ZEROS OF CERTAIN DRINFELD MODULAR FUNCTIONS

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Abstract. For every positive integer m, there is a unique Drinfeld modular function, holomorphic on the Drinfeld upper-half plane, $j_m(z)$ with the following t-expansion

$$j_m(z) = \frac{1}{t^{m(q-1)}} + \sum_{i=1}^{\infty} c_m(i) t^{i(q-1)}.$$

These functions are analogs of certain modular functions from the classical theory that have many fascinating properties. For example, they are used to prove the famous denominator formula for the Monster Lie algebra. Here we prove that (as in the classical case) the zeros of $j_m(z)$ in the fundamental domain $\mathscr{F}$ of the Drinfeld upper-half plane Ω for $\Gamma := \mathrm{GL}_2(\mathbb{F}_q[T])$

$$\mathscr{F} := \{z \in \Omega : |z| = \inf\{|z-a| : a \in \mathbb{F}_q[T]\} \geq 1\},$$

are on the unit circle $|z| = 1$. Moreover, if q is odd, the zeros are transcendental over $\mathbb{F}_q(T)$.

1. Introduction and statement of results

The modular functions $j_m(z)$, obtained by the action of the normalized mth weight zero Hecke operator on the classical j-invariant modular function, are holomorphic on $\mathbb{H}$. Therefore we can express them using integer coefficient polynomials $P_m(x)$, as $j_m(z) = P_m(j(z))$. These polynomials satisfy the beautiful identity

$$j(\tau) - j(z) = p^{-1} \exp\left(-\sum_{n=1}^{\infty} P_n(z) \cdot \frac{p^n}{n}\right), \tag{1.1}$$

which is equivalent to the famous denominator formula for the Monster Lie algebra

$$j(\tau) - j(z) = p^{-1} \prod_{m>0 \text{ and } n \in \mathbb{Z}} (1 - p^m q^n)^{c(mn)}.$$

Here $q = e^{2\pi i z}$, $p = e^{2\pi i \tau}$, and the exponents $c(n)$ are defined as the coefficients of

$$j_1(z) = j(z) - 744 = \sum_{n=-1}^{\infty} c(n) q^n.$$

K. Ono has conjectured that all the polynomials $P_m(x)$ are irreducible, and recently P. Guerzhoy proved a partial result toward this conjecture by presenting infinite families

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of these polynomials which are provably irreducible [5]. In addition, it is known that the zeros of each $j_m(z)$ in the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ are located on the unit circle $|z| = 1$ [1].

Here we study the properties of analogous polynomials in setting of Drinfeld modular functions. In this setting we don't have appropriate Hecke operators, so we are not able to derive a formula analogous to (1.1), but still, we can prove that the zeros of the modular function $j_m(z)$ in the fundamental domain are on the unit circle $|z| = 1$.

Let $A = \mathbb{F}_q[T]$ be the ring of the polynomials over the finite field $\mathbb{F}_q$, where $q = p^s$ and $K = \mathbb{F}_q(T)$. Completing K with respect to the absolute value $|\ |$ that corresponds to the degree valuation $-\deg : K \to \mathbb{Z} \cup \{\infty\}$, normalized by $|T| = q$, we obtain the field $K_\infty = \mathbb{F}_q((\frac{1}{T}))$. The completion of the algebraic closure of K_∞ with respect to the absolute value extending $|\ |$ is denoted by C. Now as an analogue of the complex upper half-plane, we define $\Omega := C - K_\infty$ to be the Drinfeld upper half plane. For any $z \in C$ there is an A-lattice $\Lambda_z := Az + A$ in C, and a corresponding Drinfeld module ϕ determined by

$$\phi_T(X) := TX + g(z)X^q + \Delta(z)X^{q^2}.$$

The j-invariant of ϕ, $j(z) := \frac{g(z)^{q+1}}{\Delta(z)}$, is a Drinfeld modular function (meromorphic, weight zero and type zero Drinfeld modular form) for $\Gamma := \mathrm{GL}_2(A)$.

