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Classifications of parabolic Dulac germs

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Dulac or almost regular germs

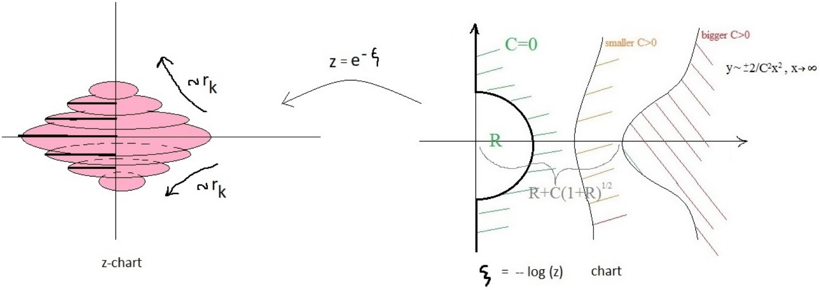
Definition [Ilyashenko].
Parabolic almost regular germ (*Dulac germ*):

- $f \in C^\infty(0, d)$
- extends to a holomorphic germ f to a *standard quadratic domain* Q:

$$Q = \Phi(\mathbb{C}_+ \setminus \overline{K(0,R)}),\ \Phi(\eta) = \eta + C(\eta+1)^{\frac{1}{2}},\ C,\ R > 0,$$

in the *logarithmic chart* $\xi = -\log z$.

Standard quadratic domain



$$r_k := r(\varphi_k) \sim e^{-C\sqrt{\frac{|k|\pi}{2}}}, \ k \to \pm\infty,$$
$$\varphi_k \in \big((k-1)\pi, \ (k+1)\pi\big)$$

- f admits the *Dulac* asymptotic expansion:

$$f(z) \sim_{z\to 0} 1\cdot z + \sum_{k=1}^{\infty} z^{\alpha_i} P_i(-\log z),$$

$$\text{i.e. } f(z) - z - \sum_{i=1}^{n} z^{\alpha_i} P_i(-\log z) = O(z^{\alpha_n}),\ n\in\mathbb{N},$$

  - $\alpha_i > 1$, strictly increasing to $+\infty$,
  - α_i finitely generated [1],
  - P_i *polynomials*.

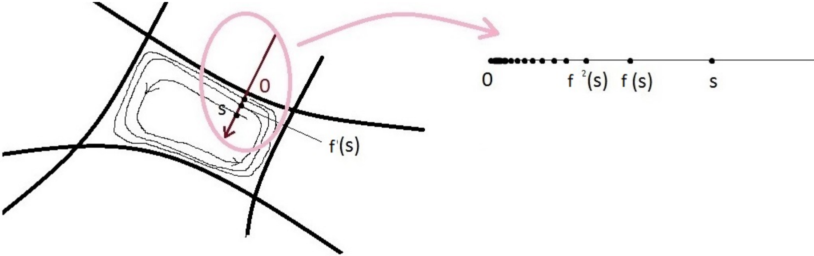
- $\mathbb{R}_+$ invariant under f (i.e. coefficients of $\widehat{f}$ real!)

[1] There exist β_k, $k = 1\dots n$, such that: $\alpha_i \in \mathbb{N}\beta_1 + \dots + \mathbb{N}\beta_n$.

Motivation and history

- *first return maps* for polycycles with hyperbolic saddle singular points – n saddle vertices with hyperbolicity ratios $\beta_i > 0$ (Dulac)
- locally at the saddle

$$\begin{cases} \dot{x} = x + \text{h.o.t.} \\ \dot{y} = -\beta_i y + \text{h.o.t.} \end{cases}$$



Motivation and history

- *Dulac's problem*: accumulation of limit cycles on a hyperbolic polycycle possible?
- limit cycles = fixed points of the first return map
- Dulac: accumulation $\Rightarrow$ trivial power-log asymptotic expansion of the first return map $\Rightarrow$ trivial germ on $\mathbb{R}_+$ (Dulac's mistake)
- the problem: Dulac asymptotic expansion does not uniquely determine f on $\mathbb{R}_+$ (add **any exponentially small term w.r.t.** x!), e.g.

$$f(x) \sim x + x^2 - \log x,\ \ f(x) + e^{-1/x} \sim x + x^2 - \log x,\ \ x \to 0$$

- Ilyashenko's solution: first return maps extendable to a SQD
- SQD *sufficiently large complex domain*: by a variant of maximum modulus principle (*Phragmen-Lindelöf*), Dulac's expansion uniquely determines the germ on a SQD!

