2. FOURIEROVA PRETVORBA

Def. $$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi\cdot x} f(x)\,dx, \quad \xi \in \mathbb{R}^d$$

$\longrightarrow$ ima smisla za $f \in L^1(\mathbb{R}^d)$

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \int_{\mathbb{R}} \underbrace{|e^{-2\pi i \xi\cdot x}|}_{=1} |f(x)|\,dx = \|f\|_{L^1(\mathbb{R}^d)}$$

$$\Rightarrow \quad \mathcal{F} : L^1(\mathbb{R}^d) \longrightarrow L^\infty(\mathbb{R}^d)$$

PRIMJER 1. $f = \chi_{[a,b]}$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} \chi_{[a,b]}(x)\,dx = \int_a^b e^{-2\pi i \xi x}\,dx = \begin{cases} b-a, & \xi = 0 \\ -\frac{1}{2\pi i \xi}\left(e^{-2\pi i \xi b} - e^{-2\pi i \xi a}\right), & \xi \neq 0 \end{cases}$$

$$= \begin{cases} b-a, & \xi = 0 \\ \frac{\sin(\pi(b-a)\xi)}{\pi\xi}\, e^{-i\pi(a+b)\xi}, & \xi \neq 0 \end{cases}$$

$$e^{-2\pi i \xi b} - e^{-2\pi i \xi a} = \cos(2\pi\xi b) - i\sin(2\pi\xi b) - \cos(2\pi\xi a) + i\sin(2\pi\xi a)$$
$$= -2\sin(\pi(b-a)\xi)\left(\sin(\pi(b+a)\xi) + i\cos(\pi(b+a)\xi)\right)$$

pomnožimo s $-\frac{1}{2\pi i\xi}$ i dobijemo što je trebalo

Primjetimo $\hat{f} \notin L^1(\mathbb{R})$. Zaista:

$$\int_{-\infty}^{+\infty} \left|\frac{\sin(\pi(b-a)\xi)}{\pi\xi}\right| d\xi = \frac{1}{\pi}\int_{-\infty}^{+\infty} \left|\frac{\sin(\pi(b-a)\xi)}{\xi}\right| d\xi = \frac{2}{\pi}\int_0^{+\infty} \left|\frac{\sin(\pi(b-a)\xi)}{\xi}\right| d\xi$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{+\infty} \underbrace{\frac{1}{k}}_{\frac{b-a}{2}} \underbrace{\int_{(k-1)\frac{2}{b-a}}^{k\frac{2}{b-a}} |\sin(\pi(b-a)\xi)|\,d\xi}_{\text{konstanta} > 0}$$

(ne ovisi o k jer je period $\frac{2}{b-a}$)

$\Rightarrow$ sve ide u beskonačnost jer harmonijski red divergira

iako imamo $\int_{-\infty}^{+\infty} \frac{\sin\xi}{\xi}\,d\xi = \pi$, tj. uvjetnu konvergenciju

ZAKLJUČAK: Ako je f s kompaktnim nosačem, tada će $\hat{f}$ biti raspršena (neće imati kompaktan nosač), a nekad čak niti integrabilna.

POSEBNO: $b = \frac{1}{2\pi}$, $a = -\frac{1}{2\pi}$

$$\frac{\sin(\xi)}{\pi\xi} = \widehat{\chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}}(\xi)$$

preskočiti

TEOREM. (Riemann – Lebesgue)

$\mathcal{F}$ je ograničen lin. op. sa $L^1(\mathbb{R}^d)$ u $C_0(\mathbb{R}^d)$.

$$\left(\lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0 \right).$$

PROP.

$$\int_{\mathbb{R}^d} \hat{f}(\xi)\, g(\xi)\, d\xi = \int_{\mathbb{R}^d} f(x)\, \hat{g}(x)\, dx, \quad f, g \in L^1(\mathbb{R}^d).$$

SVOJSTVA:

i) $f \in L^1(\mathbb{R}^d)$, $h \in \mathbb{R}^d$

$$\widehat{\tau_h f}(\xi) = \mathcal{F}(f(\cdot - h)) = e^{-2\pi i \xi \cdot h} \hat{f}(\xi)$$

$$\widehat{e^{2\pi i x \cdot h} f}(\xi) = \tau_h \hat{f}(\xi)$$

ii) $x^\alpha f \in L^1(\mathbb{R}^d)$, $|\alpha| \leq k \Rightarrow \hat{f} \in C^k(\mathbb{R}^d)$

$$\partial_j \hat{f} = (-2\pi i x_j f)^\wedge$$

$$\partial^\alpha \hat{f} = ((-2\pi i x)^\alpha f)^\wedge$$

iii) $f \in C^k(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

$$\widehat{\partial_i f} = 2\pi i \xi_i \hat{f}$$

$$\widehat{\partial^\alpha f} = (2\pi i \xi)^\alpha \hat{f}$$

iv) $f \in L^1(\mathbb{R}^d)$, $\operatorname{supp} f$ ograničen $\Rightarrow \hat{f} \in C^\infty(\mathbb{R}^d)$

