The homology of groups.

Part II : spectral sequences

These notes are an introduction to spectral sequences, illustrated by a few exercises. The exercises are chosen because they prove classical results, or because they nicely illustrate some concepts/techniques, or because they are useful in the context of the stable homology of groups or of functor categories (sometimes for all these reasons at the same time).

The reader is assumed to know basic homological algebra (such as the first two chapters of Weibel's book below), and to have a basic knowledge of algebraic topology. Material relative to spectral sequences can be found in:

- K. S. Brown, Cohomology of groups (Chap VII). Graduate Texts in Mathematics 87, Springer, 1982.
- L. Evens, The cohomology of groups (Chap 7). Oxford Mathematical Monographs. Oxford University Press, 1991.
- J. McCleary, A user's guide to spectral sequences, Cambridge studies in advanced mathematics 58, Cambridge University Press, 1994.
- C. A. Weibel, An introduction to homological algebra (Chap 5). Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.

In these notes, we shall often refer to some chapters of these books, or to specific statements that can be found in these books. When doing so, we just mention the author's name. For example, we write "see Brown, Chap VII.7" for a reference to section 7 of chapter VII of the first book.

Contents

1 A simplified overview of spectral sequences

1.1 The definitions

A *cohomology spectral sequence* of R-modules is a sequence of cochain complexes of R-modules $(E_n, d_n)_{n\geq n_0}$ such that for all $n \geq n_0$ there is an isomorphism of graded R-modules $E_{n+1} \simeq H(E_n, d_n)$.

A cohomology spectral sequence *bounded converges* to a graded R-module H (or that the graded R-module H is the abutment of the spectral sequence) if it satisfies the following two conditions:

1) **Stationnarity.** For all degrees i, there is an index $n(i)$ such that $E^i_n = E^i_{n+1}$ for all $n \geq n(i)$. The value E^i_n, $n \gg 0$, is denoted by E^i_∞.

2) **Isomorphism up to filtration.** For all degrees i, the R-module H^i has a finite filtration, and there is an isomorphism of R-modules $\mathrm{gr}H^i \simeq E^i_\infty$.

The term 'bounded" refers to the fact the the filtration on H is finite (= bounded). Bounded convergence of a spectral sequence $(E_n, d_n)_{n\geq n_0}$ to H is usually denoted by '$E_{n_0} \Rightarrow H$'. In these notes, all the spectral sequences considered will be bounded convergent, so we will drop the term 'bounded', and simply say that $(E_n, d_n)_{n\geq n_0}$ converges to H.

1.2 How to think of a convergent spectral sequence

1. **As an algorithm.** One can think of a convergent spectral sequence $E_{n_0} \Rightarrow H$ as an algorithm to compute the graded R-module H from the initial data E_{n_0} by making successive homology computations. Because of this, one sometimes speaks of 'running a spectral sequence' in the same way as one runs an algorithm.

   The main difference with a genuine algorithm is that we have no formula for the differentials d_n! Indeed, spectral sequences are typically described by theorems like (here $n_0 = 2$):

   Theorem 1. *There is a spectral sequence of R-modules*

   $$E_2 = \text{ some explicit formula } \Rightarrow H\ .$$

   Only the E_2 page is explicitly described[1]. One has to find additional information or clever tricks to compute the homology at each stage.

[1] The reason why an explicit formula of the d_n is lacking is not because the author of the theorem is lazy. It is simply that there is no nice formula for d_n one could work with.

Observe that the filtration on H is not explicit either! Thus if one succeeds in computing E_∞ and finds that $E^2_\infty = R/3R^{\oplus 2}$ then H^2 could be equal to $R/3R^{\oplus 2}$ or $R/9R$, and once again, additional information on H or clever tricks are required to decide what is the good answer.

2. **As a book.** In most examples of spectral sequences, the complexes (E_n, d_n) are actually bigraded (see section 2), so that each E_n can be represented in a two-dimensional way on a sheet of paper. Thus a spectral sequence can be thought of as a book having a picture representing E_n on page n.

   Because of this analogy, the terms (E_n, d_n) are called the *pages* of the spectral sequence, and E_{n_0} the *initial page*. We also sometimes says that one *turns a page* of the spectral sequence when one passes from page n to page $(n+1)$ by computing homology.

3. **As a generalization of cohomology long exact sequences.** If D is a cochain complex and $C \subset D$, there is a long exact sequence:

$$\cdots \xrightarrow{\partial} H^i(C) \to H^i(D) \to H^i(D/C) \xrightarrow{\partial} H^{i+1}(C) \to \cdots \qquad (1)$$

   We can view this long exact sequence as a device to compute the cohomology C from the cohomology of its pieces C and D/C.

   A spectral sequence $E_{n_0} \Rightarrow H$ is a generalization of this. Usually the graded R-module H is the cohomology of some object X (space, complex…) we are interested in, and E_{n_0} is the cohomology of its pieces. The spectral sequence is a procedure to compute the cohomology of X from the cohomology of its pieces. This procedure goes through computing the cohomology relative to successive differentials d_n, which reflect how the various pieces are pasted together to obtain X. The differentials d_n are not explicit, exactly in the same way as[2] the connecting morphism ∂ of (1) is not explicit.

Although there is no explicit formula for the differentials d_n, it is sometimes possible to prove that these differentials are zero because their source or their target is zero. If such a situation occurs, we say that the differential is zero *for lacunary reasons*. The next (rather trivial) exercise is the most extreme example of such a phenomenon. We shall see other ways of computing differentials in section 2.3.

Exercice 2. Lacunary phenomenon. Let $E_{n_0} \Rightarrow H$ be a convergent spectral sequence such that E_{n_0} is concentrated in even degrees. Show that $E_{n_0} = E_\infty$.

[2] In fact, the long exact sequence (1) is nothing but a particularly simple spectral sequence, see exercise 24.

One can also use spectral sequences to prove qualitative properties of H from the qualitative properties of E_{n_0} without making explicit computations. The next two exercises give examples of this.