A meromorphic Drinfeld modular function for $\Gamma = \mathrm{GL}_2(A)$, say $f(z)$, has t-expansion

$$f(z) = \sum_{i}^{\infty} a_f((q-1)i)t^{(q-1)i},$$

where as usual $t(z) = e_L^{-1}(\tilde{\pi}z)$. Here $L = \tilde{\pi}A$ is one dimensional lattice corresponding to the Carlitz module ρ that is defined by (see Section 4 of [4])

$$\rho_T = T\tau^0 + \tau = TX + X^q,$$

and $e_L(z)$ is the "Carlitz exponential" function related to L (see Section 2 of [4]).

In particular,

$$j(z) = -\frac{1}{t^{q-1}} + \sum_{i=0}^{\infty} c(i)t^{i(q-1)},$$

where the coefficients $c(i)$ are in A. For every positive integer m, there is a Drinfeld modular function $j_m := P_m(-j)$, where $P_m(x)$ is a degree m polynomial with coefficients in A, which has the following t-expansion

$$j_m(z) = \frac{1}{t^{m(q-1)}} + \sum_{i=1}^{\infty} c_m(i)t^{i(q-1)}.$$

We can define polynomials $P_m(x)$ recursively. Set $P_0(x) := 1$. If

$$(-j(z))^m = \frac{1}{t^{m(q-1)}} + \sum_{j=-(m-1)}^{\infty} a_j t^{j(q-1)}$$

is the t-expansion of function $(-j(z))^m$, then we define $P_m(x) := x^m - \sum_{i=0}^{m-1} a_{-i}P_i(x)$. The polynomials $P_m(x)$ also satisfy the recursive formula (3.2). Since the holomorphic Drinfeld modular functions that is zero at infinity is identically equal to zero, the j_m are well defined. In analogy with classical case, we have the following theorem.

Theorem 1.1. *The roots of the polynomials $P_m(x)$ have absolute value q^q. The zeros of $j_m(z)$ in the fundamental domain $\mathscr{F} = \{z \in \Omega : |z| = \inf\{|z-a| : a \in A\} \geq 1\}$ are on the unit circle $|z| = 1$. If q is odd, they are transcendental over K.*

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3. Proofs of Results

3.1. **Preliminaries.** A *meromorphic Drinfeld modular form for* Γ *of weight* k *and type* l (where $k \geq 0$ is an integer and l is a class in $\mathbb{Z}/(q-1)\mathbb{Z}$) is a meromorphic function $f : \Omega \to C$ that satisfies:

(i) $f(\gamma z) = (\det \gamma)^{-l}(cz+d)^k f(z)$ for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

(ii) f is meromorphic at the cusp ∞.

A Drinfeld modular function is a meromorphic Drinfeld modular form of weight zero and type zero.

For an arbitrary element $a \in A$ of degree d, let $\rho_a = \sum_{i=0}^{d} l_{d-i}(a)X^{q^i}$ be the value of the Carlitz module at a. From Section 4 of [4], we know that

$$\deg l_i(a) = iq^{d-i}, \tag{3.1}$$

and that $l_0(a)$ is equal to the leading coefficient of a. Define the $a{-}th$ inverse cyclotomic polynomial $f_a(X) \in A[X]$ by

$$f_a(X) := \rho_a(X^{-1})X^{|a|} = l_0(a) + l_1(a)X^{q^d - q^{d-1}} + \ldots + l_d(a)X^{q^d - 1}.$$

Next, denote by $t_a(z) := t(az)$. An easy computation shows that

$$t_a = \frac{t^{|a|}}{f_a(t)}$$

as a power series in t with coefficients in A. We have the following t-expansion of the normalized Eisenstein series $g(z) := \tilde{\pi}^{1-q}g(z)$ (see Section 6 of [4])

$$g(z) = 1 - (T^q - T)\sum_{a \in A \text{ monic}} t_a^{q-1}.$$

Also, the t-expansion of the normalized Delta function is given by the infinite product

$$\Delta(z) := \tilde{\pi}^{1-q^2}\Delta(z) = -t^{q-1}\prod_{a \in A \text{ monic}} f_a(t)^{(q^2-1)(q-1)}.$$

3.2. **Coefficients of the t-expansion of j-function.** Here we investigate the absolute values of the t-expansion coefficients of j-function. We are interested in the "big" coefficients, and we'll show that for our purposes we can approximate the j-function with only four terms, namely with $c(-1)t^{-(q-1)} + c(0) + c(q-1)t^{(q-1)(q-1)} + c(q)t^{q(q-1)}$.

