

MODULAR FORMS, HYPERGEOMETRIC FUNCTIONS AND CONGRUENCES

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ABSTRACT. Using the theory of Stienstra and Beukers [9], we prove various elementary congruences for the numbers

$$\sum_{\substack{i_1, i_2, \dots, i_k \geq 0 \\ i_1 + i_2 + \dots + i_k = n}} \binom{2i_1}{i_1}^2 \binom{2i_2}{i_2}^2 \cdots \binom{2i_k}{i_k}^2, \quad \text{for } k, n \in \mathbb{N}.$$

To obtain that, we study the arithmetic properties of Fourier coefficients of certain (weakly holomorphic) modular forms.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider the family of elliptic curves given by the Legendre equation

$$y^2 = x(x-1)(x-t), \quad t \in \mathbb{C},$$

whose period integrals

$$\Omega_1(t) = \int_t^1 \frac{dx}{\sqrt{x(x-1)(x-t)}}, \quad \Omega_2(t) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}},$$

are solutions of the differential equation of Picard-Fuch type

$$t(t-1)\Omega''(t) + (2t-1)\Omega'(t) + \frac{1}{4}\Omega(t) = 0.$$

One finds that $\Omega_2(t) = \pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, t\right)$, where

$${}_2F_1(a, b; c, t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n$$

is the Gauss hypergeometric function. This further gives identity

$$(1.1) \quad \theta(\tau) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 16l(\tau)\right) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 l(\tau)^n.$$

Throughout the paper, $\tau \in \mathbb{H}$, $q = e^{\pi i \tau}$, $\theta(\tau) = \left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^2$ is the classical weight 1 theta series, and $l(\tau) = q - 8q^2 + 44q^3 - 192q^4 + \dots$ is the normalized

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elliptic modular lambda function (hauptmoduln for $\Gamma(2)$). For more details see [5].

In this paper, we identify certain (weakly holomorphic) modular forms

$$f(\tau) = P(l(\tau))\theta^k(\tau)\frac{dl(\tau)}{d\tau},$$

for some $P(t) \in \mathbb{Z}[t]$, and $k \in \mathbb{N}$, whose Fourier coefficients are easily understood in terms of elementary arithmetic (e.g. a splitting behavior of primes in the quadratic extensions). Using identity (1.1), we exploit the relation (via formal group theory) between the coefficients of the power series $P(t)_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 16t\right)^k$, and Fourier coefficients of $f(\tau)$ to prove some elementary congruences for the numbers

$$A_k(n) = \sum_{\substack{i_1, i_2, \dots, i_k \geq 0 \\ i_1 + i_2 + \dots + i_k = n}} \binom{2i_1}{i_1}^2 \binom{2i_2}{i_2}^2 \cdots \binom{2i_k}{i_k}^2, \quad \text{for } k, n \in \mathbb{N}.$$

Beukers and Stienstra [9] invented this approach to study congruence properties of Apéry numbers

$$B(n) = \sum_{k=0}^{\infty} \binom{n+k}{k} \binom{n}{k}^2.$$

Using the formal Brauer group of some elliptic K3-surface, they proved that for all primes p , and $m, r \in \mathbb{N}$ with m odd

$$B\left(\frac{mp^r - 1}{2}\right) - a(p)B\left(\frac{mp^{r-1} - 1}{2}\right) + (-1)^{\frac{p-1}{2}} p^2 B\left(\frac{mp^{r-2} - 1}{2}\right) \equiv 0 \pmod{p^r},$$

where $\eta^6(4\tau) = \sum_{n=1}^{\infty} a(n)q^{2n}$.

Many authors have subsequently studied arithmetic properties of $B(n)$'s and discovered similar three term congruence relations for other Apéry like numbers. For related work see [1, 3, 6, 7, 10, 11].