Exercice 3. Euler characteristics and spectral sequences. Let $\Bbbk$ be a field, and let $(E_n, d_n)_{n\geq n_0}$ be a spectral sequence of $\Bbbk$-vector spaces converging to H. Assume that there is an m such that the total dimension of E_m is finite. Show that the pages E_n for $n \geq m$, E_∞ and H all have finite total dimension, and if χ stands for the Euler characteristic of a complex of finite total dimension then:

$$\chi(E_m) = \chi(E_{m+1}) = \cdots = \chi(E_\infty) = \chi(H) \; .$$

Exercice 4. Serre classes. A *Serre class* of R-modules is a class $\mathcal{C}$ of modules such that for all short exact sequences of modules

$$0 \to M' \to M \to M'' \to 0 \; ,$$

M belongs to $\mathcal{C}$ if and only if M' and M'' both belong to $\mathcal{C}$. For example, R-modules of finite cardinal form a Serre class. If R is noetherian, finitely generated R-modules also form a Serre class.

Assume given a Serre class $\mathcal{C}$ and a spectral sequence $E_{n_0} \Rightarrow H$, such that in each degree i, $E^i_{n_0} \in \mathcal{C}$. Show that for all degrees i, $H^i \in \mathcal{C}$.

1.3 Morphisms of convergent spectral sequences

Before defining morphisms of spectral sequences, we must go back to convergent spectral sequences and define them in more precise way. A *convergent spectral sequence* is a triple $E = ((E_n, d_n)_{n\geq n_0}, H, \phi)$ where:

- $(E_n, d_n)_{n\geq n_0}$ is a cohomology spectral sequence of R-modules which is stationnary (hence E_∞ exists),
- H is a filtered graded R-module, with finite filtration in each degree,
- ϕ is an isomorphism of graded modules between E_∞ and gr$H$.

In particular, we insist on the fact that the filtration on H and the isomorphism ϕ are part of the data of a convergent spectral sequence. The precise meaning of the notation $E_{n_0} \Rightarrow H$ is that such a triple exists.

Assume given two convergent spectral sequences $((E_n)_{n\geq n_0}, H, \phi)$ and $((E'_n)_{n\geq n_0}, H', \phi')$. A morphism of convergent spectral sequences is a pair $((f_n)_{n\geq n_0}, f)$ where:

- Each $f_n : (E_n, d_n) \to (E'_n, d'_n)$ is morphism of cochain complexes, such that $f_{n+1} = H(f_n)$ for all $n \geq n_0$. Note that this implies that the f_n induce for all degrees i a morphism of R-modules $f^i_\infty : E^i_\infty \to E'^i_\infty$.

- $f : H \to H'$ is a graded morphism, which preserves the filtrations (hence $\mathrm{gr}f$ is well-defined), and there is a commutative square:

$$\begin{tikzcd} E_\infty \arrow[r, "\phi", "\simeq"'] \arrow[d, "f_\infty"] & \mathrm{gr}H \arrow[d, "\mathrm{gr}f"] \\ E'_\infty \arrow[r, "\phi'", "\simeq"'] & \mathrm{gr}H' \end{tikzcd} .$$

Exercice 5. Propagation of connectivity. Let E and E' be two convergent spectral sequences, and let $((f_n), f)$ be a morphism of spectral sequences from E to E'. Assume that there is an index m such that f_m is k-connected, that is, $f^i_{m+1} : E^i_{m+1} \to E'^i_{m+1}$ is an isomorphism if $i < k$, and injective if $i = k$. Show that the maps: f_n for $n \geq m$, f_∞, and f, are all k-connected.

1.4 Homology spectral sequences.

Everything so far dealt with cohomology spectral sequences. The notion of a *homology spectral sequence* is similar, except that the pages of the spectral sequence are *chain complexes* instead of cochain complexes. In order to avoid notational conflicts between page numbers and homological degrees, the page numbers of homological spectral sequences are denoted as exponents. In particular $E^{n_0} \Rightarrow H$ denotes a homology spectral sequence converging to a graded R-module H.

Exercises 2, 3 and 4 work without change for homology spectral sequences, but in exercise 5, one has to replace the cohomology version of k-connectedness by its homology counterpart, namely a map $g : C \to D$ between chain complexes is k-connected if $H_i(g)$ is an isomorphism for $i < k$ and surjective for $i = k$.

1.5 Towards first quadrant spectral sequences.

Many spectral sequences, and all the spectral sequences that we will consider in theses notes, are *first quadrant spectral sequences.*

Such spectral sequences have additional structure. In particular their pages are bigraded, with nonzero terms placed in nonnegative bidegrees. The differentials of the spectral sequence and the filtration of the abutment must satisfy some compatibility relations with bidegrees. We make these conditions explicit in the next section.

When we restrict our attention to the total degree on each page (and when we forget the compatibility relations with bidegrees) these first quadrant spectral sequences become spectral sequences in the sense of this section. It is good to think in terms of the total degree because it notably simplifies the picture. However, the information given by the bidegree is very useful to compute differentials, and to reconstruct H from $\mathrm{gr}H$ (among other things).

To illustrate this, let us give the example of the Leray-Serre spectral spectral sequence. Let $F \hookrightarrow X \twoheadrightarrow B$ be a fibration[3] with path connected fiber F and simply connected base space B, and let $\Bbbk$ be a field. Then the homology Leray-Serre spectral sequence allows to compute the singular homology of X from the singular homologies of B and F. Restricting our attention to the total degree, we can view the Leray-Serre spectral sequence as a spectral sequence:

$$E^2_* = H_*(B,\Bbbk) \otimes H_*(F,\Bbbk) \Rightarrow H_*(X,\Bbbk)\ . \tag{2}$$

This is sufficient to conclude that the Euler characteristic[4] $\chi(X)$ is equal to $\chi(B)\chi(F)$. However, in the simple case where $F = S^3$ and $B = S^2$, the spectral sequence (2) does not allow us to determine $H_*(X,\Bbbk)$: the initial page equals $\Bbbk$ in degrees 0, 2, 3, 5 and there could very well be a differential d_n inducing an isomorphism between the vector space of degree 3 and that of degree 2. The bigraded version of the Serre spectral sequence reads:

$$E^2_{p,q} = H_p(B,\Bbbk) \otimes H_q(F,\Bbbk) \Rightarrow H_{p+q}(X,\Bbbk)\ . \tag{3}$$

The reader can go to the definition of a first quadrant homology spectral sequence in the next section, inspect how the differentials behave with respect to the bidegree, and conclude that if $F = S^3$ and $B = S^2$, then $H_*(X,\Bbbk)$ must be isomorphic to $H_*(B,\Bbbk) \otimes H_*(F,\Bbbk)$.

