeta_{A,\Omega}$ can be meromorphically extended to $W \subseteq \mathbb{C}$.
The **set of complex dimensions** of A **visible in** W:

$$\mathcal{P}(\zeta_{A,\Omega}, W) := \Big\{\omega \in W : \omega \text{ is a pole of } \zeta_{A,\Omega}\Big\}.$$

Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function** $t \mapsto |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

Theorem (Simplified pointwise formula with error term)

- $\alpha < \overline{\dim}_B(A,\Omega) < N$; $\zeta_{A,\Omega}$ satisfies suitable rational decay (d-**languidity**) on the half-plane $\mathbf{W} := \{\mathrm{Re}\, s > \alpha\}$, then:

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega},\mathbf{W})} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s),\omega\right) + O(t^{N-\alpha}).$$

- if we allow polynomial growth of $\zeta_{A,\Omega}$, in general, we get a tube formula in the sense of Schwartz distributions

The Minkowski measurability criterion

Theorem (Minkowski measurability criterion)

- (A, Ω) is such that $\exists D := \dim_B(A, \Omega)$ and $D < N$
- $\zeta_{A,\Omega}$ is d-languid on a suitable domain $W \supset \{\operatorname{Re} s = D\}$

Then, the following is equivalent:

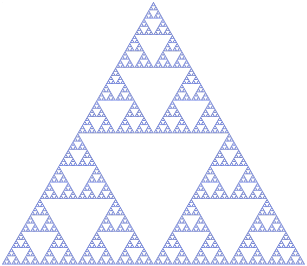
(a) (A, Ω) is Minkowski measurable.

(b) D is the only pole of $\zeta_{A,\Omega}$ located on the critical line $\{\operatorname{Re} s = D\}$ and it is simple.

In that case:

$$\mathcal{M}^D(A, \Omega) = \frac{\operatorname{res}(\zeta_{A,\Omega}, D)}{N - D}$$

Figure: The Sierpiński gasket



- an example of a **self-similar fractal spray** with a generator G being an open equilateral triangle and with **scaling ratios** $r_1 = r_2 = r_3 = 1/2$
- $(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{3} (r_j A, r_j \Omega)$

Fractal tube formula for The Sierpiński gasket

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$$

$$\mathcal{P}(\zeta_A) = \{0,1\} \cup \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right)$$

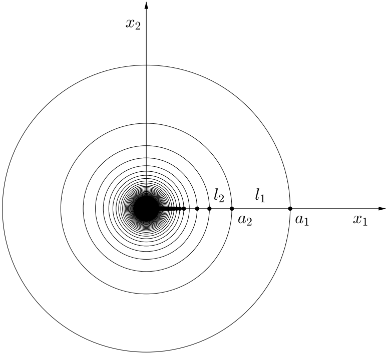
By letting $\omega_k := \log_2 3 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 2$ we have that

$$|A_t| = \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta),\omega\right)$$

$$= t^{2-\log_2 3}\,\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{+\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k} + \left(\frac{3\sqrt{3}}{2}+\pi\right)t^2,$$

valid pointwise for all $t \in (0, 1/2\sqrt{3})$.

The fractal nest generated by the a-string



$a > 0$, $a_j := j^{-a}$, $\ell_j := j^{-a} - (j+1)^{-a}$, $\Omega := B_{a_1}(0)$

$$\zeta_{A_a,\Omega}(s) = \frac{2^{2-s}\pi}{s-1} \sum_{j=1}^{\infty} \ell_j^{s-1}(a_j + a_{j+1})$$

Fractal tube formula for the fractal nest generated by the a-string

Example

$$\mathcal{P}(\zeta_{A_a,\Omega}) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}$$

$a \neq 1,\ D := \frac{2}{1+a} \Rightarrow$

$$|(A_a)_t \cap \Omega| = \frac{2^{2-D} D\pi}{(2-D)(D-1)} a^{D-1} t^{2-D} + 2\pi\big(2\zeta(a) - 1\big) t + O\big(t^{2-\frac{1}{a+1}}\big), \text{ as } t \to 0^+$$

$$|(A_1)_t \cap \Omega| = \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1,\Omega}(s), 1\right) + o(t) = 2\pi t(-\log t) + \text{const} \cdot t + o(t) \quad \text{as } t \to 0^+$$

- a pole ω of order m generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \ldots, m$ in the fractal tube formula

Further research directions

- Riemann surfaces generated by relative fractal drums
- Extending the notion of complex dimensions to include complicated "mixed" singularities/branching points and connecting them with various gauge functions
- Obtaining corresponding tube formulas and gauge-Minkowski measurability criteria
- Applying the theory to problems from dynamical systems

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