An overview of the theory of complex dimensions and fractal zeta functions

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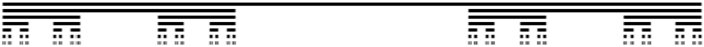

Figure: The middle-third Cantor set C.

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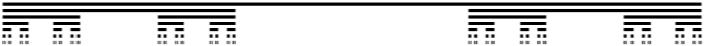

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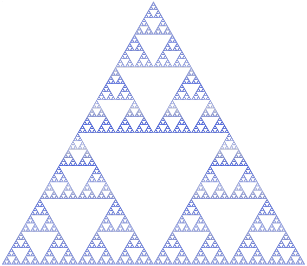

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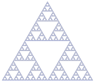

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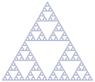

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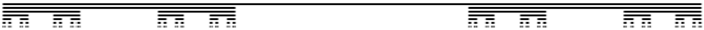

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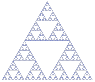

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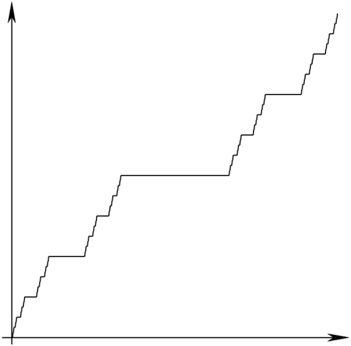
- Mandelbrot: A set is fractal if its fractal dimension exceeds its topological dimension.
- None of the above dimensions give a completely satisfactory definition of a fractal.

Some more examples



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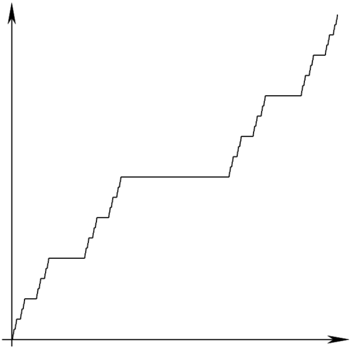

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All of the known fractal dimensions are equal to 1, i.e., to its topological dimension.

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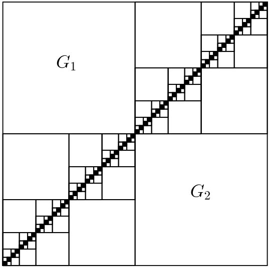

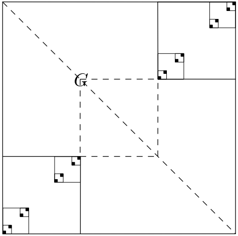

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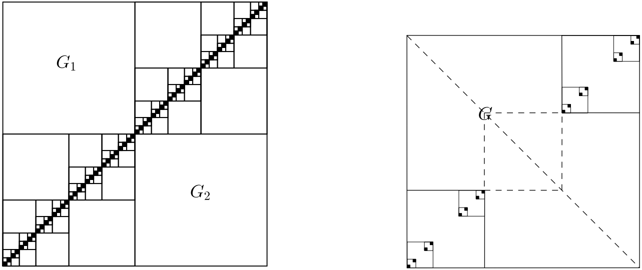

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The Hausdorff and Minkowski dimensions equal to 1 which is also their topological dimension.

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The set of complex dimensions: $\left\{\log_3 2 + \frac{2\pi \mathbb{i}\mathbb{Z}}{\log 3}\right\}$.

The Distance Zeta Function - generalization to higher dimensions

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- $\zeta_{A_\mathcal{L}}(s) = \frac{2^{1-s}}{s}\zeta_\mathcal{L}(s) + \frac{2\delta^s}{s}$, given a large enough $\delta > 0$

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Assume ζ_A can be meromorphically extended to $W \subseteq \mathbb{C}$.
The **set of complex dimensions** of A **visible in** W:

$$\mathcal{P}(\zeta_A, W) := \Big\{\omega \in W : \omega \text{ is a pole of } \zeta_A\Big\}.$$

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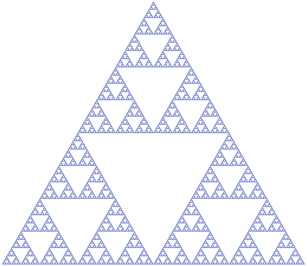
Definition (A new proposed definition of fractality)

The set A is fractal if it has at least one nonreal complex dimension.