[3]See section 3.1 for more details on fibrations. For the moment it is sufficient to think that X is a certain topological space constructed from B and F.

[4]The Euler characteristic of a space is by definition the Euler characteristic of its homology. This number does not depend on the field $\Bbbk$ considered.

2 First quadrant spectral sequences

All the spectral sequences that we will consider in these notes are first quadrant spectral sequences. Such spectral sequences are instances of the spectral sequences of the previous section, but they bear more structure. In particular, each page is bigraded, and the differentials and the filtrations satisfy some compatibility relations with the bidegrees. One recovers the notion of spectral sequence as described in section 1 by forgetting these compatibility conditions and considering only the total degree.

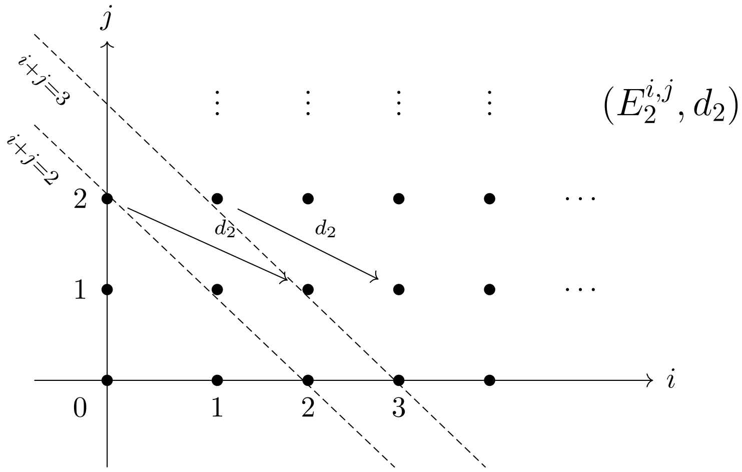
2.1 Definitions

A *cohomology first quadrant spectral sequence* is a sequence $(E_n, d_n)_{n\geq n_0}$ satisfying the following conditions.

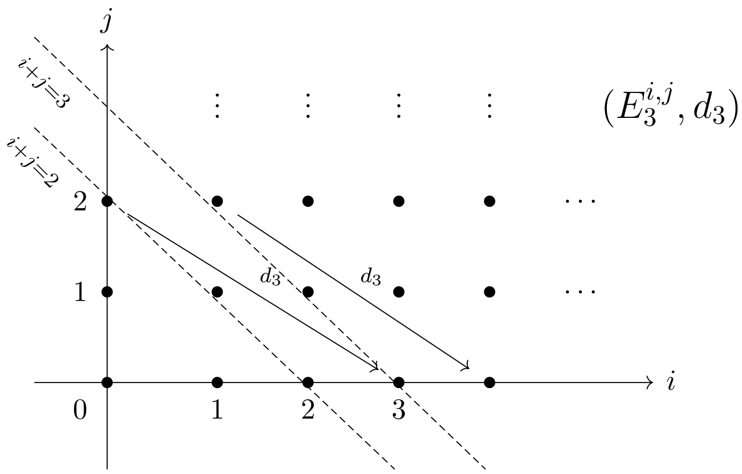
- Each E_n is a bigraded R-module, with $E_n^{i,j} = 0$ as soon as i or j is negative.
- The differential d_n is compatible with the bidegree in the sense that it restricts to maps $d_n : E_n^{i,j} \to E_n^{i+n,j+1-n}$.

  This implies that the cohomology of (E_n, d_n) is bigraded, namely
  $$H^{i,j}(E_n, d_n) = \frac{\mathrm{Ker}\left(d_n : E_n^{i,j} \to E_n^{i+n,j+1-n}\right)}{\mathrm{Im}\left(d_n : E_n^{i-n,j+n-1} \to E_n^{i,j}\right)}\;.$$
- Each term of the sequence is related to the following one by: $E_{n+1}^{i,j} = H^{i,j}(E_n, d_n)$ for all i, j.

Visualization of the pages. The pages of a first quadrant spectral sequence can be visualized as a diagram in the plane of the following form. Each dot in the diagram represents a potential nonzero term: the dot at position (i, j) stands for the term $E_2^{i,j}$. The total degree is constant along each dashed line, and a few differentials d_2 are represented.



Instead of a dot, one can also write the actual value of $E_2^{i,j}$ in position (i,j). Here is a similar drawing for page 3.



This way of drawing spectral sequences explains the term 'first quadrant spectral sequence'. Indeed the nonzero terms in the pages of the spectral sequence occupy the north-east region of the plane, aka the first quadrant of the plane.

Convergence. First quadrant cohomology spectral sequences are automatically stationnary. Indeed, for all pairs (i,j), the differential d_n starting from $E_n^{i,j}$ or landing in $E_n^{i,j}$ are zero for lacunary reasons if $n \geq i+j+2$ (these differentials are "too long" and their other extremity is outside the first quadrant). Thus,

$$E_{i+j+2}^{i,j} = E_{i+j+3}^{i,j} = E_{i+j+4}^{i,j} = \cdots =: E_\infty^{i,j} \ .$$

A first quadrant cohomology spectral sequence $(E_n, d_n)_{n\geq 0}$ converges to H if in each degree k, H^k has a finite filtration

$$0 = F^{k+1}H^k \subset F^k H^k \cdots \subset F^2 H^k \subset F^1 H^k \subset F^0 H^k = H^k$$

and if there is a graded isomorphism

$$\phi : \frac{F^i H^k}{F^{i+1}H^k} \simeq E_\infty^{i,k-i} \;.$$

Convergence is usually denoted by $E_{n_0}^{i,j} \Rightarrow H^{i+j}$.