(posljedica ii))

v) $f(ax) \xmapsto{\mathcal{F}} \frac{1}{|a|^d} \hat{f}\left(\frac{\xi}{a}\right)$, $a \in \mathbb{R}$

vi) $\widehat{f_\sigma} = (\hat{f})_\sigma$, gdje je $f_\sigma(x) := f(-x)$.

ZAD 1. Izračunajte Fourierovu pretvorbu f-je $f(x) = e^{-ax^2}$, $a>0$.

Rj: $\underline{a=1}$

$$f'(x) = -2x f(x) \quad /\,\wedge$$

$$2\pi i \xi \hat{f}(\xi) = -2\,\widehat{xf}(\xi) = \frac{1}{i\pi}(-2\pi i x f)^{\wedge}(\xi) = \frac{1}{i\pi}(\hat{f})'(\xi)$$

$$\Rightarrow \quad (\hat{f})' + 2\pi^2 \xi \hat{f} = 0$$

$$\Rightarrow \quad \hat{f}(\xi) = C e^{-\pi^2 \xi^2}$$

PRIMJETITI DA JE ZA $a=\pi$ $\hat{f} = f$

$\underline{C=?}$

$$\hat{f}(0) = \int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi} \quad \Rightarrow \quad \hat{f}(\xi) = \sqrt{\pi}\, e^{-\pi^2 \xi^2}$$

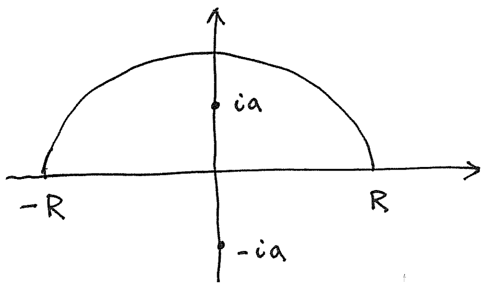
$a>0$ (koristimo svojstvo v)) $\hat{f}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2}{a}\xi^2}$

$a<0 \rightarrow$ funkcija nije iz L^1 pa F. pretvorba nije definirana.

ZAD.2. $f(x) = \frac{1}{a^2+x^2}$, $\hat{f}(\xi) = ?$ $\quad (a>0)$

Rj:

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-2\pi i \xi x} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{e^{-2\pi i \xi x}}{a^2+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{e^{-2\pi i \xi x}}{(x-ia)(x+ia)}\,dx$$



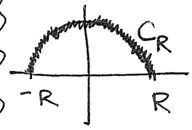
KORISTIMO JORDANOVU LEMU PA MORAMO RAZDOJITI NA SLUČAJEVE $\xi<0$ i $\xi>0$.

$\underline{\xi<0}$ (tu je sve dobro)

$$\Rightarrow \quad \hat{f}(\xi) = 2\pi i \operatorname{Res}\left(\frac{e^{-2\pi i x \xi}}{a^2+x^2}, ia\right) = 2\pi i\left((x-ai)\frac{e^{-2\pi i x \xi}}{a^2+x^2}\Big|_{x=ia}\right)$$

$$= 2\pi \cancel{i}\, \frac{e^{2\pi a \xi}}{\cancel{2}\cancel{i}a} = \frac{\pi}{a} e^{2\pi a \xi}$$

JORDANOVA LEMA



$f(z) = e^{iaz} g(z)$, $z \in C_R$, $a>0$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi}{a} \max_{\theta\in[0,\pi]} |g(Re^{i\theta})|$$

• za $a<0$ se gleda donja polukružnica

NAPOMENA f parna (neparna) $\Rightarrow$ $\hat{f}$ parna (neparna).