Questions

⋆ goal: theory like the standard theory of Birkhoff, Ecalle, Voronin, Kimura, Leau etc. for parabolic analytic germs $\mathrm{Diff}(\mathbb{C}, 0)$

- **formal classification** of parabolic Dulac germs – by a **sequence** (!!! not necesarily convergent) of *formal power-logarithmic changes of variables*

$$\widehat{g} = \widehat{\varphi}^{-1} \circ \widehat{f} \circ \widehat{\varphi},$$

  $\widehat{f}, \widehat{g}$ Dulac expansions,
  $\widehat{\varphi}(z) = z + h.o.t.$ diffeo- with power-log asymptotic expansion

- **analytic classification** of parabolic Dulac germs

$$g = \varphi^{-1} \circ f \circ \varphi,$$

  $f,\ g$ Dulac germs on Q, $\varphi(z) = z + o(z)$ analytic on Q

- φ admits $\widehat{\varphi}$ as its asymptotic expansion?

- simpler question: is a Dulac germ **analytically embeddable** in a flow of an analytic vector field $\xi(z)\frac{d}{dz}$ defined on a standard quadratic domain? (= describe *trivial* analytic class)

$$g = \varphi^{-1} \circ f_0 \circ \varphi,$$

$f, \ f_0$ Dulac germs,
f_0 time-one map of an analytic vector field,
φ analytic.

Example

$f(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z}$ time-one map of $z^2\frac{d}{dy}$.

Historical results - germs of *parabolic analytic diffeomorphisms*

(Fatou $\sim$ end of 19^{th} century; Birkhoff~ 1950; Ecalle, Voronin$\sim 1980, \ldots$)

$f \in \mathrm{Diff}(\mathbb{C}, 0)$, $f(z) = z + a_1 z^{k+1} + a_2 z^{k+2} + \ldots, \quad k \in \mathbb{N}$

- Formal embedding

= formal reduction to a **time-one map of a vector field**:

$$f_0(z) = \mathrm{Exp}\Big(\frac{z^{k+1}}{1+\rho z^k}\frac{d}{dx}\Big).\mathrm{id} = z + z^{k+1} + \Big(\rho + \frac{k+1}{2}\Big)z^{2k+1} + \ldots$$

Step-by-step elimination of monomials from f:

$$\varphi_\ell(z) = \begin{cases} az, \ a \neq 1, \\ z + cz^\ell, \ \ell \in \mathbb{N} \end{cases} \leftrightarrow \widehat{\varphi}(z) = az + \sum_{k=2}^{\infty} c_k z^k \in \mathbb{C}[[z]]$$

(formal changes of variables)

$\Rightarrow \ (k, \rho), \ k \in \mathbb{N}, \ \rho \in \mathbb{C} \ldots$ formal invariants for f.

Historical results - germs of *analytic diffeomorphisms*

- Is f analytically embeddable, or just formally?

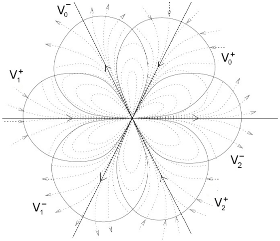
$\leftrightarrow$ Does $\widehat{\varphi}$ converge to an analytic function at 0?

Leau-Fatou flower theorem (1987):

$\star$ $2k$ analytic conjugacies φ_i of f to f_0, all expanding in $\widehat{\varphi}$

$\star$ defined on $2k$ *petals* invariant under local discrete dynamics

$\star$ k attracting directions: $(-a_1)^{-\frac{1}{k}}$; k repelling directions: $a_1^{-\frac{1}{k}}$



$k = 3 \rightarrow$ 6 petals, $f(z) = z + z^4 + \ldots$

$\rightarrow$ in general, analytic embedding in a flow **only on open sectors**

$\rightarrow$ the analytic class of f in direct relation with this question

FORMAL CLASSIFICATION OF DULAC GERMS

Formal embedding into flows for Dulac germs (non-analytic at 0)

• elimination **term-by-term** by an *adapted* 'sequence' of non-analytic *elementary changes of variables*:

$$\varphi(z) = az;\ \ \varphi_{\alpha,m}(z) = z + cz^\alpha \boldsymbol{\ell}^m,\ m \in \mathbb{Z},\ \alpha > 0,\ (\alpha, m) \succ (1, 0).$$