Exercice 6. The five terms exact sequence. Let $E_2^{i,j} \Rightarrow H^{i+j}$ be a convergent first quadrant spectral sequence. Show that there is a five term exact sequence:

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{0,2} \to H^2 \;.$$

Exercice 7. Gysin and Wang exact sequences.[5] Let $E_2^{i,j} \Rightarrow H^{i+j}$ be a convergent first quadrant spectral sequence.

1. Assume that $E_2^{i,j} = 0$ if $j \notin \{0, k\}$ (i.e. only two rows of the initial page are nonzero, the 0-th one and the k-th one). Show that there is a Gysin long exact sequence:

$$\cdots \to H^{i+k+1} \to E_2^{i+1,k} \xrightarrow{d_{k+1}} E_2^{i+k+2,0} \to H^{i+k+2} \to E_2^{i+2,k} \xrightarrow{d_{k+1}} \ldots$$

2. Assume that only the 0-th column and the k-th column are nonzero. Write down a long exact sequence relating E_2 and H. (This long exact sequence is the Wang sequence).

Morphisms. A morphism between two first quadrant spectral sequences is just a morphism of spectral sequences defined as in section 1.3, which in addition preserves the bigrading at each page.

[5]When applying exercise 7 to the Leray-Serre spectral sequence of a fibration whose fibre, resp. base, is a sphere, one obtains the traditional Gysin sequence, resp. Wang sequence, in singular cohomology.

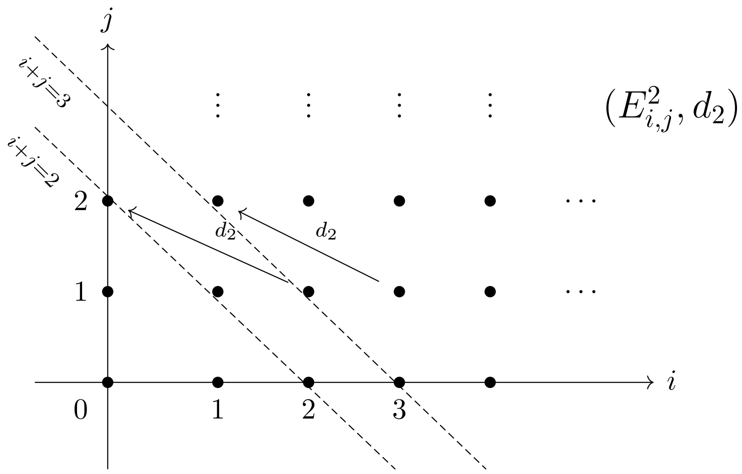
2.2 Homology first quadrant spectral sequences.

Everything so far dealt with cohomology spectral sequences. The notion of a *homology first quadrant spectral sequence* is similar. However, bigradings are a bit complicated, so we take time to repeat here the definitions.

A homology first quadrant spectral sequence is a sequence $(E^n, d_n)_{n\geq 0}$ such that

- the pages are first quadrant bigraded R-module, i.e. $E^n_{i,j}$ is zero if i or j is negative
- the differentials d_n restrict to maps $d_n : E^n_{i,j} \to E^n_{i-n,j+n-1}$.

Homology first quadrant spectral sequences are then drawn in the following way (compare with the cohomology version):



For lacunary reasons, homology first quadrant spectral sequences are stationnary. One says that (E^n, d_n) converges to the graded R-module H, and one writes $E^n_{i,j} \Rightarrow H_{i+j}$, if there is a finite filtration on each H_k:

$$0 = F_{-1}H_k \subset F_0H_k \subset \cdots \subset F_{k-1}H_k \subset F_kH_k = H_k$$

and a graded isomorphism

$$\phi : \frac{F_iH_k}{F_{i-1}H_k} \simeq E^\infty_{i,k-i} \,.$$

2.3 How to compute with spectral sequences

As already pointed out, the problem for computations is that the differentials d_n and the filtration on H^* are not explicit. In particular H^* cannot be computed from the knowledge of the initial page only (see e.g. exercise 14). We now review briefly three classical ways of obtaining additional information, which may be of great help for computations.

Edge morphisms. The two edge morphisms of a convergent first quadrant cohomology spectral sequence $E_{n_0}^{i,j} \Rightarrow H^{i+j}$ are the composite morphisms

$$H^* = F^0H^* \twoheadrightarrow F^0H^*/F^1H^* \simeq E_\infty^{0,*} \hookrightarrow E_{n_0}^{0,*}\ , \tag{4}$$
$$E_{n_0}^{*,0} \twoheadrightarrow E_\infty^{*,0} \simeq F^*H^*/F^{*+1}H^* = F^*H^* \hookrightarrow H^*\ . \tag{5}$$

Such edge morphisms are specific to first quadrant spectral sequences: the maps $E_\infty^{0,*} \hookrightarrow E_{n_0}^{0,*}$ and $E_{n_0}^{0,*} \twoheadrightarrow E_\infty^{0,*}$ come from lacunary phenomena due to the first quadrant shape of the spectral sequence.

These edge morphisms can often be interpreted as familiar maps constructed independently of the spectral sequence, which make them precious for computations (see e.g. exercise 10 below).

Exercice 8. Write down the two edge morphisms in the case of a homology spectral sequence.

Functoriality. Another technique consists in comparing the spectral sequence we are interested in, with another spectral sequence that we understand well, via a morphism of spectral sequences. The morphism of spectral sequences then allows to transport information on differentials of the well-understood spectral sequence to the less well-understood one. The next exercise is a very extreme instance of this technique.