Dz. Iz definicije F. pretvorbe slijedi $\hat{\tilde{f}} = \tilde{\hat{f}}$ pa $\tilde{f} = f \Rightarrow \tilde{\hat{f}} = \hat{f}$

$$\Rightarrow \underbrace{\hat{f}(\xi) = \hat{f}(-\xi)}_{\xi > 0} = \frac{\pi}{a} e^{-2\pi a \xi}$$

$$\Rightarrow \boxed{\hat{f}(\xi) = \frac{\pi}{a} e^{-2\pi a |\xi|}}$$

ZAD. 3. Izračunajte Fourierovu pretvorbu sljedećih funkcija:

a) $f_1(x) = e^{-ax} H(x)$, $Re(a) > 0$,

b) $f_2(x) = e^{ax} H(-x)$, $Re(a) > 0$,

c) $f_3(x) = \frac{x^k}{k!} e^{-ax} H(x)$, $R(a) > 0$,

d) $f_4(x) = \frac{x^k}{k!} e^{ax} H(-x)$, $Re(a) > 0$,

e) $f_5(x) = e^{-a|x|}$, $Re(a) > 0$,

f) $f_6(x) = \operatorname{sign}(x) e^{-a|x|}$, $Re(a) > 0$,

gdje je H Heavisideova funkcija.

Rj.

a) $$\hat{f}_1(\xi) = \int_0^{+\infty} e^{-2\pi i x \xi} e^{-ax} dx = \int_0^{+\infty} e^{-(2\pi i \xi + a)x} dx = \lim_{b \to +\infty} \frac{-e^{-(2\pi i \xi + a)x}}{a + 2\pi i \xi} \Bigg|_0^b$$

$$= \frac{1}{a + 2\pi i \xi} \quad \left(\text{jer je } \lim_{b \to +\infty} e^{-Re(a)x} = 0 \right)$$

b) Primjetimo $f_2 = (f_1)_b$ $\quad (f_2 = \tilde{f}_1)$

$$\Rightarrow \hat{f}_2(\xi) = \hat{f}_{1_b}(\xi) = (\hat{f}_1)_b(\xi) = \hat{f}_1(-\xi) = \frac{1}{a - 2\pi i \xi}$$

c) Koristimo ii)

$$f_3(x) = (-2\pi i x)^k f_1(x) \frac{1}{(-2\pi i)^k} \frac{1}{k!}$$

$$\Rightarrow \hat{f}_3(\xi) = \frac{1}{k!} \frac{1}{(-2\pi i)^k} \hat{f}_1^{(k)}(\xi) = \frac{1}{\cancel{k!}} \frac{1}{\cancel{(-2\pi i)^k}} \frac{\cancel{(-1)^k k! (2\pi i)^k}}{(a + 2\pi i \xi)^{k+1}}$$

$$= \frac{1}{(a + 2\pi i \xi)^{k+1}}$$

d) Analogno kao (c) dobivamo:

$$f_4(x) = \frac{1}{k!}\,\frac{1}{(-2\pi i)^k}\,(-2\pi i x)^k f_2(x)$$

$$\Rightarrow \hat{f}_4(\xi) = \frac{1}{k!}\,\frac{1}{(-2\pi i)^k}\,\hat{f}_2^{(k)}(\xi) = \frac{1}{\cancel{k!}}\,\frac{1}{\underset{(-1)^k}{\cancel{(-2\pi i)^k}}}\,\frac{\cancel{k!\,(2\pi i)^k}}{(a-2\pi i\xi)^{k+1}}$$

$$= \frac{(-1)^k}{(a-2\pi i\xi)^{k+1}} = \frac{-1}{(-a+2\pi i\xi)^{k+1}}$$

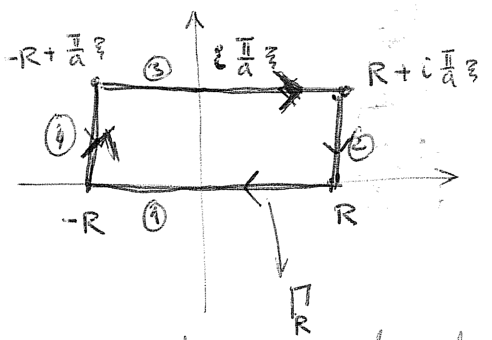
e) $f_5(x) = f_1(x) + f_2(x)$ s.s. (za $x=0$ jednakost ne vrijedi)

$$\Rightarrow \hat{f}_5(\xi) = \hat{f}_1(\xi) + \hat{f}_2(\xi) = \frac{a - 2\pi i\xi + a + 2\pi i\xi}{a^2+4\pi^2\xi^2} = \frac{2a}{a^2+4\pi^2\xi^2}$$

f) $f_6(x) = f_1(x) - f_2(x)$

$$\Rightarrow \hat{f}_6(\xi) = \frac{-4\pi i\xi}{a^2+4\pi^2\xi^2}.$$

NAPOMENA: $f(x) = e^{-ax^2}$, $a>0$. Fourierova pretvorba f-je f alternativno se može dobiti kompleksnom integracijom.