Example (MRRZ, 2016)

0. $f(z) = z - z^2\boldsymbol{\ell}^{-1} + z^2 + z^3$,

1. $\varphi_1(z) = z + c_1 z\boldsymbol{\ell},\ c_1 \in \mathbb{C}$,

   $f_1(z) = \varphi_1^{-1} \circ f \circ \varphi_1(z) = z - z^2\boldsymbol{\ell}^{-1} + a_1 z^2 \boldsymbol{\ell} + h.o.t,$

2. $\varphi_2(z) = z + c_2 z\boldsymbol{\ell}^2,\ c_2 \in \mathbb{R}$,

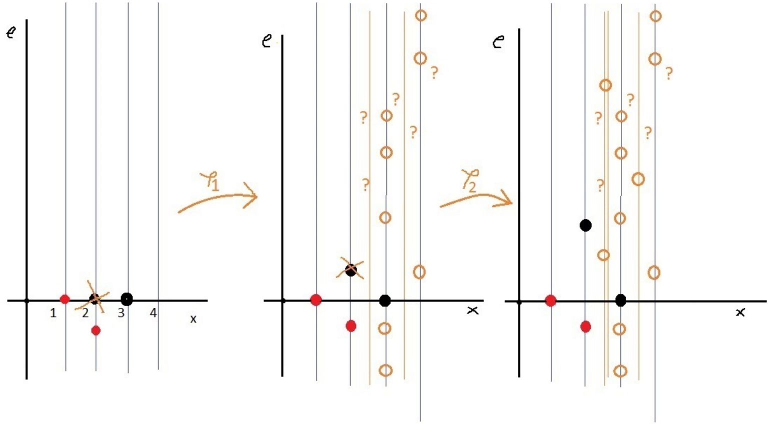
   $f_2(z) = \varphi_2^{-1} \circ f \circ \varphi_2(z) = z + z^2\boldsymbol{\ell}^{-1} \qquad + a_2 z^2 \boldsymbol{\ell}^2 + h.o.t,$

3. $\varphi_3(z) = z + c_3 z\boldsymbol{\ell}^3,\ c_3 \in \mathbb{R}$,

   $f_3(z) = \varphi_3^{-1} \circ f \circ \varphi_3(z) = z + z^2\boldsymbol{\ell}^{-1} \qquad\qquad + a_2 z^2 \boldsymbol{\ell}^3 + h.o.t,$

   $\vdots$

The visualisation of the reduction procedure



the control of the support!

The description of the formal change of variables

• more than just a *formal series composition* of changes of variables: a transfinite composition, $\to$ produces a **transseries** $\widehat{\varphi}$:

⋆ in the process, prove that *every change has its successor change*

⋆ prove the *formal convergence* of composition of changes of variables: by **transfinite induction**[1] in the *formal topology*[2]

---

[1] a generalization of the mathematical induction from $\mathbb{N}$ to ordinal numbers: existence of a *successor element* and a *limit element*,
[2] i.e. in each step of composition the support remains well-ordered; the coefficient of each monomial in the support stabilizes in the course of composition.

A broader class closed to embeddings: the class of power-log transseries $\widehat{\mathcal{L}}$

...contains both the Dulac germ expansions $f \mapsto \widehat{f}$ and the *formal changes of variables*

$$\widehat{\mathcal{L}} \ldots \ \widehat{f}(z) = \sum_{\alpha \in S} \sum_{k=N_\alpha}^{\infty} a_{\alpha,k} z^\alpha \boldsymbol{\ell}^k, \ a_{\alpha,k} \in \mathbb{R}, \ N_\alpha \in \mathbb{Z},$$

$S \subseteq (0, \infty)$ well-ordered (here: finitely gen.)

Similarly we define $\widehat{\mathcal{L}}_2$, $\widehat{\mathcal{L}}_3$, etc. and

$$\widehat{\mathfrak{L}} := \cup_{k \in \mathbb{N}} \widehat{\mathcal{L}}_k.$$

(iterated logarithms admitted!)

Theorem (Formal embedding theorem for Dulac germs, MRRZ 2016)

$\widehat{f}(z) = z - az^{\alpha}\boldsymbol{\ell}^m + h.o.t.$ *parabolic Dulac*, $a > 0$, $\alpha > 1$, $m \in \mathbb{N}_-$.