Exercice 9. Let $f : E \to E'$ be a morphism of convergent spectral sequences. Assume that $f_{n_0} : E_{n_0}^{*,*} \to E_{n_0}'^{*,*}$ is injective and that E' collapses at page n_0 (i.e. $E_{n_0}'^{*,*} = E_\infty'^{*,*}$). Show that E collapses at page n_0 as well.

Algebra structure. A spectral sequence of algebras is a spectral sequence $E = ((E_n, d_n)_{n\geq 0}, \phi, H)$ such that:

1. Each $E_n^{*,*}$ is equipped with a bigraded algebra structure, and the differential satisfies $d_n(xy) = d_n(x)y + (-1)^{\deg x} x d_n(y)$ where $\deg x$ denotes the total degree of x. Each $E_{n+1}^{*,*}$ equals the homology of the previous page as a bigraded algebra.

2. The abutment H is a filtered graded algebra, i.e. a graded algebra such that the product takes $F^iH^k \otimes F^jH^\ell$ to $F^{i+j}H^{k+\ell}$. In particular, $\mathrm{gr}H$ is a bigraded algebra.

3. The graded isomorphism $\phi : E_\infty^{i,j} \simeq \mathrm{gr}^i H^{i+j}$ is an isomorphism of algebras.

Algebra structures are precious for computations, since the behaviour of the differentials on the generators of the algebras determine their behaviour on all the elements of the algebra. Here is a classical application.

Exercice 10. The Leray-Hirsch theorem[6]. Let B^* and F^* be two connected graded $\Bbbk$-algebras over a field $\Bbbk$, and let $E_2^{p,q} = B^p \otimes F^q \Rightarrow H^{p+q}$ be a spectral sequence of $\Bbbk$-algebras. Assume that the edge map $H^* \to E_2^{0,*}$ is surjective.

1. Show that $H^* \simeq B^* \otimes F^*$ as graded vector spaces.
2. The edge map $B^* = E_2^{*,0} \to H^*$ defines a morphism of graded algebras, hence H^* is a B^*-module. Show that the isomorphism of the previous question is actually an isomorphism of B^*-modules.

Working backwards. The filtration on H is a source of nuisance. The situation is better for spectral sequences of $\Bbbk$-vector spaces, since we always have $\mathrm{gr}H \simeq H$ as graded $\Bbbk$-vector spaces. But it may be difficult to reconstruct the algebra structure on H from that on $\mathrm{gr}H$. To bypass this difficulty one sometimes work 'backwards', i.e. use spectral sequences with known abutment and deduce information on the initial page. The next exercise is a typical illustration of this approach (see also exercise 16).

Exercice 11. An elementary case of a theorem of Borel[7]. Let B^* and F^* be two connected graded $\Bbbk$-algebras over a field $\Bbbk$, and let $E_2^{p,q} = B^p \otimes F^q \Rightarrow H^{p+q}$ be a spectral sequence of $\Bbbk$-algebras. Assume that $H^{>0} = 0$ and that $H^0 \simeq \Bbbk$, and that $F^* = \Lambda(x)$, an exterior algebra on a generator of degree n. Show that $B^* = k[y]$, a polynomial algebra on a generator y of degree $n+1$.

[6] One recovers the classical Leray-Hirsch theorem in singular cohomology by applying this exercise to the Leray-Serre spectral sequence.

[7] Borel's theorem treat the case where F^* is an exterior algebra on possibly more than one generators, see e.g. McCleary, thm 3.27.

3 Examples of spectral sequences.

In this section we present a few classical spectral sequences. We describe the spectral sequences and give some of their applications in the exercises, but we don't explain their construction. We refer the reader to McCleary for the construction of these spectral sequences, or Chap. 7 of Evens, for the Lyndon-Hochschild-Serre spectral sequence. (In practice, the details of the construction of a spectral sequence doesn't help for concrete computations). The three subsections can be read independently.

3.1 The Leray-Serre spectral sequence.

Recall that a (Serre) fibration is a map $f : X \to B$ satisfying the homotopy lifting property for CW-complexes. If B is arcwise connected, then all the fibers of f have the same homotopy type. A large class of geometric examples is provided by locally trivial fibrations with fiber F, i.e. maps $f : X \to B$ such that every $b \in B$ is contained in an open set $\mathcal{U}$ such that there exists an homeomorphism $\phi_\mathcal{U}$ making the following triangle commutative (where pr denote the projection onto the first factor)

$$\begin{tikzcd} \mathcal{U} \times F \arrow[r, "\phi_\mathcal{U}", "\sim"'] \arrow[dr, "\mathrm{pr}"'] & p^{-1}(\mathcal{U}) \arrow[d, "p"] \\ & B \end{tikzcd} .$$

One may think of a locally fibration as a twisted product of B and F. One can also think of it as the analogue for topological spaces of group extensions. Here are some examples.

1. The canonical projection $B \times F \to B$ is a locally trivial fibration.

2. If F is a discrete space, a locally trivial fibration is nothing but a covering map.

3. Fix a unital vector $u \in \mathbb{R}^n$. Let S^{2n-1} be the unit sphere of $\mathbb{C}^n$. The map $\pi : SU_n \to S^{2n-1}$ such that $\pi(g) = gu$ is a locally trivial fibration with fiber SU_{n-1}.

4. Every vector bundle of dimension n over $\pi : E \to B$ is a locally trivial fibration with fiber $\mathbb{R}^n$. If the fiber bundle is equipped with a riemanian metric, then restricting π to the vectors of norm 1 yields a locally trivial fibration $\pi : SE \to B$ with fiber S^{n-1}.

A morphism of fibrations is a commutative square:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f_F"] & X \arrow[r, "p"] \arrow[d, "f_X"] & B \arrow[d, "f_B"] \\ F' \arrow[r] & X' \arrow[r, "p'"] & B' \end{tikzcd} .$$

The Leray-Serre spectral sequence gives a way to compute the singular cohomology of the total space X of a fibration $\pi : X \to B$ from that of the fiber F and of the base B. For simplicity, we state the theorem over a field, when the base is simply connected. (See MacCleary, Chap. 5 for the general statement.)