- računamo za $\xi>0$, a $\xi<0$ ćemo dobiti preko parnosti

$$\int_{\Gamma_R} e^{-az^2}\,dz = 0 \quad \text{jer je } e^{-az^2} \text{ holomorfna f-ja.}$$

Tada rastavimo integrale i pustimo $R \to +\infty$.
(dobije se da integrali ② i ④ idu u nulu, ① je Gaussian, a u ③ prepoznamo F. pretvorbu).

- $② = -\int_0^{\frac{\pi}{a}\xi} e^{-a(R+it)^2}\,dt = -\underset{\downarrow\, 0}{e^{-aR^2}} \underbrace{\int_0^{\frac{\pi}{a}\xi} e^{-a(2Rit-t^2)}\,dt}_{\text{omeđeno}} \to 0$
- Analogno i ④
- $① = -\int_{-R}^{R} e^{-ax^2}\,dx \to -\sqrt{\frac{\pi}{a}}$
- $③ = \int_{-R}^{R} e^{-a(x+i\frac{\pi}{a}\xi)^2}\,dx = e^{\frac{\pi^2}{a}\xi^2}\int_{-R}^{R} e^{-2\pi i x\xi}\, e^{-ax^2}\,dx \to e^{\frac{\pi^2}{a}\xi^2}\,\widehat{e^{-ax^2}}(\xi)$

$$e^{\frac{\pi^2\xi^2}{a}}\hat{f} - \sqrt{\frac{\pi}{a}} = 0 \Leftarrow$$

$$\Downarrow$$

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2\xi^2}{a}} \checkmark$$

Inverzna Fourierova pretvorba

Def. $(\overline{\mathcal{F}}f)(\xi) = \check{f}(\xi) = \int_{\mathbb{R}^d} e^{+2\pi i \xi \cdot x} f(x)\, dx$, $\xi \in \mathbb{R}^d$.

Lako se vidi da vrijedi $\overline{\mathcal{F}f(\xi)} = (\overline{\mathcal{F}}\overline{f})(\xi)$.

TEOREM. $f, \hat{f} \in L^1(\mathbb{R}^d) \Rightarrow \overline{\mathcal{F}}\hat{f}(t) = f(t)$ u svim točkama gdje je f neprekinuta.

PROPOZICIJA. Ako je f neprekinuta i integrabilna, te $\hat{f}$ je također integrabilna, tada za svaki $x \in \mathbb{R}^d$ vrijedi

$$\hat{\hat{f}}(x) = \check{f}_\sigma(x) = f(-x).$$

Dz.

$$g := \hat{f}, \quad \hat{g}(-x) = \int e^{2\pi i \xi \cdot x} \hat{f}(\xi)\, d\xi = \overline{\mathcal{F}}\hat{f}(x)$$

$$\overset{\text{TM.}}{\Longrightarrow} \hat{g}(-x) = f(x)$$

$$\Rightarrow \mathcal{F}\hat{f}(x) = f_\sigma(x).$$

Vratimo se sad na prethodni zadatak i primijenimo Propoziciju.

PRIMJER 2.

a) $f_3(x) = \frac{x^k}{k!} e^{-ax} H(x) \rightarrow \hat{f}_3(\xi) = \frac{1}{(a+2\pi i \xi)^{k+1}} \in L^1(\mathbb{R})$ za $k \geq 1$

$\hookrightarrow \left|\frac{1}{(a+2\pi i \xi)^{k+1}}\right| = \frac{1}{(\sqrt{a^2+4\pi^2\xi^2})^{k+1}}$

