LINEAR RELATIONS FOR COEFFICIENTS OF DRINFELD MODULAR FORMS

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Abstract. Choie, Kohnen and Ono have recently classified the linear relations among the initial Fourier coefficients of weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$, and they employed these results to obtain particular p-divisibility properties of some p-power Fourier coefficients that are common to all modular forms of certain weights. Using this, they reproduced some famous results of Hida on non-ordinary primes. Here we generalize these results to Drinfeld modular forms.

1. Introduction and statement of results

In a recent paper, Y. Choie, W. Kohnen and K. Ono [1] determined all the linear relations among the initial Fourier coefficients of weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$. As a consequence, they identified spaces M_k in which there are universal p-divisibility properties for certain p-power coefficients. As an application, they gave a new proof of a famous theorem of Hida on non-ordinary primes (see Corollary 1.3 of [1]). Here we generalize these results to Drinfeld modular forms.

Let $A = \mathbb{F}_q[T]$ be the ring of the polynomials over the finite field $\mathbb{F}_q$ where $q = p^s$ and $K = \mathbb{F}_q(T)$. Completing K with respect to the absolute value $|\ |$ that corresponds to the degree valuation $\deg : K \to \mathbb{Z} \cup \{-\infty\}$, normalized by $|T| = q$, we obtain the field $K_\infty = \mathbb{F}_q((\frac{1}{T}))$. The completion of the algebraic closure of K_∞ with respect to the absolute value extending $|\ |$ is denoted by C. Now as an analogue of the complex upper half-plane, we define $\Omega := C - K_\infty$ to be the Drinfeld upper half plane.

Throughout, if $k > 0$ is an integer, we denote by $\mathscr{M}_k^l$ the vector space of Drinfeld modular forms for $\Gamma = \mathrm{GL}_2(A)$ of weight k and type l. By the Theorem 2.1.3 of [3], if $q-1|k$, then its dimension is

$$d(k) := \dim_C(\mathscr{M}_k^0) = \left\lfloor \frac{k}{q^2-1} \right\rfloor + 1. \tag{1.1}$$

Every Drinfeld modular form $f(z) \in \mathscr{M}_k^0$ has a t-expansion

$$f(z) = \sum_{i=0}^{\infty} a_f((q-1)i) t^{(q-1)i},$$

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where as usual $t(z) = e_L^{-1}(\tilde{\pi} z)$. Here $L = \tilde{\pi} A$ is the one dimensional lattice corresponding to the Carlitz module ρ that is defined by (see Section 4 of [2])

$$\rho_T = T\tau^0 + \tau = TX + X^q,$$

and $e_L(z)$ is the "Carlitz exponential" function related to L (see Section 2 of [2]). The A-algebra of weight k and type l modular forms with t-expansion coefficients in A is denoted by $\mathscr{M}_k^l(A)$.

As in the classical case, we have Eisenstein series $E^{(k)} \in \mathscr{M}_k^0$ (for convenience set $E^{(0)} = 1$), a Delta-function $\Delta(z) \in \mathscr{M}_{q^2-1}^0$, and a Poincare series $h(z) := P_{q+1,1}(z) \in \mathscr{M}_{q+1}^1$. We also make use of the normalized weight $q-1$ Eisenstein series $g(z) = \tilde{\pi}^{1-q}[1]E^{(q-1)} \in \mathscr{M}_{q-1}^0$. Here we use the notation $[d] := T^{q^d} - T$, for integers $d > 0$. For more information about t-expansions of these functions see [2].

Define $\sigma(k) \in \{0, q-1, 2(q-1), \ldots, q(q-1)\}$ by the relation $k \equiv \sigma(k) (\operatorname{mod} q^2 - 1)$, and for positive integers N we define

(1.2)

$$L_{k,N} := \{(c_0, \ldots, c_{N+d(k)} \in C^{d(k)+N+1}) : \sum_{\nu=0}^{N+d(k)} c_\nu a_f((q-1)\nu) = 0, \text{ for every } f \in \mathscr{M}_k^0\}.$$

This is the space of linear relations satisfied by the first $d(k)+N+1$ Fourier coefficients of all the forms $f(z) \in \mathscr{M}_k^0$. For each $G \in \mathscr{M}_{(q^2-1)N}^0$, define numbers $b(k, N, G; \nu)$ by

$$\frac{Gg^q h}{E^{(\sigma(k))}\Delta^{N+d(k)}} = \sum_{\nu=0}^{N+d(k)} b(k, N, G; \nu) t^{-\nu(q-1)+1} + \sum_{\nu=1}^{\infty} c(k, N, G; \nu) t^{\nu(q-1)+1}. \tag{1.3}$$

Generalizing the work of Choie, Kohnen and Ono (see Theorem 1.1. of [1]), we have the following description of the $L_{k,N}$.

Theorem 1.1. *The map $\phi_{k,N} : \mathscr{M}_{(q^2-1)N}^0 \to L_{k,N}$ defined by*

$$\phi_{k,N}(G(z)) = (b(k, N, G; \nu) : \nu = 0, 1, \ldots, d(k) + N)$$

defines a linear isomorphism between $\mathscr{M}_{(q^2-1)N}^0$ and $L_{k,N}$.