Theorem 12 (Leray-Serre spectral sequence)**.** *Let $\pi : X \to B$ be a fibration with arcwise connected fiber F, and with simply connected base B. Let $\Bbbk$ be a field. There is a homology spectral sequence of $\Bbbk$-vector spaces:*

$$E^2_{p,q} = H_p(B,\Bbbk) \otimes H_q(F,\Bbbk) \Rightarrow H_{p+q}(X,\Bbbk)$$

whose edge maps $H_*(F,\Bbbk) \to H_*(X,\Bbbk)$ and $H_*(X,\Bbbk) \to H_*(B,\Bbbk)$ equal the maps induced by the inclusion $F \hookrightarrow X$ and f respectively. Moreover, this spectral sequence is natural with respect to morphisms fibrations, the morphism induced by (f_F, f_X, f_B) coincides with $H_*(f_B) \otimes H_*(f_F)$ on the second page, and with $H_*(f_X)$ on the abutment.

There is also a cohomology spectral sequence of $\Bbbk$-algebras:

$$E_2^{p,q} = H^p(B,\Bbbk) \otimes H^q(F,\Bbbk) \Rightarrow H^{p+q}(X,\Bbbk) .$$

The two edge maps $H^*(B,\Bbbk) \to H^*(X,\Bbbk)$ and $H^*(X,\Bbbk) \to H^*(F,\Bbbk)$ equal the maps induced by f and by the inclusion $F \hookrightarrow X$ respectively, and the spectral sequence is natural with respect to morphisms of fibrations.

Exercice 13. Show that the statement on the edge maps of the Leray-Serre spectral sequence is a consequence of the naturality statement.

Exercice 14. Find two fibrations giving rise to Leray-Serre spectral sequences with the same initial pages but non isomorphic abutments.

Exercice 15. Let $M \xrightarrow{\pi} M''$ be a fibration between two manifolds, with fiber a manifold M'. Assume that M'' is simply connected and that M' is connected. Show that M has trivial singular homology (with field coefficients) if and only if M' and M'' both have trivial singular homology (with field coefficients).

Exercice 16. Complex projective spaces. Use the fibration $S^{2n+1} \to \mathbb{C}P^n$ with fiber S^1 to compute the graded ring $H^*(\mathbb{C}P^n, \mathbb{Z})$.

Exercice 17. Special unitary groups. Let $n \geq 2$. Use the fibration $SU_n \to S^{2n-1}$ with fibre SU_{n-1} to prove that the singular cohomology ring $H^*(SU_n, \mathbb{Z})$ is an exterior algebra on generators $x_3, \dots, x_{2n-1}$, each x_k being of degree k.

3.2 The Lyndon-Hochschild-Serre spectral sequence

Let H be a normal subgroup of G. The Lyndon-Hochschild-Serre spectral sequence computes the homology of G from the homology of H and G/H. In order to formulate the theorem, we first recall a few facts on the homology of groups.

Let M be a representation of G. The quotient G/H acts on $H_*(H,M)$ and $H^*(H,M)$, hence

$$H_p(G/H, H_q(H,M)) \qquad \text{and} \qquad H^p(G/H, H^q(H,M))$$

are well-defined. If $M = \Bbbk$, the latter has an algebra structure. Namely (letting $\mathbb{H}^* := H^*(H,\Bbbk)$ for reasons of space), the product is the composition

$$H^p(G/H,\mathbb{H}^q)\otimes H^r(G/H,\mathbb{H}^s) \to H^{p+r}(G/H,\mathbb{H}^r\otimes\mathbb{H}^s) \to H^{p+r}(G/H,\mathbb{H}^{q+s})\ ,$$

where the first map is the cup product on the cohomology of G/H and the second one is induced by the algebra structure on $\mathbb{H}^*$. Finally, a morphism of extensions of groups is a commutative diagram of groups

$$\begin{tikzcd} 1 \arrow[r] & H' \arrow[r] \arrow[d, "f_H"] & G' \arrow[r] \arrow[d, "f_G"] & G'/H' \arrow[r] \arrow[d, "f_{G/H}"] & 1 \\ 1 \arrow[r] & H \arrow[r] & G \arrow[r] & G/H \arrow[r] & 1 \end{tikzcd}\ .$$

Theorem 18. *Let $1 \to H \to G \to G/H \to 1$ be an extension of groups, and let M be a representation of G. There is a homology spectral sequence*

$$E^2_{p,q} = H_p(G/H, H_q(H,M)) \Rightarrow H_{p+q}(G,M)\ .$$

This spectral sequence is natural with respect to the representation M as well as with respect to morphisms of extensions of groups. The composition:

$$H_*(H,M) \twoheadrightarrow H_*(H,M)_{G/H} \to H_*(G,M)$$

where the second map is the edge map equals the map induced in homology by the inclusion $H \hookrightarrow G$. The edge map $H_*(G,M) \to H_*(G/H, M_H)$ equals the following composition (induced by the quotients $M \twoheadrightarrow M_H$ and $G \to G/H$):

$$H_*(G,M) \to H_*(G,M_H) \to H_*(G/H,M_H)\ .$$

There is also a cohomological spectral sequence

$$E_2^{p,q} = H^p(G/H, H^q(H,M)) \Rightarrow H^{p+q}(G,M)\ .$$

If $M = \Bbbk$ this is a spectral sequence of algebras. This spectral sequence is natural with respect to the representation M as well as with respect to *morphisms of extensions of groups. The composition of the edge map with the canonical inclusion:*

$$H^*(G,M) \to H^*(H,M)^{G/H} \hookrightarrow H^*(H,M)$$

is induced by the inclusion $H \hookrightarrow G$, while the edge map $H^*(G/H, M^H) \to H^*(G,M)$ is induced by restriction along $G \twoheadrightarrow G/H$ and by the G-equivariant quotient map $M^H \hookrightarrow M$.