Definition 3.1. *We say that the power series $f(t) = \sum a_i t^i$, with coefficients in A, is (c,e)-small if it has the property that the degree of every non-zero coefficient a_i is less than or equal to $e + ic$, where c and e are real numbers. The coefficient a_i of a (c, e)-small power series $f(t)$ is called maximal, if its degree is $e + ic$.*

Similar to the Lemma 2.6.9 of [2], we have the following proposition.

Proposition 3.2. *The series $j(t)$ is $(\frac{q}{q-1}, q)$-small (i.e. $\deg c(i) \leq (i+1)q$), and the only maximal coefficients of $j(t)$ are $c(-1), c(0), c(q-1)$ and $c(q)$, with leading coefficients $-1, 1, 1$ and -1.*

Proof of Proposition 3.2. If $f(t), g(t) \in A[[t]]$ are both $(c, 0)$-small, than it is easy to see that the series $f(t)g(t)$, $f(t) + g(t)$ and $\frac{1}{f(t)}$ are also $(c, 0)$-small. The formula (3.1) implies that the inverse cyclotomic polynomial $f_a(t)$ is $(1/(q-1), 0)$-small, so the same is true for the series

$$(T^q - T)t^{(q-1)|a|} \left(\frac{1}{f_a(t)}\right)^{q-1},$$

when a is of degree greater than zero (i.e. if $|a| \geq q$). In other words, the only "big" coefficients of

$$g(z) = 1 - (T^q - T) \sum_{a \in A \text{ monic}} t_a^{q-1}$$

are the first two nonzero one, i.e. the $g(z)$ is $(q/(q-1), 0)$-small, and its maximal coefficients are 1 and $-(T^q - T)t_1^{q-1} = -(T^q - T)t^{q-1}$. Now, $g(z)^{q+1}$ is also $(q/(q-1), 0)$-small, and its maximal terms are

$$\begin{aligned}
(1 - (T^q - T)t^{q-1})^{q+1} &= (1 - (T^q - T)t^{q-1})^q (1 - (T^q - T)t^{q-1}) \\
&= (1 + (-T^q + T)^q \, t^{q(q-1)})(1 - (T^q - T)t^{q-1}) \\
&= 1 - (T^q - T)t^{q-1} + (-T^q + T)^q \, t^{q(q-1)} + (-T^q + T)^{q+1} t^{(q-1)(q+1)}.
\end{aligned}$$

Using the same reasoning, $\prod_{a \in A \text{ monic}} f_a(t)^{(q^2-1)(q-1)}$ is $(1/(q-1), 0)$-small, hence

$$\frac{g(z)^{q+1}}{\prod_{\substack{a \in A \\ \text{monic}}} f_a(t)^{(q^2-1)(q-1)}}$$

is $(q/(q-1), 0)$-small with the same maximal part as $g(z)^{q+1}$, and finally

$$j(z) = \frac{g(z)^{q+1}}{-t^{q-1} \prod_{\substack{a \in A \\ \text{monic}}} f_a(t)^{(q^2-1)(q-1)}}$$

is $(q/(q-1), q)$-small with the maximal part

$$-t^{-(q-1)} + (T^q - T) - (-T^q + T)^q t^{(q-1)(q-1)} - (-T^q + T)^{q+1} t^{q(q-1)}.$$

□

3.3. **The leading coefficients of the maximal coefficients of j_m.** From the t-expansion

$$j_{m-1}(z)(-j(z)) = t^{-m(q-1)} - c(0)t^{-(m-1)(q-1)} - \ldots - c(m-1) + c_{m-1}(1) + \sum_{i=1}^{\infty} b(i)t^{i(q-1)},$$

it follows that the polynomials $P_m(x)$ satisfy the following recursive relation

(3.2)

$$P_m(x) = P_{m-1}(x)x + (c(0)P_{m-1}(x) + c(1)P_{m-2}(x) + \ldots + c(m-1)P_0(x)) - c_{m-1}(1),$$

where $m \geq 1$ and $P_0(x) := 1$.

To calculate the degrees of the constant coefficients of the polynomials $P_m(x)$, we need to understand the degrees of coefficients $c_{m-1}(1)$, and this can be done by studying relations between the maximal coefficients of all the j_k functions.