$\Rightarrow$ *formally in* $\widehat{\mathcal{L}}$ *conjugated to:*

$$f_0(z) = \exp\Big(\frac{-z^{\alpha}\boldsymbol{\ell}^m}{1 - \frac{\alpha}{2}z^{\alpha-1}\boldsymbol{\ell}^k + \big(\frac{k}{2} - \rho\big)z^{\alpha-1}\boldsymbol{\ell}^{k+1}}\frac{d}{dz}\Big).\mathrm{id} =$$

$$= z - z^{\alpha}\boldsymbol{\ell}^m + \rho z^{2\alpha-1}\boldsymbol{\ell}^{2m+1} + h.o.t.$$

- $\star$ (α, m, ρ), $\rho \in \mathbb{R}\ldots$ formal invariants for Dulac germ
- $\star$ $f_0(z)$ a time-one map of an analytic vector field on SQD $(\mathbb{Q}_+)$

Example continued

Example (continued)

$$f_0(z) = \exp\Big(-\frac{z^2\boldsymbol{\ell}^{-1}}{1 - z\boldsymbol{\ell}^{-1} + \big(b - \frac{1}{2}\big)z}\Big).\mathrm{id} =$$

$$= z - z^2\boldsymbol{\ell}^{-1} + bz^3\boldsymbol{\ell}^{-1} + h.o.t.,$$

$f_0 = \widehat{\varphi}^{-1} \circ \widehat{f} \circ \widehat{\varphi}, \quad \widehat{\varphi} \in \widehat{\mathcal{L}}$ – a *transfinite* change of variables

ANALYTIC CLASSIFICATION OF DULAC GERMS

Choice of analytic conjugacy - analytic on standard quadratic domain

Definition [MRR, in progress] f and g Dulac on SQD Q are *analytically conjugated* if there exists

- $\varphi(z) = z + o(z)$ analytic on Q
- $g = \varphi^{-1} \circ f \circ \varphi$ on Q.

$\Rightarrow$ φ admits asymptotic expansion in $\widehat{\mathfrak{L}}$
$\Rightarrow$ f and g formally conjugated in $\widehat{\mathcal{L}}$ $\Rightarrow$ expansion in $\widehat{\mathcal{L}} \subset \widehat{\mathfrak{L}}$.

Another possible classification: $\varphi \in \mathbb{R}\{z\}$ (non-ramified)

The (formal) Fatou coordinate and Abel equation " = " (formal) embedding in a vector field

'Equivalent' problems:

1. (formal) conjugation of f to f_0 (time-one map of an analytic vector field)
2. (formal) Fatou coordinate for f

$$\Psi(f(z)) - \Psi(z) = 1 \quad \text{(Abel equation)}$$
$$\widehat{\Psi}(\widehat{f}(z)) - \widehat{\Psi}(z) = 1 \quad \text{(formal Abel equation)}$$

$\Psi = \Psi_0 \circ \varphi$, $\widehat{\Psi} = \Psi_0 \circ \widehat{\varphi}$

Historical results - construction of the Ecalle-Voronin moduli of analytic classification for Diff$(\mathbb{C}, 0)$

- ⋆ simplest formal class $(k = 1, \rho = 0)$;
  $f_0(z) = \mathrm{Exp}(z^2 \frac{d}{dz}) = \frac{z}{1-z}$
- ⋆ $f \in$ Diff$(\mathbb{C}, 0)$, $f(z) = z + z^2 + z^3 + o(z^3)$

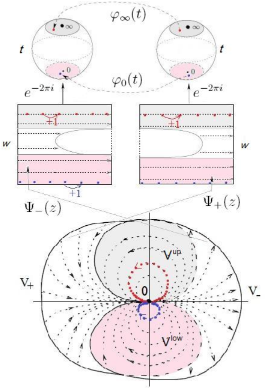
$\Psi(f(z)) - \Psi(z) = 1$ (Abel equation)

Fatou, 1919:

- unique (up to aditive constant) formal solution $\widehat{\Psi}(z) \in -1/z + z\mathbb{C}[[z]]$,
- unique (up to aditive constant) analytic solutions $\Psi_\pm(z)$ on petals $V_\pm$
- $\Psi_\pm$ admit $\widehat{\Psi}(z)$ as asymptotic expansion