Exercice 19. Cyclic groups. From the knowledge of the graded algebras $H^*(C_n,\mathbb{F}_2)$, with $n = 2, k, 2k$, determine all the differentials in the LHS spectral sequence of the group extension $1 \to C_k \to C_{2k} \to C_2 \to 1$. Deduce from this analysis the morphisms $H^*(C_2,\mathbb{F}_2) \to H^*(C_{2k},\mathbb{F}_2)$ and $H^*(C_{2k},\mathbb{F}_2) \to H^*(C_k,\mathbb{F}_2)$ respectively induced by the morphisms of groups $C_{2k} \to C_2$ and $C_k \to C_{2k}$.

Exercice 20. The Heisenberg group. Let $G \subset M_3(\mathbb{Z})$ be the group of upper triangular matrices with 1s on the diagonal (this is the Heisenberg group). Let Z be the center of G.

1. Show that Z is an infinite cyclic group, and that the action of G on $H_*(Z,\mathbb{Z})$ is trivial. Compute the E^2-page of the LHS spectral sequence associated to the extension $1 \to Z \to G \to G/Z \to 1$.

2. Compute $H^*(G,\mathbb{Z})$. [Hint: use a direct computation of $H^1(G,\mathbb{Z})$.]

Exercice 21. Dihedral groups. Let $D_{2n} = \langle a, b \mid a^n = 1 = b^2\ ab = ba^{-1}\rangle$ be the dihedral group of order $2n$. Use the LHS spectral sequence of the extension $1 \to C_n \to D_{2n} \to C_2 \to 1$ to compute the graded vector space $H^*(D_{2n},\mathbb{F}_p)$.

[Hint: there are three cases. The easiest case is $p = 2$ and n odd. The case p odd is slightly more difficult. The hardest case is $p = 2$ and n even. One approach for this case is to use our knowledge of the cohomology of D_{2n} in low degrees (see the exercises in part I).]

3.3 Spectral sequences associated to double complexes

A *cohomological first quadrant bicomplex* is a bigraded R-module $(C^{p,q})_{p,q\geq 0}$ equipped with horizontal differentials $d : C^{p,q} \to C^{p+1,q}$ and vertical differentials $\partial : C^{p,q} \to C^{p,q+1}$ which commute $d \circ \partial = \partial \circ d$. The *associated total complex* $\operatorname{Tot} C$ has $\bigoplus_{p+q=n} C^{p,q}$ in degree n, and its differential sends $x \in C^{p,q}$ to $dx + (-1)^p \partial x$.

In order to compute the cohomology of $\mathrm{Tot}\, C$ one may first ignore the horizontal differential, and compute the cohomology according to the vertical differential only. Unless the horizontal differential is zero, the result obtained is very far from the cohomology of $\mathrm{Tot}\, C$. To take the horizontal differential d into account, we observe that d can be viewed as a morphism of complexes $d : (C^{p,*}, \partial) \to (C^{p+1,*}, \partial)$ hence it induces maps $d_1 : H^q(C^{p,*}, \partial) \to H^q(C^{p+1,*}, \partial)$. In this way we obtain a bigraded complex $(E^1_{p,q}, d_1)$ which is vizualized as follows.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & (E_1^{p,q}, d_1) \\
& & & & \\
H^2(C^{0,*},\partial) & \xrightarrow{d_1} & H^2(C^{1,*},\partial) & \xrightarrow{\quad d_1 \quad} & \cdots \\
& & & & \\
H^1(C^{0,*},\partial) & \xrightarrow{d_1} & H^1(C^{1,*},\partial) & \xrightarrow{\quad d_1 \quad} & \cdots \\
& & & & \\
H^0(C^{0,*},\partial) & \xrightarrow{d_1} & H^0(C^{1,*},\partial) & \xrightarrow{\quad d_1 \quad} & \cdots
\end{array}
$$

One can compute the homology of (E_1, d_1). This is a better approximation of $\mathrm{Tot}\, C$, but it is usually different[8] from $\mathrm{Tot}\, C$. To get closer to the cohomology of $\mathrm{Tot}\, C$, there are higher differentials to be computed, resulting in a spectral sequence.

Note that we started computing the homology according to the vertical differential, but we could have as well started with the horizontal differential. This would lead to another spectral sequence, converging to $\mathrm{Tot}\, C$.

In general these two spectral sequences are very different, and much information on $\mathrm{Tot}\, C$ can be obtained by comparing them.

Theorem 22. *Let $(C^{p,q}, d, \partial)$ be a cohomology first quadrant bicomplex. There is a cohomology spectral sequence*

$$E_1^{p,q} = H^q(C^{p,*}, \partial) \Rightarrow H^{p+q}(\mathrm{Tot}\, C)$$

whose first differential d_1 is the map induced in cohomology by d. Moreover, this spectral sequence is natural with respect to the bicomplex $(C^{p,q}, d, \partial)$.

There is also another cohomology spectral sequence

$$E_1^{p,q} = H^p(C^{q,*}, d) \Rightarrow H^{p+q}(\mathrm{Tot}\, C)$$

whose first differential d_1 is the map induced in cohomology by ∂. Moreover, this spectral sequence is natural with respect to the bicomplex $(C^{p,q}, d, \partial)$.

[8] Otherwise, by exercise 24, homology of complexes would be an exact functor, and you would certainly not be reading these lines, as homological algebra would not exist.

Exercice 23. Describe the edge maps of the spectral sequences associated to $(C^{p,q}, d, \partial)$. [Hint: use naturality.]

Exercice 24. The cohomology long exact sequence. Let C be a non-negatively graded cochain complex, and let C' be a subcomplex of C. Consider the short exact sequence $0 \to C' \to C \to C'' \to 0$ as a double complex, and derive from the associated spectral sequences the usual cohomology long exact sequence associated to a short exact sequence of complexes.