In contrast to these result, the novelty of this paper is that we use families of modular forms, as well as the weakly holomorphic modular form to extract information about congruence properties of numbers $A_k(n)$. In particular, even though Fourier coefficients of weakly holomorphic modular forms do not satisfy three term relation satisfied by coefficients of Hecke eigenforms, they sometimes satisfy three term congruence relation of Atkin and Swinnerton-Dyer type (see [4]), which then give rise to the three term congruence relations satisfied by corresponding Apéry like numbers.

Let Δ be the free subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Note that $\Gamma(2) = \{\pm I\}\Delta$, and that $\Gamma_1(4) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \Delta \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$. Divisors of $\theta(\tau)$, $l(\tau)$, and $1 - 16l(\tau)$ are supported at cusps. For an integer

k , we denote by $M_k(\Delta)$ and $S_k(\Delta)$ the spaces of modular forms and cusp forms of weight k for group Δ .

First we identify some (weakly holomorphic) modular forms for Δ that have “simple” Fourier coefficients.

Theorem 1.1.

For $n \in \mathbb{N}$, let

$$h_n(\tau) = \theta(\tau)^{6n+1} (1 - 16l(\tau))^{\lfloor \frac{n+1}{2} \rfloor} l(\tau)^{2n} = \sum_{m=1}^{\infty} a_n(m) q^m.$$

Then $h_n(\tau) \in S_{6n+1}(\Delta)$, and for a prime $p \equiv (-1)^{n+1} \pmod{4}$, we have that $a_n(p) = 0$.

Theorem 1.2.

a) The modular form

$$f_1(\tau) = l(\tau)(1 - 16l(\tau))\theta(\tau)^5 = \sum_{n=1}^{\infty} b_1(n) q^n \in M_5(\Delta)$$

is the cusp form with complex multiplication by $\mathbb{Q}(i)$. In particular, for a prime $p > 2$ we have

$$b_1(p) = \begin{cases} 2x^4 - 12x^2y^2 + 2y^4 & \text{if } p \equiv 1 \pmod{4}, \text{ and } p = x^2 + y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

b) For an integer $n > 2$, the Fourier coefficients $c_1(n)$ of the weakly holomorphic modular form $g_1(\tau) = l(\tau)^2(1 - 16l(\tau))^2\theta(\tau)^5 = \sum_{n=2}^{\infty} c_1(n) q^n$ satisfy the following congruence relation

$$b_1(n) \equiv 108c_1(n) \pmod{n^3}.$$

c) For integer $n > 1$, let

$$f_n(\tau) = \theta(\tau)^{6n-6} (1 - 16l(\tau))^{\lfloor \frac{n-1}{2} \rfloor} l(\tau)^{2n-2} f_1(\tau) = \sum_{m=1}^{\infty} b_n(m) q^m.$$

Then for a prime $p \equiv (-1)^n \pmod{4}$, we have that $b_n(p) = 0$.

Corollary 1.3. Let $p > 3$ be a prime, and $r \in \mathbb{N}$. If $p \equiv 1 \pmod{4}$, let x and y be integers such that $p = x^2 + y^2$. Denote by $D_3(n) = A_3(n - 1) - 16A_3(n - 2)$. Then the following congruences hold

(1.2)

$$A_3(mp^r - 1) - b_1(p)A_3(mp^{r-1} - 1) + \left(\frac{-1}{p}\right) p^4 A_3(mp^{r-2} - 1) \equiv 0 \pmod{p^r},$$

(1.3)

$$D_3(mp^r-1)-b_1(p)D_3(mp^{r-1}-1)+\left(\frac{-1}{p}\right)p^4D_3(mp^{r-2}-1)\equiv 0 \pmod{p^{r-\frac{(-1)}{2}+1}}.$$

In particular, we have

$$(1.4) \quad A_3(p-1)\equiv \begin{cases} 16x^4 \pmod{p} & \text{if } p\equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p\equiv 3 \pmod{4}. \end{cases}$$

$$(1.5) \quad D_3(p-1)\equiv \begin{cases} \frac{4}{27}x^4 \pmod{p} & \text{if } p\equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p\equiv 3 \pmod{4}. \end{cases}$$