Complex dimensions of the Sierpiński gasket



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$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$$

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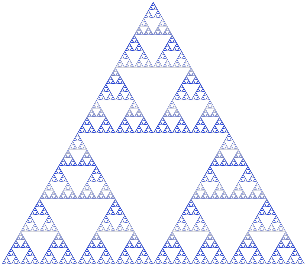

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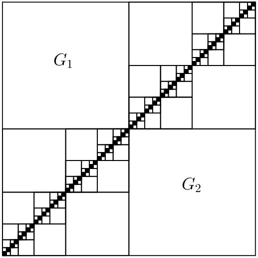

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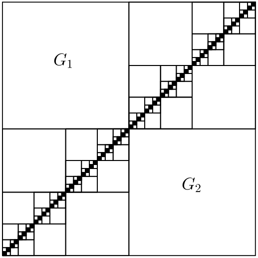

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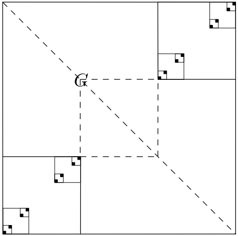

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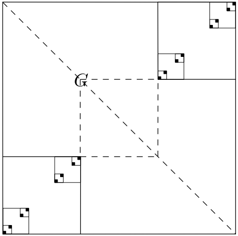

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Relative fractal drum (A, Ω)

- $\emptyset \neq A \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, Lebesgue measurable, i.e., $|\Omega| < \infty$
- **upper r-dimensional Minkowski content of** (A, Ω):

$$\overline{\mathcal{M}}^r(A, \Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

- **upper Minkowski dimension of** (A, Ω):

$$\overline{\dim}_B(A, \Omega) = \inf\{r \in \mathbb{R} \ : \ \overline{\mathcal{M}}^r(A, \Omega) = 0\}$$

- **lower Minkowski content** and **dimension** defined via lim inf

Minkowski measurability

- $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega) \Rightarrow \exists \dim_B(A,\Omega)$
- if $\exists D \in \mathbb{R}$ such that
$$0 < \underline{\mathcal{M}}^D(A,\Omega) = \overline{\mathcal{M}}^D(A,\Omega) < \infty,$$
we say (A,Ω) is **Minkowski measurable**; in that case $D = \dim_B(A,\Omega)$
- if the above inequalities are not satisfied for D, we call (A,Ω) **Minkowski degenerated**

The relative distance zeta function

- (A, Ω) RFD in $\mathbb{R}^N$, $s \in \mathbb{C}$ and **fix** $\delta > 0$
- the **distance zeta function of** (A, Ω):

$$\zeta_{A,\Omega}(s;\delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N}\, dx$$

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- the **complex dimensions of** (A, Ω) are defined as the poles of $\zeta_{A,\Omega}$
- take Ω to be an open neighborhood of A in order to recover the classical ζ_A

Embeddings in higher dimensions

Theorem

- (A,Ω) such that $\overline{D} := \overline{\dim}_B(A,\Omega) < N$ and fix $a > 0$

Then, the following functional equation is valid:

$$\zeta_{A\times\{0\},\Omega\times[-a,a]}(s) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_{A,\Omega}(s) + E(s;a). \tag{7}$$

$E(s;a)$ is meromorphic on $\mathbb{C}$ with a set of simple poles contained in $\{N+2k : k\in\mathbb{N}_0\}$.

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- complex dimensions of an RFD are independent of the ambient space
- determine complex dimensions of RFDs by decomposing them into relative fractal subdrums

Figure: The Cantor dust

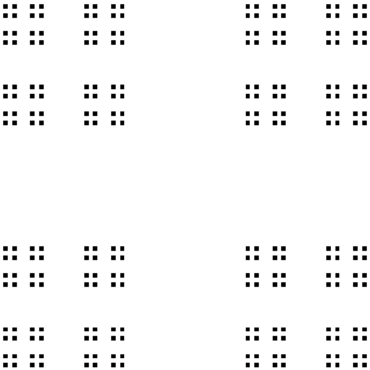

Figure: $C \times C$ where C is the middle-third Cantor set.

Complex dimensions of the Cantor dust

Example

Let $A := C^{(1/3)} \times C^{(1/3)}$ be the Cantor dust and $\Omega := [0,1]^2$. Then,

$$\zeta_{A,\Omega}(s) = \frac{8}{s(3^s-4)}\left(\frac{I(s)}{6^s} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1})\right),$$

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$$\mathcal{P}(\zeta_{A,\Omega}) \subseteq \left(\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{0\}.$$

Complex dimensions of the Cantor dust

Example

Let $A := C^{(1/3)} \times C^{(1/3)}$ be the Cantor dust and $\Omega := [0,1]^2$. Then,

$$\zeta_{A,\Omega}(s) = \frac{8}{s(3^s - 4)} \left(\frac{I(s)}{6^s} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)} \frac{\sqrt{\pi}}{6^s s(3^s - 2)} + E(s; 6^{-1}) \right),$$

where $I(s) = 2^{-1}\mathrm{B}_{1/2}\left(1/2, (1-s)/2\right)$ is entire.

$$\mathcal{P}(\zeta_{A,\Omega}) \subseteq \left(\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \right) \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \right) \cup \{0\}.$$

- $\mathrm{B}_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$; the incomplete beta func.