There is an analogue of these spectral sequences for homology first quadrant bicomplexes, i.e. bigraded R-modules $(C_{p,q})_{p,q\geq 0}$ equipped with differentials $d : C_{p,q} \to C_{p-1,q}$ and $\partial : C_{p,q} \to C_{p,q-1}$.

Theorem 25. *Let $(C_{p,q}, d, \partial)$ be a homology first quadrant bicomplex. There is a homology spectral sequence*

$$E^1_{p,q} = H^q(C_{p,*}, \partial) \Rightarrow H_{p+q}(\mathrm{Tot}\, C)$$

whose first differential d_1 is the map induced in cohomology by d. Moreover, this spectral sequence is natural with respect to the bicomplex $(C_{p,q}, d, \partial)$.

There is also another homology spectral sequence

$$E^1_{p,q} = H^p(C_{q,*}, d) \Rightarrow H_{p+q}(\mathrm{Tot}\, C)$$

whose first differential d_1 is the map induced in cohomology by ∂. Moreover, this spectral sequence is natural with respect to the bicomplex $(C_{p,q}, d, \partial)$.

Many spectral sequences can be constructed as special cases of a spectral sequence of a bicomplex. This is the case for the Leray-Serre spectral sequence (see McCleary, Chap 6, p. 225-229) and for the Lyndon-Hochschild-Serre spectral sequence (see Evens, Chap 7 or Brown, Chap 7 section 6). The next exercises give further examples.

Exercice 26. The Cartan-Leray spectral sequence. Let G be a group acting freely on a space X and assume that the quotient map $X \twoheadrightarrow X/G$ is a covering map. Let $\Bbbk$ be a commutative ring.

1. Show that there is a spectral sequence[9] (called Cartan-Leray spectral sequence)

$$H_p(G, H_q(X, \Bbbk)) \Rightarrow H_{p+q}(X/G, \Bbbk)\ .$$

[Hint: Analyse the bicomplex $P_q \otimes_{\Bbbk G} C^{\mathrm{sing}}_q(X, \Bbbk)$ where P_* is a projective resolution of the trivial representation $\Bbbk$.]

[9] The Leray-Serre spectral sequence can only be applied to fibrations with path connected fibers. The Cartan-Leray spectral sequence is a replacement of the Leray-Serre spectral sequence for covering maps.

2. Assume that X is n-connected, and let $Y = X/G$. Show that there is a 5-term exact sequence:

$$H_{n+1}(Y, \Bbbk) \to H_{n+1}(G, \Bbbk) \to H_n(Y, \Bbbk)_G \to H_n(Y, \Bbbk) \to H_n(G, \Bbbk) \to 0 \ .$$

Exercice 27. Equivariant homology. Let X be a G-complex[10] Let $\Bbbk$ be a commutative ring. The cellular chain complex $C^{\mathrm{cell}}_*(X, \Bbbk)$ is a complex of $\Bbbk G$-modules. Let P_* be a projective resolution of the trivial representation $\Bbbk$. The G-*equivariant homology* $H^G_*(X)$ is the homology of $P_* \otimes_{\Bbbk G} C^{\mathrm{cell}}_*(X, \Bbbk)$.

1. Show that if the homology groups $H^i(X, \Bbbk)$ are zero for $0 < i < n$, then $\overline{H}^G_*(X) \simeq H_*(G, \Bbbk)$ for $i < n$.
2. For all $p \geq 0$ denote by Σ_p the set of representatives of the G-orbits of p-cells of X. If e is a p-cell, we denote by G_e the stabilizer of the cell, and by $\Bbbk_e$ the representation by the $\Bbbk$-module $\Bbbk$ on which each $g \in G$ acts trivially if g preserves the orientation of e and by -1 if it reverses the orientation of e. Show that there is a spectral sequence

$$E^1_{p,q} = \bigoplus_{e \in \Sigma_p} H_q(G_e, \Bbbk_e) \Rightarrow H^G_{p+q}(X, \Bbbk) \ .$$

[Hint: Analyse the bicomplex $P_q \otimes_{\Bbbk G} C^{\mathrm{cell}}_q(X, \Bbbk)$. You may use the induction formula described at the end of part I.]

For the next exercise, one needs the notion of a Cartan-Eilenberg projective resolution of a chain complex. If C is a nonnegatively graded chain complex, a Cartan-Eilenberg resolution of C is a first quadrant double complex $(P_{p,q}, d, \partial)$, together with a chain map $\epsilon : (P_{0,*}, \partial) \to C_*$ such that:

- For all $q \geq 0$, $\epsilon : (P_{*,q}, d) \to C_q$ is a projective resolution of C_q,
- For all $q \geq 0$, the map induced on boundaries $B_q(\epsilon) : B_q(P, \partial) \to B_q(C)$ is a projective resolution of $B_q(C)$,
- For all $q \geq 0$, the map induced on homology $H_q(\epsilon) : H_q(P, \partial) \to H_q(C)$ is a projective resolution of $H_q(C)$.

Such Cartan-Eilenberg projective resolutions exist (!), see Weibel, Chap 5, section 5.7.

Exercice 28. The Grothendieck spectral sequence. Let $F : S{-}\mathrm{Mod} \to T{-}\mathrm{Mod}$ and $G : R{-}\mathrm{Mod} \to S{-}\mathrm{Mod}$ be two right exact functors. Assume

[10] Recall from part I that a G-complex is a CW-complex endowed with an action of G which permutes the cells.

that for all projective modules P, $G(P)$ is F-acyclic, i.e. the derived functors of F satisfy $\mathbf{L}_iF(G(P)) = 0$ for $i > 0$. Show that there is a spectral sequence, natural with respect to the R-module M:

$$E^2_{p,q} = \mathbf{L}_pF(\ \mathbf{L}_qG(M)\ ) \Rightarrow \mathbf{L}_{p+q}(F \circ G)(M)\ .$$

[Hint: Let Q be a projective resolution of M. Consider the bicomplex $F(P)$, where P is a Cartan-Eilenberg resolution of $G(Q)$.]