$\rightarrow$ **Fatou coordinates, sectorial trivialisations**

Ecalle-Voronin moduli of analytic classification for $\mathrm{Diff}(\mathbb{C}, 0)$



Ecalle, Voronin: spaces of attr./repelling orbits (spheres!) "glued" at closed orbits (poles!) by 2 germs of diffeomorphisms:

$$\varphi_0(t) := e^{-2\pi i \Psi_- \circ (\Psi^+)^{-1}(-\frac{\log t}{2\pi i})},\ t \approx 0,$$

$$\varphi_\infty(t) := e^{-2\pi i \Psi_+ \circ (\Psi^-)^{-1}(-\frac{\log t}{2\pi i})},\ t \approx \infty$$

Ecalle-Voronin moduli of analytic classification for $\mathrm{Diff}(\mathbb{C},0)$

Identifications:

$$\Big(\varphi_0(t),\varphi_\infty(t)\Big)\equiv\big(a\varphi_0(bt),\frac{1}{b}\varphi_\infty(\frac{t}{a})\big),\ a,b\in\mathbb{C}^*$$

(choice of constant in $\Psi_\pm$, i.e. coordinates on spheres)

Theorem Ecalle-Voronin: After identifications, $(\varphi_0,\varphi_\infty)$ are analytic invariants.

Realisation theorem: Each pair $(\varphi_0,\varphi_\infty)$ tangent to identity can be *realized as E-V modulus* of a germ from the model formal class.

Trivial modulus $(\mathrm{id},\mathrm{id})\leftrightarrow$ analytically embeddable germs

Invariant domains (petals) for the local dynamics of a parabolic Dulac germ

L-F-like theorem, Dulac germs [MRR, in progress].

$f(z) = z + az^{\alpha}\boldsymbol{\ell}^m + \ldots$ Dulac germ on a SQD Q, $a \in \mathbb{R}$, $\alpha > 1$, $m \in \mathbb{N}_-$.

$\Rightarrow$ countably many overlapping attracting/repelling petals $V_i^{\pm}$, $i \in \mathbb{Z}$, of opening $\frac{2\pi}{\alpha-1}$

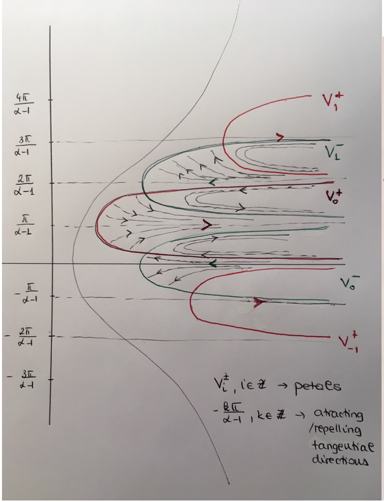
$\Rightarrow$ centered at complex directions

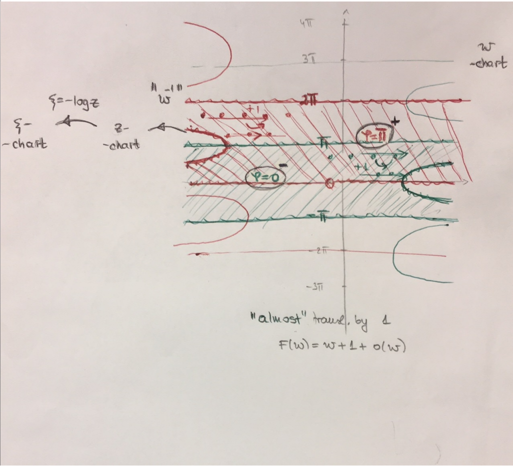
$$(-\mathrm{sgn}(a))^{\frac{1}{\alpha-1}}(\text{attracting}),\ (\mathrm{sgn}(a))^{\frac{1}{\alpha-1}}(\text{repelling})$$

(invariant lines for f tangential to these directions at 0)

Sketch of the proof. In the chart $w = -\frac{1}{a(\alpha-1)} z^{-\alpha+1}\boldsymbol{\ell}^{-m}$ f *almost translation by* 1, *easier construction of invariant domain.*

Dynamics of a Dulac germ (logarithmic chart)





$$f(z) = z + az^{\alpha}\boldsymbol{\ell}^{m} + \ldots, \ a < 0$$

(Formal) Fatou coordinate of a Dulac germ

Theorem [MRRZ2 (2019), MRRp (in progress)]
f Dulac on SQD Q, $\widehat{f}$ its Dulac expansion.