Proposition 3.3. *For every positive integer m, the $j_m(t)$ is $(\frac{q}{q-1}, mq)$-small.*

Proof of Proposition 3.3. Follows by induction immediately from Proposition 3.2, the recursive formula (3.2), and the fact that the product of (c, e)-small power series and (c, f)-small power series is $(c, e+f)$-small power series. □

Definition 3.4. *Define $H(x, y) = \sum_{m,n\in\mathbb{Z}, m\geq 0} a_{m,n}x^m y^n$ to be the generating function of the leading coefficients of the maximal coefficients of j_m functions, i.e. $a_{m,n}$ is the leading coefficient of $c_m(n)$ when $c_m(n)$ is maximal coefficient of j_m, zero otherwise. For convenience let $a_{m,n}$ be zero for $m < 0$.*

Proposition 3.5. *The generating function $H(x, y)$ is equal to $\frac{1-x^q}{1-v(x,y)}$, where $v(x, y) = x^q - x^{q+1} + x(y^q - y^{q-1} + \frac{1}{y})$.*

Proof of Proposition 3.5. Comparing the t-expansion coefficients of the both sides of the recursive formula (3.2), after setting x to be $-j$, for $m \geq 1$ and $n \neq 0$, we get the following identity

$$\begin{aligned}(3.3)\ \widetilde{c(-1)}a_{m,n} + \widetilde{c(0)}a_{m-1,n} &+ \widetilde{c(q-1)}a_{m-q,n} + \widetilde{c(q)}a_{m-q-1,n} = \\ \widetilde{c(-1)}a_{m-1,n+1} &+ \widetilde{c(0)}a_{m-1,n} + \widetilde{c(q-1)}a_{m-1,n-q+1} + \widetilde{c(q)}a_{m-1,n-q},\end{aligned}$$

where $\widetilde{c(i)}$ is the leading coefficient of $c(i)$. The generating function $H(x, y)$ is uniquely determined by the relation (3.3) and the "initial condition"

$$a_{0,n} := \begin{cases} 0 & \text{if } n \neq 0, \\ \widetilde{c_0(0)} = 1 & \text{if } n = 0 \end{cases} \tag{3.4}$$

and

$$a_{m,0} := \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases} \tag{3.5}$$

Since $\widetilde{c(-1)} = -1$, $\widetilde{c(q-1)} = 1$ and $\widetilde{c(q)} = -1$, by comparing the coefficients of the both sides of identity $1 + v(x,y)\frac{1}{1-v(x,y)} = \frac{1}{1-v(x,y)}$, we see that the coefficients of the power series $\frac{1}{1-v(x,y)} = \sum_{j=0}^{\infty} v(x,y)^j = \sum_{m,n\in\mathbb{Z},\, m\geq 0} b_{m,n}x^m y^n$(the series converges as a power series in x) satisfy relation (3.3). The coefficients of $\frac{1-x^q}{1-v(x,y)}$ satisfy the same relation, so to finish the proof, we need to check that this function also fulfills the "initial condition". Condition (3.4) is trivially satisfied. The constant coefficient of the denominator (we consider denominator to be a polynomial in x) of the "regular part"

$$R(x,y) := \frac{1-x^q}{1-v(x,y)} - \frac{y}{y-x} = \frac{xy(y-1)(x^q - y^q)}{(x-y)^2(-1 + xy(\frac{y^{q-1}-x^{q-1}}{x-y} + \frac{x^q - y^q}{x-y}))}$$

of the function $\frac{1-x^q}{1-v(x,y)}$ is -1 (since in characteristic $p|\, q$, $x^q - y^q$ is divisible by $(x-y)^2$). This implies that the power expansion of $R(x,y)$(in variable x) does not contain negative powers of y. On the other hand, since the numerator of $R(x,y)$ is divisible by y, it is easy to see that each term of the power expansion is divisible by y, so the condition (3.5) is satisfied (because the only term of the forms $ax^m y^0$ of the power series $\frac{y}{y-x} = \sum_{i=0}^{\infty}(\frac{x}{y})^i$ is $(\frac{y}{x})^0 = 1$). □

Corollary 3.6. *The only maximal coefficient of the form $c_m(1)$ (i.e. $\deg c_m(1) = (m+1)q$), for non-negative integer m, is $c_{q-1}(1)$, and its leading coefficient is* 1.