For integers $n \geq 1$ and $m \geq 0$, define the sequences $B_n(m)$ and $C_n(m)$ by the following identities

$$(1-16t)^{\lfloor \frac{n-1}{2} \rfloor} t^{2n-1} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 16t\right)^{6n-1} = \sum_{m=0}^{\infty} B_n(m)t^m,$$

$$(1-16t)^{\lfloor \frac{n-1}{2} \rfloor} t^{2n-2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 16t\right)^{6n-3} = \sum_{m=0}^{\infty} C_n(m)t^m.$$

Corollary 1.4. For $n \in \mathbb{N}$ and a prime $p > 2$, we have that $B_n(p-1) \equiv 0 \pmod{p}$, if $p \equiv (-1)^{n+1} \pmod{4}$. Moreover, $C_n(p-1) \equiv 0 \pmod{p}$ if $p \equiv (-1)^n \pmod{4}$.

Example. Let $p > 2$ be a prime. Consider the coefficients of ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 16t\right)^2$. The corresponding modular form

$$l(\tau)(1-16l(\tau)){}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 16l(\tau)\right)^4 = \sum_{k=1}^{\infty} (-1)^k (\sigma_3(k/2) - \sigma_3(k)) q^k$$

is an Eisenstein series whose p^{th} Fourier coefficient is $\equiv 1 \pmod{p}$ (if k is odd, we define $\sigma_3(k/2)$ to be 0). Hence Lemma 4.2 implies

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{2(p-1-k)}{p-1-k}^2 \equiv 1 \pmod{p},$$

or equivalently $\binom{p-1}{\frac{p-1}{2}}^4 \equiv 1 \pmod{p}$.

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3. MODULAR FORMS FOR Δ

3.1. **Preliminaries.** Let

$$(3.1) \quad \eta(\tau) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n})$$

be a Dedekind eta function (recall $q = e^{\pi i \tau}$). As we mentioned in the introduction

$$(3.2) \quad l(\tau) = \frac{\eta(2\tau)^{16} \eta(\tau/2)^8}{\eta(\tau)^{24}} \quad \text{and}$$

$$(3.3) \quad 1 - 16l(\tau) = \frac{\eta(\tau/2)^{16} \eta(2\tau)^8}{\eta(\tau)^{24}}$$

are modular functions for Δ . They are holomorphic on \mathbb{H} , and $l(\tau) \neq 0, 1/16$ for all $\tau \in \mathbb{H}$.

Modular curve $X(2)$ has three cusps: 0, 1, and ∞ . As functions on $X(2)$, $l(\tau)$ and $1 - 16l(\tau)$ have simple poles at ∞ and zeros of order 1 at cusps 0 and 1 respectively.

It is well known that

$$(3.4) \quad \theta(\tau) = \frac{\eta(\tau)^{10}}{\eta(\tau/2)^4 \eta(2\tau)^4}$$

is a modular form of weight 1 for Δ . It has a zero at cusp 1 of order 1/2 (cusp 1 is irregular).

We will later need the following dimension formula for the spaces of cusp forms. Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of genus g such that $-I \notin \Gamma$. For k odd, [8, Theorem 2.25] gives the following formula for the dimension of $S_k(\Gamma)$

$$(3.5) \quad \dim S_k(\Gamma) = (k-1)(g-1) + \frac{1}{2}(k-2)r_1 + \frac{1}{2}(k-1)r_2 + \sum_{i=1}^j \frac{e_i - 1}{2e_i},$$

where r_1 is the number of regular cusp, r_2 is the number of irregular cusps, and e_i 's are the orders of elliptic points. Since Δ has no elliptic points (it is a free group), it follows that $\dim S_5(\Delta) = 1$.

3.2. **Identities.** In this subsection we list some technical facts and identities which will be used later in the proofs of the theorems. The proofs of these statements are straightforward, so detail are omitted.