& LEMA 6 Predavanje (Fourier)

$$g_1(\xi) := \hat{f}_3(\xi)$$

$$\Rightarrow \hat{g}_1(x) = \frac{(-x)^k}{k!} e^{ax} H(-x) = (-1)^k f_4(x)$$

b) $f_4(x) = \frac{x^k}{k!} e^{ax} H(-x) \longrightarrow \hat{f}_4(\xi) = \frac{-1}{(-a+2\pi i \xi)^{k+1}} \in L^1(\mathbb{R})$ za $k \geq 1$

$$g_2(\xi) := \hat{f}_4(\xi)$$

$$\Rightarrow \hat{g}_2(\xi) = (-1)^k f_3(x)$$

c) $f_5(x) \longrightarrow \hat{f}_5(\xi) = \frac{2a}{a^2+4\pi^2\xi^2} \in L^1(\mathbb{R})$

$$g_3(\xi) := \hat{f}_5(\xi) \Rightarrow \hat{g}_3(x) = e^{-a|x|} = f_5(x), \text{ tj. } \hat{\hat{f}}_5 = f_5$$

d) $f_6(x) = \operatorname{sign}(x)\, e^{-a|x|} \longrightarrow \hat{f}_6(\xi) = \dfrac{-4i\pi\xi}{a^2+4\pi^2\xi^2} \notin L^1(\mathbb{R})$

(u beskonačnosti se ponaša kao $\frac{1}{\xi}$)

e) $f(x) = e^{-ax^2} \longrightarrow \hat{f}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2}{a}\xi^2} =: g(\xi) \in L^1(\mathbb{R})$

$\Rightarrow \hat{g}(x) = e^{-ax^2}$, tj. $\hat{\hat{f}} = f$.

Također bismo htjeli definirati F. pretvorbu f-je $\hat{f}_1, \hat{f}_2 \notin L^1(\mathbb{R})$ iz ZAD 3. $\longrightarrow$

htjeli bismo moći napisati (definirati) $\hat{\hat{f}}_6(x) = f_6(-x)$ pa za to trebamo proširiti djelovanje F. pretvorbe na veće prostore

Schwartzov prostor $\mathcal{S}$

$$\mathcal{S}(\mathbb{R}^d) := \left\{ u \in C^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}_0^d,\ \| x^\alpha \partial^\beta u \|_{L^\infty(\mathbb{R}^d)} < +\infty \right\}$$

$\to$ brzo opadajuće f-je

$\|u\|_{\alpha,\beta} := \| x^\alpha \partial^\beta u \|_{L^\infty(\mathbb{R}^d)}$

$\|u\|_k := \max\limits_{|\alpha+\beta| \le k} \|u\|_{\alpha,\beta}$ ... polunorma

$$d(u,v) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|u-v\|_k}{1+\|u-v\|_k} \longrightarrow \text{metrika u } \mathcal{S}$$

Čak imamo i više: $\mathcal{S}$ je Fréchetov prostor

(Iako je $\mathcal{S}$ lijep prostor, treba biti svjestan da nije normiran)

$\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow (\forall \alpha, \beta \in \mathbb{N}_0^d)\ x^\alpha \partial^\beta \varphi \in \mathcal{D}(\mathbb{R}^d)$

$\Rightarrow \mathcal{D}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$

Vrijedi i: $\mathcal{D}(\mathbb{R}^d)$ je gust u $\mathcal{S}(\mathbb{R}^d)$ u topologiji prostora $\mathcal{S}(\mathbb{R}^d)$.

SVOJSTVA:

i) $f \in \mathcal{S} \Rightarrow \partial^\alpha f \in \mathcal{S}$

ii) $\mathcal{S} \subseteq L^p$, $p \ge 1$

iii) $\mathcal{F}(\mathcal{S}) \subseteq \mathcal{S}$.

iii) ima smisla zato što iz ii) imamo $\mathcal{S} \subseteq L^1$. Međutim, vrijedi i više:

TEOREM. $\mathcal{F}$ je linearna bijekcija s $\mathcal{S}(\mathbb{R}^d)$ u samog sebe. Inverz od $\mathcal{F}$ je $\bar{\mathcal{F}}$.

Konvolucija

2013. sam to odradio ranije

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy$$

SVOJSTVA:

i) $\|f*g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$

ii) $\frac{1}{p} + \frac{1}{p'} = 1$, $\|f*g\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p'}(\mathbb{R}^d)}$

iii) $\|f*g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}$

iv) $\partial^\alpha (f*g) = (\partial^\alpha f) * g = f * (\partial^\alpha g)$

PRIMJER 3. $f = g = \chi_{[0,1]}$

$$(f*g)(x) = \int_{\mathbb{R}} f(x-t)\, g(t)\, dt = \int_0^1 \chi_{[0,1]}(x-t)\, dt = \mu([0,1] \cap [x-1,x])$$

$$\Rightarrow (f*g)(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x \leq 1 \\ 2-x & , 1 < x \leq 2 \\ 0 & , x > 2 \end{cases}$$

LEMA. $\operatorname{supp}(f*g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$.