- unique (up to an additive constant) formal Fatou coordinate $\widehat{\Psi}$ for $\widehat{f}$ in class $\widehat{\mathfrak{L}}$ (in $\widehat{\mathcal{L}}_2$)
- unique (up to additive constants) analytic Fatou coordinates $\Psi_j^{\pm}$, $j \in \mathbb{Z}$, on attracting/repelling petals $V_j^{\pm}$
- $\Psi_j^{\pm}$ admit $\widehat{\Psi}$ as transserial asymptotic expansion with respect to integral sums on limit ordinal steps as $z \to 0$ on $V_j^{\pm}$

Caution! Transserial asymptotic expansion is not well-defined (unique), if we do not prescribe a canonical summation method on limit ordinal steps (dictated here by Abel equation)!

Non-uniqueness of asymptotic expansion of a germ in $\widehat{\mathcal{L}}$

$\rightarrow$ ambiguity: choice of the sum in $\boldsymbol{\ell}$ at limit ordinal steps

Example

$f(z) = z + z^2 \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}} + z^5$

Some possible asymptotic expansions:

$$\widehat{f}_1(z) = z + z^2(\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots) + z^5$$
$$\widehat{f}_2(z) = z + z^2(\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots) - z^3 + z^5, \text{ etc.}$$

- $\widehat{f}_1$: canonical (convergent sum) at the first limit ordinal step:

$$\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots \mapsto \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}}$$

- $\widehat{f}_2$: $\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots \mapsto \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}} + e^{-\frac{3}{\boldsymbol{\ell}}} \qquad \left(z = e^{-1/\boldsymbol{\ell}}\right)$

Moreover: (?) canonical choice if series in $\boldsymbol{\ell}$ was **divergent** (the case in the Fatou coordinate)

Sketch of the proof / method of summation

$f(z) \sim \widehat{f}(z) = z + z^{\alpha_1} P_1(-\log z) + z^{\alpha_2} P_2(-\log z) + \dots$

- solve (formal) Abel equation by *blocks*

$$\widehat{\Psi}(z + z^{\alpha_1} P_1(\boldsymbol{\ell}^{-1}) + \dots) - \widehat{\Psi}(z) = 1$$

- $\widehat{\Psi}(z) := \sum z^{\beta_i} \widehat{T}_i(\boldsymbol{\ell})$
- In each step, $\widehat{T}_i$ obtained solving one differential equation:

$$\frac{d}{dz}\Big(z^{\beta_i} \widehat{T}_i(\boldsymbol{\ell})\Big) := z^{\beta_i - 1} R(\boldsymbol{\ell}),$$

$$(*)\ \widehat{T}_i(\boldsymbol{\ell}) = z^{-\beta_i} \int z^{\beta_i - 1} R(\boldsymbol{\ell}) dz,$$

  β_i a finite combination of α_i; R a rational function in $\boldsymbol{\ell}$.

- $(*)$ solvable analytically (T_i analytic on Q) as well as formally ($\widehat{T}_i \in \mathbb{C}[[z]]$) by partial integration
  $\to$ principle of summation at limit ordinal steps: $\widehat{T}_i \mapsto T_i$
  (integral sum)

- $\widehat{\Psi} := \Psi_\infty + \widehat{R}$, where Ψ_∞ contains *only finitely many* infinite blocks

- analytic Fatou coordinate on petals:
  iterative summation of the Abel equation along the orbit of f/f^{-1}, after subtracting sufficiently many blocks:

  $$R(f(z)) - R(z) = \delta(z),$$

  $\delta(z)$ of arbitrarily small order.

  $$\Rightarrow \; R_\pm^j(z) := -\sum_{k=0}^{\infty} \delta(f^{\circ(\pm)k}(z)),\; j \in \mathbb{Z}.$$

  Converges *locally uniformly on petals* $V_\pm^j$.