Proof of Corollary 3.6. This follows from the fact that the derivative of the "regular part" $R(x,y)$ of $H(x,y)$ with respect to y, evaluated at $y = 0$ is x^{q-1} (it is easy to see that we can differentiate with respect to y series $H(x,y) = \frac{1-x^q}{1-v(x,y)} - \frac{y}{y-x}$ "term by term"). □

3.4. **Proof of the Theorem 1.1.** Using the results developed in previous sections, here we show that the Newton polygon of the polynomial $P_m(x)$, with respect to the valuation $-\deg$, is a line of slope q, and than we conclude that all its zeros are of degree q. The theorem then easily follows from the results of Cornelissen, Brown and Yu (see [3], [2] and [7]).

Proof of the Theorem 1.1. First we prove by induction that the constant coefficient of the polynomial $P_m(x)$ has degree mq and the leading coefficient 1. By Proposition 3.2, $\deg c(i) \leq (i+1)q$ so the induction assumption implies that the degree of the constant coefficient of $P_k(x)c(m-1-k)$ is less than or equal to mq. The only terms on the right side of the recursive formula (3.2) that are affecting the degree of the constant coefficient of $P_m(x)$ are the ones corresponding to the maximal coefficients of $j(t)$ (i.e. $\deg c(i) = (i+1)q$) and $c_{m-1}(1)$. According to the Corollary 3.6, $c_{m-1}(1)$ has degree

mq only if $m = q-1$ (in other cases the degree is smaller). We consider the following three cases:

Case 1: Assume that $1 \leq m \leq q-1$.
The only maximal coefficient appearing in the formula (3.2) is $c(0)$ (and its leading coefficient is 1), so the constant coefficient of $P_m(x)$ in this case is equal to the constant coefficient of $P_{m-1}(x)c(0)$, which has degree mq and the leading coefficient 1.

Case 2: Assume that $m = q$.
Now the constant coefficient is equal to the constant coefficient of $P_{q-1}(x)c(0) + P_0(x)c(q-1) - c_{q-1}(1)$. The degree is again mq, and a leading coefficient (Proposition 3.2, Corollary 3.6) is equal to $1 \cdot 1 + 1 \cdot 1 - 1 = 1$.

Case 3: Assume that $m \geq q+1$.
The leading coefficient of the constant coefficient of $P_{m-1}(x)c(0) + P_{m-q}(x)c(q-1) + P_{m-q-1}(x)c(q)$ is equal to $1 \cdot 1 + 1 \cdot 1 + 1 \cdot (-1) = 1$, so the claim follows.

Next, we prove by induction that for $P_m(x) = x^m + a_1 x^{m-1} + \ldots + a_{m-1}x + a_m$ the degree of the coefficient a_i is less than or equal to iq. This follows from the recursive formula (3.2); the induction assumption and the fact that $\deg c(i) \leq (i+1)q$ and $\deg c_m(1) \leq (m+1)q$ (Proposition 3.2, Corollary 3.6).

Now, since $\deg a_i \leq iq$ and $\deg a_m = mq$, we see that the Newton polygon of the polynomial $P_m(x) = x^m + a_1 x^{m-1} + \ldots + a_{m-1}x + a_m$ with respect to the valuation $-\deg$, is a straight line of slope q. Hence the roots of the polynomial $P_m(x)$ have the absolute value q^q (see IV.3 of [6]).

If $z_0 \in \mathscr{F} = \{z \in \Omega : |z| = \inf\{|z-a| : a \in A\} \geq 1\}$ is the zero of j_m, then $-j(z_0)$ is a zero of polynomial $P_m(x)$, hence $|j(z_0)| = q^q$, and the estimates relating $|j(z_0)|$ and $|z_0|$

$$|j(z_0)| = |t(z_0)|^{-(q-1)},$$

$$|t(z_0)| = |\tilde{\pi} z_0 / T^n|^{-q^n},$$

where $n = \lceil \log_q |z_0| \rceil$ and $|\tilde{\pi}| = q^{q/(q-1)}$ (see (2.6.11), (2.6.3) and (2.5.2) of [2]), imply that $|z_0| = 1$ (see the proof of Theorem 3 of [3]).

As in the Corollary 2 of [3], Theorem 5.6 of [7] implies that non-transcendental zeros of $j_m(z)$ are quadratic over K, and hence have complex multiplication. For q odd, an estimate of j-invariant of CM-points (see Theorem 2.8.2 of [2]) shows that these invariants never have absolute value q^q.

□

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