Lemma 3.1. *Let*

$$\Psi(\tau) = (1 - 16l(\tau))^{1/2}l^2(\tau)\theta(\tau)^6 = q^2 + \sum_{m=2}^{\infty} a(m)q^{2m}.$$

Then, $\Psi(\tau) \in S_6(\Gamma_0(4))$ is a newform. Moreover $a(2) = 0$, hence the coefficients of the Fourier expansion of $\Psi(\tau)$ in q are supported at integers congruent to 2 mod 4.

Lemma 3.2. *The following identity holds*

$$\frac{\eta(\tau/2 + 1/2)}{\eta(\tau/2)} = \frac{\eta(\tau)^3}{\eta(\tau/2)^2\eta(2\tau)}.$$

Proof. It follows from the product formula (3.1) for Dedekind eta function. \square

The following lemma follows from the previous lemma and equations (3.3) and (3.4).

Lemma 3.3. *We have that*

$$\frac{\theta(\tau + 1)}{\theta(\tau)} = (1 - 16l(\tau))^{1/2}.$$

Denote by $D = \frac{1}{\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$. We will need the following curious identities.

Lemma 3.4. *The following equality holds*

$$-\frac{1}{12}D^4 \left(\frac{1}{\theta(\tau)^3} \right) = (1 - 16l(\tau))l(\tau)\theta(\tau)^5 - 108(1 - 16l(\tau))^2l(\tau)^2\theta(\tau)^5.$$

Proof. By Bol's theorem [2], the lefthand side of the equality is a weakly holomorphic modular form of weight 5 for group Δ . One checks that the initial Fourier coefficients of the both sides of the equality agree, hence the lemma follows. \square

Lemma 3.5. *The following identity holds*

$$D(l(\tau)) = l(\tau)(1 - 16l(\tau))\theta(\tau)^2.$$

Proof. By Bol's theorem [2], $D(l(\tau))$ is a weakly holomorphic modular form of weight 2. The initial Fourier coefficients of both forms agree, hence the claim follows. \square

4. PROOFS OF THE THEOREMS

4.1. Proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. It is easy to check that $h_n(\tau)$ vanishes at cusps (see Section 3.1). Denote by $\nu(\tau) = \theta(\tau)^{12}(1 - 16l(\tau))l(\tau)^4$. If $n = 2k$ is even, then $h_n(\tau) = \theta(\tau)\nu(\tau)^k$. From Lemma 3.1, it follows that $\nu(\tau)$ has Fourier coefficients supported at integers that are congruent 0 mod 4. Since $\theta(\tau) = \sum_{i=0}^{\infty} r_2(i)q^i$ has a property that $r_2(4j-1) = 0$ for $j \in \mathbb{N}$, the claim follows.

If $n = 2k + 1$ is odd, then $h_n(\tau) = h_1(\tau)\nu(\tau)^k$. Therefore, it is enough to prove the statement of the theorem for $h_1(\tau)$. Since, by Lemma 3.1, the Fourier coefficients of $\theta(\tau)^6(1 - 16l(\tau))^{1/2}l(\tau)^2$ are supported at integers that are congruent to 2 mod 4, it follows that the coefficients of $\theta(\tau)^7(1 - 16l(\tau))^{1/2}l^2(\tau)$ are supported at integers congruent to 3 mod 4. The same is true for the coefficients of $h_1(\tau)$ since Lemma 3.1 and Lemma 3.3 imply that

$$h_1(\tau) = \theta(\tau + 1)^7(1 - 16l(\tau + 1))^{1/2}l(\tau + 1)^2.$$

□

Proof of Theorem 1.2. a) It is easy to check that $f_1(\tau)$ is a cusp form (see Section 3.1). The space of cusp forms $S_5(\Delta)$ is one dimensional (see equation (3.5)). Since it contains a CM modular form with the stated properties, the claim follows.

b) It follows from Lemma 3.4.

c) Same as the proof of Theorem 1.1.