The relative tube zeta function

(A, Ω) an RFD in $\mathbb{R}^N$ and fix $\delta > 0$

- the **tube zeta function of** (A, Ω):

$$\widetilde{\zeta}_{A,\Omega}(s;\delta) := \int_0^\delta t^{s-N-1} |A_t \cap \Omega| \, \mathrm{d}t$$

- dependence on δ is inessential

The relative tube zeta function

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$$\widetilde{\zeta}_{A,\Omega}(s; \delta) := \int_0^\delta t^{s-N-1} |A_t \cap \Omega| \, \mathrm{d}t$$

- dependence on δ is inessential
- analogous holomorphicity theorem holds for $\widetilde{\zeta}_{A,\Omega}(s; \delta)$
- a functional equation connecting the two zeta functions:

$$\zeta_{A,\Omega}(s; \delta) = \delta^{s-N} |A_\delta \cap \Omega| + (N - s) \widetilde{\zeta}_{A,\Omega}(s; \delta)$$

Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function** $t \mapsto |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

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Theorem (Simplified pointwise formula with error term)

• $\alpha < \overline{\dim}_B(A,\Omega) < N$; $\zeta_{A,\Omega}$ satisfies suitable rational decay (d-**languidity**) on the half-plane $\mathbf{W} := \{\mathrm{Re}\, s > \alpha\}$, then:

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega},\mathbf{W})} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \omega\right) + O(t^{N-\alpha}).$$

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- if we allow polynomial growth of $\zeta_{A,\Omega}$, in general, we get a tube formula in the sense of Schwartz distributions

The Minkowski measurability criterion

Theorem (Minkowski measurability criterion)

- (A, Ω) is such that $\exists D := \dim_B(A, \Omega)$ and $D < N$
- $\zeta_{A,\Omega}$ is d-languid on a suitable domain $W \supset \{\operatorname{Re} s = D\}$

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(a) (A, Ω) is Minkowski measurable.

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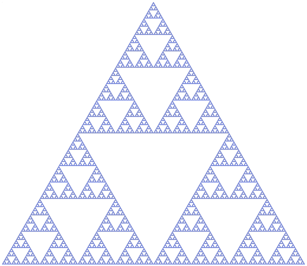
In that case:

$$\mathcal{M}^D(A, \Omega) = \frac{\operatorname{res}(\zeta_{A,\Omega}, D)}{N - D}$$

The Minkowski measurability criterion

- $(a) \Rightarrow (b)$: from the distributional tube formula and the **Uniqueness theorem for almost periodic distributions** due to **Schwartz**
- $(b) \Rightarrow (a)$: a consequence of a **Tauberian theorem** due to **Wiener** and **Pitt** (conditions can be considerably weakened)
- the assumption $D < N$ can be removed by appropriately embedding the RFD in $\mathbb{R}^{N+1}$

Figure: The Sierpiński gasket



- an example of a **self-similar fractal spray** with a generator G being an open equilateral triangle and with **scaling ratios** $r_1 = r_2 = r_3 = 1/2$
- $(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{3} (r_j A, r_j \Omega)$

Fractal tube formula for The Sierpiński gasket

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$$

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$$|A_t| = \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta), \omega\right)$$

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$$= t^{2-\log_2 3}\,\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{+\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k} + \left(\frac{3\sqrt{3}}{2}+\pi\right)t^2,$$

valid pointwise for all $t \in (0, 1/2\sqrt{3})$.

The devil's staircase RFD

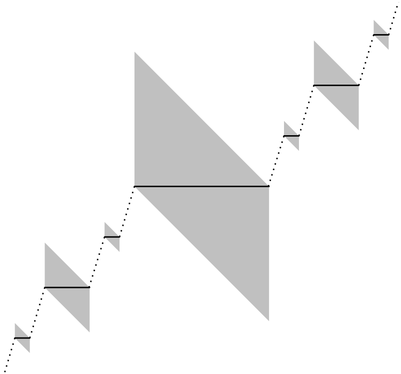

Figure: The third step in the construction of the **Cantor graph relative fractal drum** (A, Ω). One can see, in particular, the sets B_k, $\triangle_k$ and $\widetilde{\triangle}_k$ for $k = 1, 2, 3$.