We recall that in the classical case we say that a prime number p is non-ordinary for a normalized Hecke eigenform $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k$ if $a_f(p) \equiv 0 (\operatorname{mod} p)$. Generalizing Theorem 1.2 of [1], which gives a result on non-ordinary primes, we obtain condition which determine Fourier coefficient with a divisibility by generic $[d]$.

Theorem 1.2. *Let d be a positive integer and $f \in \mathscr{M}_k^0(A)$. If $a \geq 0$ and $b > 0$ are integers such that*

$$(q^2 + 1 - \sigma(k))(q^b - 1) + (q^d - 1)a + k - \sigma(k) = (q^b - 1)(q^2 - 1), \tag{1.4}$$

then

$$a_f(q^b(q-2) + 1) \equiv 0 (\operatorname{mod} [d]).$$

Example. Define $g_d := (-1)^{d+1}\tilde{\pi}^{1-q^d} L_d E^{(q^d-1)} \in \mathscr{M}^0_{q^d-1}(A)$, where $L_d = [d][d-1]\dots[1]$. The constant coefficient of the t-expansion of g_d is 1 and $g_d \equiv 1(\mathrm{mod}\,[d])$ (see Section 6 of [2]). Using the notation of Theorem 1.2, for even d and $q \geq 4$, set $b = d$, $k = (q-1)^2 + q^d - 1$ and $a = q^2 - 2q - 2$. Now $\sigma(k) = (q-1)^2$, and the condition (1.4) is satisfied. It follows that for $f(z) = g_d(z)g(z)^{q-1} \in \mathscr{M}^0_{(q-1)^2+q^d-1}(A)$, we have $a_f(q^d(q-2)+1) \equiv 0(\mathrm{mod}\,[d])$. Since $g_d(z) \equiv 1(\mathrm{mod}\,[d])$, we conclude that $a_{g^{q-1}}(q^d(q-2)+1) \equiv 0(\mathrm{mod}\,[d])$.

2. Acknowledgements

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3. Proofs of Results

3.1. **Preliminaries.** A meromorphic *Drinfeld modular form for* Γ *of weight* k *and type* l (where $k \geq 0$ is an integer and l is a class in $\mathbb{Z}/(q-1)$) is a meromorphic function $f : \Omega \to C$ that satisfies:

(i) $f(\gamma z) = (\det \gamma)^{-l}(cz+d)^k f(z)$ for every $\gamma \in \Gamma$,

(ii) f is meromorphic at the cusp ∞.

If f is a meromorphic Drinfeld modular form of weight k and type l, then the t-expansion of f is of the form

$$f = \sum_i a_f((q-1)i + l)t^{(q-1)i+l}.$$

Moreover, if f is a holomorphic (on Ω and at the cusp ∞), then f is called *Drinfeld modular form for* Γ. The space of all Drinfeld modular forms (resp. Drinfeld modular forms with t-expansion coefficients in A) of weight k and type l is denoted by $\mathscr{M}^l_k$ (resp. $\mathscr{M}^l_k(A)$).

We will need the valence formula for meromorphic modular forms (see Section 5 of [2]):

$$\sum_{z\in\Gamma\backslash\Omega}{}' v_z(f) + \frac{v_0(f)}{q+1} + \frac{v_\infty(f)}{q-1} = \frac{k}{q^2-1}, \tag{3.1}$$

where we are summing over the non-elliptic equivalence classes of $z \in \Omega$, and v_z (resp. v_0, resp. v_∞) is the order of f at z (resp. at the elliptic points, resp. at ∞).

For every meromorphic weight two type one Drinfeld modular form $f(z)$, $\omega := f(z)dz$ is a 1-form on the compactification $\overline{\Gamma\backslash\Omega}$ of $\Gamma\backslash\Omega$. If $f(z) = \sum_{n=n_0}^{\infty} a(n)t^n$ is the t-expansion of $f(z)$, and if $\pi : \Omega \to \Gamma\backslash\Omega$ is the quotient map, then we have the following lemma.

Lemma 3.1. *Assuming the notation above, the following is true:*

a) $\mathrm{Res}_\infty \omega = -a(1)/\tilde{\pi}$

b) $\mathrm{Res}_\tau f(z) = \mathrm{Res}_{\pi(\tau)}\omega$ *for each* $\tau \in \Omega$
c) $\sum_{\gamma \in \overline{\Gamma\backslash\Omega}} \mathrm{Res}_\gamma \omega = 0.$

Hence if $f(z)$ is holomorphic on Ω, then we have $a(1) = 0$.

3.2. **Proof of Theorem 1.1.** Here we prove Theorem 1.1.