□

4.2. Proof of Corollary 1.3 and Corollary 1.4. We need the following result of Beukers [1].

Proposition 4.1 (Beukers). *Let p be a prime and*

$$\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1} dt$$

a differential form with $b_n \in \mathbb{Z}_p$. Let $t(q) = \sum_{n=1}^{\infty} A_n q^n, A_n \in \mathbb{Z}_p$, and suppose

$$\omega(t(q)) = \sum_{n=1}^{\infty} c_n q^{n-1} dq.$$

Suppose there exist $\alpha_p, \beta_p \in \mathbb{Z}_p$ with $p|\beta_p$ such that

$$b_{mp^r} - \alpha_p b_{mp^{r-1}} + \beta_p b_{mp^{r-2}} \equiv 0 \pmod{p^r}, \quad \forall m, r \in \mathbb{N}.$$

Then

$$c_{mp^r} - \alpha_p c_{mp^{r-1}} + \beta_p c_{mp^{r-2}} \equiv 0 \pmod{p^r}, \quad \forall m, r \in \mathbb{N}.$$

Moreover, if A_1 is p -adic unit then the second congruence implies the first.

Lemma 4.2. *Define a differential form*

$$\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1} dt,$$

where $b_n \in \mathbb{Z}_p$. Let $t(q) = \sum_{n=1}^{\infty} A_n q^n$, with $A_n \in \mathbb{Z}_p$ and $A_1 = 1$. Define

$$\omega(t(q)) = \sum_{n=1}^{\infty} c_n q^{n-1} dq.$$

For a prime p , we have that $b_p \equiv c_p \pmod{p}$.

Proof. We have that

$$\sum_{n=1}^{\infty} c_n q^{n-1} = \frac{d}{dq} \left(\sum_{n=1}^{\infty} \frac{b_n}{n} t(q)^n \right).$$

Hence $c_p \pmod{p}$ is equal to the $(p-1)^{th}$ coefficient of $\frac{d}{dq} \left(\frac{b_p}{p} t(q)^p \right)$, which is b_p . \square

Proof of Corollary 1.3. Since $f_1(\tau)$ is an eigenform for the Hecke operator T_p , the assumptions of Proposition 4.1 are satisfied for $c_n = b_1(n)$, $b_n = A_3(n-1)$, $\alpha_p = b_1(p)$, $\beta_p = \left(\frac{-1}{p}\right) p^4$, and $t(q) = 16l(\tau)$. The formula (1.2) follow from identity

$$D(l(\tau)) {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1, 16l(\tau) \right)^3 = f_1(\tau),$$

(which is a consequence of equation (1.1) and Lemma 3.5).

To prove (1.3) we use Theorem 1.2 b) to establish three term congruence relation from Proposition 4.1 between coefficients $c_1(np^r)$ (because there is one satisfied by $b_1(np^r)$'s). Note that assumptions of Proposition 4.1 are satisfied if we take, when $p \equiv 3 \pmod{4}$, $c_n = c_1(n)$, $b_n = D_3(n-1)$, $\alpha_p = b_1(p)$, $\beta_p = \left(\frac{-1}{p}\right) p^4$, and $t(q) = 16l(\tau)$ (since then $b(p) = 0$).

If $p \equiv 1 \pmod{4}$, we need to take $c_n = pc_1(n)$, $b_n = pD_3(n-1)$, $\alpha_p = b_1(p)$, $\beta_p = \left(\frac{-1}{p}\right) p^4$, and $t(q) = 16l(\tau)$, hence we get a congruence relation weaker for one power of p .

Formulas (1.4) and (1.5) follow from Lemma 4.2 with the choice of parameters as above. \square

Proof of Corollary 1.4. We apply Lemma 4.2 to cusp forms $f_n(\tau)$ and $h_n(\tau)$, where we choose $t(q)$ to be the inverse of $l(q)$ under composition. The claim follows from $a_n(p) = b_n(p) = 0$. \square

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