$Q.E.D.$

Example of blocks computation in the Fatou coordinate of a Dulac germ

Example

$f(z) = z + z^2\boldsymbol{\ell}^{-1} + z^3 \;\Rightarrow\; \Psi(z + z^2\boldsymbol{\ell}^{-1} + z^3) - \Psi(z) = 1.\ (*)$

Computation of the first block of Ψ by formal T. expansion of $(*)$:

$$\Psi_0'(z) z^2 \boldsymbol{\ell}^{-1} = 1 \;\Rightarrow\; \Psi_0(z) = \int z^{-2}\boldsymbol{\ell}\, dz$$

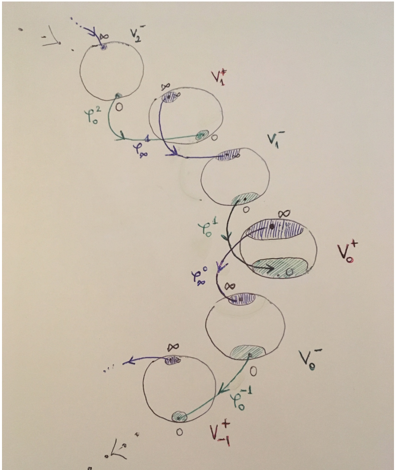
- Integration by parts: $\widehat{\Psi}_0(z) = z^{-1}\sum_{n\in\mathbb{N}} n!\boldsymbol{\ell}^n$ (divergent series in $\boldsymbol{\ell}$ in the first block!)
- Analytic integration on SQD: $\Psi_0(z) = \int_*^z y^{-2}\boldsymbol{\ell}(y)\, dy$

? appropriate sum of divergent series above ? integral sum

$$\sum_n n!\boldsymbol{\ell}^n \mapsto \frac{\int_*^z y^{-2}\boldsymbol{\ell}(y)\, dy}{z^{-1}}.$$

Ecalle-Voronin moduli for Dulac germs

- infinitely many attracting/repelling petals indexed by $\mathbb{Z}$
- neighboring spheres glued at closed orbits by a germ of a diffeomorphism
- infinite necklace of spheres (spaces of orbits on petals), not closed



Ecalle-Voronin moduli for Dulac germs

Theorem E-V for Dulac maps (MRRp)
f and g Dulac in the same $\widehat{\mathcal{L}}$-formal class (α, m, ρ).

- analytic invariants given by a sequence of diffeomorphisms of 0 and ∞ tangent to the identity, up to identifications $(*)$

$$\varphi_0^i(t) := e^{-2\pi i \Psi_+^{i-1} \circ (\Psi_-^i)^{-1}(-\frac{\log t}{2\pi i})}, \ t \approx 0$$

$$\varphi_\infty^i(t) := e^{-2\pi i \Psi_-^{i} \circ (\Psi_+^i)^{-1}(-\frac{\log t}{2\pi i})}, \ t \approx \infty, \ i \in \mathbb{Z}$$

- radii of definition (at least)

$$|t| < R_i \sim K_1 e^{-K e^{C\sqrt{i}}}, \ i \to \infty \quad \text{(SQD)}$$

- identifications $(*)$

$$(\varphi_0^i, \varphi_\infty^i; \ R_i)_{i \in \mathbb{Z}} \equiv (\psi_0^i, \psi_\infty^i; \ \tilde{R}_i)_{i \in \mathbb{Z}}$$

  if $R_i, \ \tilde{R}_i$ bounded as above (possibly different constants) and there exist sequences $(a_i)_{i \in \mathbb{Z}}, \ (b_i)_{i \in \mathbb{Z}}$ in $\mathbb{C}^*$ such that

$$\varphi_0^i(t) = a_{i-1} \cdot \psi_0^i\left(\frac{t}{b_i}\right), \ \varphi_\infty^i(t) = b_i \cdot \psi_\infty^i\left(\frac{t}{a_i}\right), \ i \in \mathbb{Z}.$$

- necklace symmetric w.r.t. $\mathbb{R}_+$-axis
  Proof: Schwarz's reflection lemma,
  $f(\mathbb{R}_+) \subseteq \mathbb{R}_+ \;\Rightarrow \overline{f(\overline{z})} = f(z)$.

$\star$ f embeddable analytically on SQD in a vector field $\leftrightarrow$ modulus trivial, $(\dots, \mathrm{id}, \mathrm{id}, \dots)$

Perspectives and comments

- realization of moduli in wider *generalized Dulac* class
- what can be realized really by Dulac corner maps of one saddle or by first return maps of more saddle polycycles (expected: *periodicity* of modules after finitely many)

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Thank you for the attention!