The devil's staircase RFD

Let A be the devil's staircase and Ω.

$$\zeta_{A,\Omega}(s) = \frac{2}{s(3^s - 2)(s-1)}, \quad \text{for all } s \in \mathbb{C}. \tag{8}$$

$$\mathcal{P}(\zeta_{A,\Omega}) := \mathcal{P}(\zeta_{A,\Omega}, \mathbb{C}) = \{0,1\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right), \tag{9}$$

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$$\begin{aligned} |A_t \cap \Omega| &= \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega})} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A,\Omega}(s), \omega\right) \\ &= 2t^{2-D_{CF}} + t^{2-D_{CS}} G_{CF}\left(\log_3 t^{-1}\right) + t^2, \end{aligned} \tag{10}$$

where $\omega_k := \log_3 2 + \mathbb{i}k\mathbf{p}$ (for each $k \in \mathbb{Z}$),
$D_{CF} = \dim_B(A, \Omega) = 1$, $D_{CS} = \log_3 2$ and $\mathbf{p} := 2\pi/\log 3$.
G_{CF} is a nonconstant 1-periodic function on $\mathbb{R}$, which is bounded away from zero and infinity.

Gauge Minkowski content [HeLap]

If (A,Ω) is Minkowski degenerate, $\exists D := \dim_B(A,\Omega)$ and

$$|A_t \cap \Omega| = t^{N-D}(F(t) + o(1)) \quad \text{as } t \to 0^+, \tag{11}$$

where $F(t) = h(t)$ or $F(t) = 1/h(t)$ for $h : (0,\varepsilon_0) \to (0,+\infty)$, $h(t) \to +\infty$ as $t \to 0^+$ and $h \in O(t^\beta)$ for $\forall \beta < 0$.

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- h is called a **gauge function of slow growth to** $+\infty$ **at** 0^+
- $1/h$ is called a **gauge function of slow decay to** 0 **at** 0^+
- typical gauge functions: $\left(\log^{\circ k} t^{-1}\right)^a$ for $a \in \mathbb{R}^*, k \in \mathbb{N}$

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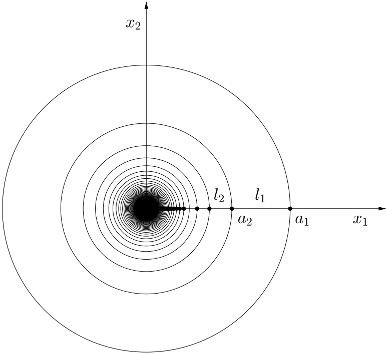
- h-**Minkowski content**: $\mathcal{M}^D(A,\Omega,h) = \lim_{t\to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D}h(t)}$.

The fractal nest generated by the a-string



$a > 0$, $a_j := j^{-a}$, $\ell_j := j^{-a} - (j+1)^{-a}$, $\Omega := B_{a_1}(0)$

$$\zeta_{A_a,\Omega}(s) = \frac{2^{2-s}\pi}{s-1} \sum_{j=1}^{\infty} \ell_j^{s-1}(a_j + a_{j+1})$$

Fractal tube formula for the fractal nest generated by the a-string

Example

$$\mathcal{P}(\zeta_{A_a,\Omega}) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}$$

$a \neq 1,\ D := \frac{2}{1+a} \Rightarrow$

$$|(A_a)_t \cap \Omega| = \frac{2^{2-D} D\pi}{(2-D)(D-1)} a^{D-1} t^{2-D} + 2\pi\big(2\zeta(a) - 1\big) t + O\big(t^{2-\frac{1}{a+1}}\big),\ \text{as } t \to 0^+$$

Fractal tube formula for the fractal nest generated by the a-string

Example

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$$|(A_1)_t \cap \Omega| = \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1,\Omega}(s), 1\right) + o(t) = 2\pi t(-\log t) + \text{const} \cdot t + o(t) \quad \text{as } t \to 0^+$$

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- a pole ω of order m generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \ldots, m$ in the fractal tube formula

Fractal tube formula for the $1/2$-square fractal

$$\zeta_A(s) = \frac{2^{-s}}{s(s-1)(2^s-2)} + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{12}$$

$$D(\zeta_A) = 1, \quad \mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup \left(1 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right). \tag{13}$$

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$$\begin{aligned} |A_t| &= \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s), \omega\right) \\ &= \frac{1}{4\log 2} t \log t^{-1} + t\, G\left(\log_2 (4t)^{-1}\right) + \frac{1+2\pi}{2} t^2, \end{aligned} \tag{14}$$

valid for all $t \in (0, 1/2)$, where G is a nonconstant 1-periodic function on $\mathbb{R}$ bounded away from zero and ∞.