Proof of Theorem 1.1. First we are going to show that

$$\sum_{\nu=0}^{N+d(k)} b(k, N, G; \nu) a_f((q-1)\nu) = 0 \tag{3.2}$$

for all $G \in \mathscr{M}^0_{(q^2-1)N}$ and all $f(z) = \sum_{\nu=0}^{\infty}((q-1)\nu)t^{(q-1)\nu} \in \mathscr{M}^0_k$. Let us define $V(z) := \dfrac{g^q hG}{E^{(\sigma(k))}\Delta^{N+d(k)}}$. Then (3.2) is equivalent to the statement that the coefficient of t in the t-expansion of $V(z)f(z)$ is zero. A simple calculation shows that $V(z)f(z)$ is a weight two, level one meromorphic Drinfeld modular form, so by Lemma 3.1 it is enough to prove that $\dfrac{g^q hfG}{E^{(\sigma(k))}\Delta^{N+d(k)}}$ is holomorphic on Ω. According to the valence formula (3.1), the zeros of $E^{(\sigma(k))}$ are at elliptic points of Ω with multiplicity $\dfrac{\sigma(k)}{q-1} \le q$. The only zeros of g are also at elliptic points, with multiplicity 1, and so $\dfrac{g^q}{E^{(\sigma(k))}}$ is holomorphic on Ω. Also, Δ has no zeros besides infinity so the claim follows.

The map $\phi_{k,N}$ is obviously linear, and it is also injective since $\phi_{k,N}(G) = (0)$ implies that $V(z)f(z)$ is the holomorphic modular form of weight 2 that vanishes at infinity, hence is 0. Since $d(k)$ functionals $\{a_f(0), a_f(q-1), \ldots, a_f((q-1)(d(k)-1))\}$ form the basis for the dual space $(\mathscr{M}^0_k)^*$, we conclude that $\dim_C L_{k,N} = N+1 = \dim \mathscr{M}_{(q^2-1)N}$ so $\phi_{k,N}$ is isomorphism. ☐

3.3. **Proof of the Theorem 1.2.** We use the normalized Eisenstein series $E^{(\sigma(k))} := -\tilde{\pi}^{-k}(-[1])^{\frac{\sigma(k)}{q-1}} E^{(\sigma(k))} \in \mathscr{M}^0_{\sigma(k)}(A)$ (see Section 6 of [2], we employ the fact that $\sigma(k) < q^2-1$), and the normalized Delta-function $\Delta := \tilde{\pi}^{(1-q^2)}\Delta \in \mathscr{M}^0_{q^2-1}(A)$(see Section 6 of [2]). The t-expansion coefficients of the functions $g(z)$ and $h(z)$ are already the elements of A, and the t-coefficient of the t-expansion of $h(z)$ is -1 (see Section 9 of [2]).

Proof of the Theorem 1.2. Let $u(z) := \dfrac{g(z)^q h(z)}{E^{(\sigma(k))}(z)}$. From the proof of the Theorem 1.1, $u(z)$ is holomorphic on Ω. Define

$$G(z) = u(z)^{q^b-1} g_d(z)^a.$$

Since $k \equiv \sigma(k) (\mathrm{mod}\, (q^2-1))$, (1.4) implies that the weight of G, $(q^2+1-\sigma(k))(q^b-1) + (q^d-1)a$, is of the form $N(q^2-1)$, where N is a positive integer. Thus $G \in \mathscr{M}^0_{(q^2-1)N}$.

An easy calculation shows that $N + d(k) = q^b$, so as in the proof of Theorem 1.1, the t-coefficient of t-expansion of function

$$\frac{Gg^q hf}{E^{(\sigma(k))}\Delta^{N+d(k)}} = \frac{u^{q^b} g_d^a f}{\Delta^{q^b}}$$

is zero. Now from the t-expansions

$$\frac{1}{\Delta(z)} = -t^{-(q-1)} + b_0 + b_1 t^{q-1} + \ldots,$$

$$u(z) = -t + a_2 t^{(q-1)+1} + \ldots,$$

we derive t-expansions

$$\frac{1}{\Delta^{q^b}(z)} = (-1)^{q^b} t^{-q^b(q-1)} + b_0^{q^b} + b_1^{q^b} t^{(q-1)q^b} + \ldots,$$

$$u^{q^b}(z) = (-1)^{q^b} t^{q^b} + a_2^{q^b} t^{q^{b+1}} + \ldots$$

and

$$\frac{u^{q^b}(z)}{\Delta^{q^b}(z)} = (-1)^{q^b}(-1)^{q^b} t^{-(q-2)q^b} + ((-b_0)^{q^b} + (-a_2)^{q^b}) t^{g^b} + \ldots.$$

Since $\Delta(z)$ and $E^{(\sigma(k))}$ are both normalized with coefficients in A, the coefficients of $u(z)$ and $\frac{1}{\Delta(z)}$ are also in A. Finally, from $g_d \equiv 1(\mathrm{mod}\,[d])$ it follows

$$\frac{u^{q^b} g_d^a f}{\Delta^{q^b}} \equiv \ldots + a_f(q^b(q-2)+1)t + \ldots (\mathrm{mod}\,[d]).$$

Hence $a_f(q^b(q-2)+1) \equiv 0(\mathrm{mod}\,[d])$.

$\square$

Remark. It came to our knowledge that the similar results have been obtained independently by S. Choi in [4].

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