The $1/2$-square fractal is **critically fractal** in dimension 1.

The $1/3$-square fractal

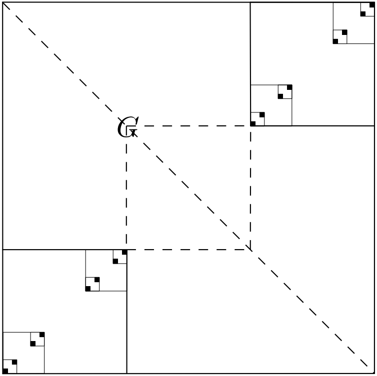

Figure: Here, G is the single generator of the corresponding self-similar spray or RFD (A, Ω), where $\Omega := (0,1)^2$.

Fractal tube formula for the $1/3$-square fractal

$$\zeta_A(s) = \frac{2}{s(3^s-2)}\left(\frac{6}{s-1} + Z(s)\right) + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{15}$$

$$\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) \subseteq \{0\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{1\}, \tag{16}$$

$$\begin{aligned} |A_t| &= \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A, \omega\right) \\ &= 16t + t^{2-\log_3 2} G\left(\log_3(3t)^{-1}\right) + \frac{12+\pi}{2}t^2. \end{aligned} \tag{17}$$

valid for all $t \in (0, 1/\sqrt{2})$, where G is a nonconstant 1-periodic function on $\mathbb{R}$ bounded away from zero and infinity.
The $1/3$-square fractal is **subcritically fractal** in dimension $\omega = \log_3 2 < \dim_B A = 1$.

The Cantor set of second order

Example

C the standard middle-third Cantor set in $[0,1]$, $\Omega := (0,1)$.
$G := \Omega \setminus C$; scaling ratios $r_1 = r_2 = 1/3$.

$$\zeta_{C_2,\Omega_2}(s) = \frac{3^s}{3^s - 2}\,\zeta_{C,\Omega}(s) = \frac{3^s}{2^{s-1}s(3^s-2)^2}$$

$$\mathcal{P}(\zeta_{C_2,\Omega_2}) = \{0\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right)$$

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$G, H\colon \mathbb{R} \to \mathbb{R}$ nonconstant, periodic with $T = \log 3$.

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$G, H \colon \mathbb{R} \to \mathbb{R}$ nonconstant, periodic with $T = \log 3$.

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Higher order Cantor sets

Example (The Cantor set of n-th order)

Define (C_n, Ω_n) as a fractal spray generated by (C_{n-1}, Ω_{n-1}) and scaling ratios $r_1 = r_2 = 1/3$ for $n \geq 2$.

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$$|(C_n)_t \cap \Omega_n| = t^{1-\log_3 2} \sum_{k=0}^{n-1} (\log t^{-1})^k G_k(\log t^{-1}) + 2 \cdot (-1)^n t$$

$G_k \colon \mathbb{R} \to \mathbb{R}$ nonconstant, periodic with $T = \log 3$.

The Cantor set of infinite order

Example

Let $(C_1, \Omega_1) := (C, \Omega)$ and

$$(C_\infty, \Omega_\infty) := \bigsqcup_{n=1}^{\infty} \frac{1}{3^n\, n!}(C_n, \Omega_n).$$

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Holomorphic on $\{\operatorname{Re} s > 0\} \setminus \left(\log_3 2 + \frac{2\pi \mathbb{i}}{\log 3} \mathbb{Z} \right)$.

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$$\zeta_{C_\infty, \Omega_\infty}(s) = \frac{2}{6^s s} \sum_{n=1}^{\infty} \frac{1}{(n!)^s (3^s - 2)^n}$$

Holomorphic on $\{\operatorname{Re} s > 0\} \setminus \left(\log_3 2 + \frac{2\pi \mathbb{i}}{\log 3}\mathbb{Z}\right)$.

$$|(C_\infty)_t \cap \Omega_\infty| = t^{1-\log_3 2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (\log t^{-1})^k G_{k,n}(\log t^{-1}) + O(t)$$

$G_{k,n} \colon \mathbb{R} \to \mathbb{R}$ nonconstant, periodic with $T = \log 3$.

Further research directions

- Riemann surfaces generated by relative fractal drums
- Extending the notion of complex dimensions to include complicated “mixed” singularities/branching points and connecting them with various gauge functions
- Obtaining corresponding tube formulas and gauge-Minkowski measurability criteria
- Applying the theory to problems from dynamical systems

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