ZAD. 4.

$$\chi_{[-a,a]} * \sin x = \int_{\mathbb{R}} \sin(x-y)\, \chi_{[-a,a]}(y)\, dy = \int_{-a}^{a} \sin(x-y)\, dy = 2 \sin a \sin x.$$

SAŽETAK:

- $L^1 * L^1 \subseteq L^1$ (s ovom operacijom L^1 postaje algebra)
- $L^1 * L^\infty \subseteq L^\infty \cap C^0$
- $L^2 * L^2 \subseteq L^\infty \cap C^0$
- $L^r * L^1 \subseteq L^r$
- $\mathcal{D} * \mathcal{D} \subseteq \mathcal{D}$
- $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$

Fourierova pretvorba na $L^2(\mathbb{R}^d)$

TEOREM. Fourierova pretvorba se na jedinstven način proširuje sa $\mathcal{S}(\mathbb{R}^d)$ do unitarnog operatora na $L^2(\mathbb{R}^d)$ i vrijedi Parsevalova formula:

$$\int_{\mathbb{R}^d} \hat{f}(\xi)\,\hat{g}(\xi)\,d\xi = \int_{\mathbb{R}^d} f(x)\,g(x)\,dx \quad .$$

Nadalje imamo $(\hat{f})^\vee = (\check{f})^\wedge = f$ s.s. i $\|f\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(\mathbb{R}^d)}$.

PROPOZICIJA. $f, g \in L^2(\mathbb{R}^d)$,

i) $\widehat{f*g} = \hat{f}\cdot\hat{g}$

ii) $(f\cdot g)^\wedge = \hat{f} * \hat{g}$

ZAD 5. $f = \chi_{[0,1]}$, $\widehat{f*f} = ?$

i) $\Rightarrow \widehat{f*f}(\xi) = \hat{f}(\xi)\cdot\hat{f}(\xi)$

$$\hat{f}(\xi) = \int_0^1 e^{-2\pi i \xi x}\,dx = -\frac{e^{-2\pi i \xi x}}{2\pi i \xi}\Big|_0^1 = \frac{1}{2\pi i \xi} - \frac{e^{-2\pi i \xi}}{2\pi i \xi}$$

$$\Rightarrow \widehat{f*f}(\xi) = \left(\frac{1}{2\pi i\xi} - \frac{e^{-2\pi i \xi}}{2\pi i \xi}\right)^2$$

KOMENTAR: Iako smo ranije već izračunali $f*f$, puno bi bilo kompliciranije računati direktno $\widehat{f*f}$ jer se javljaju integrali koje treba računati parcijalnom integracijom

DEFINICIJA. <u>Temperirana distribucija</u> je neprekinut linearan funkcional na $\mathcal{S}(\mathbb{R}^d)$. Oznaka: $\mathcal{S}'(\mathbb{R}^d)$.

$\mathcal{S}'(\mathbb{R}^d) \subseteq \mathcal{D}'(\mathbb{R}^d)$

[$\mathcal{D} \subseteq \mathcal{S}$, $f \in \mathcal{S}' \Rightarrow f$ je definirana i na $\mathcal{D}$
Budući da je topologija na $\mathcal{D}$ jača od topologije na $\mathcal{S}$, zaključujemo $f|_{\mathcal{D}} \in \mathcal{D}'$. Sada nam je još preostalo vidjeti je li preslikavanje $f \mapsto f|_{\mathcal{D}}$ injektivno. Uistinu jest jer je $\mathcal{D}$ gust u $\mathcal{S}$ pa je $f|_{\mathcal{D}}$ jedinstveno proširiv do f.
Uočiti da ovdje nije riječ o normiranim prostorima pa se ne možemo pozvati na rezultate iz ~~e~~normiranih prostora.

DEFINICIJA. $T \in \mathcal{S}'(\mathbb{R}^d)$, $\langle \hat{T}, f \rangle = \langle \tilde{T}, \hat{f} \rangle$
$\hat{T}$ ... Fourierova pretvorba temperirane distribucije T

NAP. Ne možemo definirati F. pretvorbu na $\mathcal{D}'(\mathbb{R}^d)$ jer $\mathcal{F}(\mathcal{D}(\mathbb{R}^d)) \not\subseteq \mathcal{D}(\mathbb{R}^d)$ pa $\langle \hat{T}, \hat{f} \rangle$ ne bi bilo def.

TEOREM. $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ je izomorfizam, $\mathcal{F}^{-1} = \overline{\mathcal{F}}$.

(i dalje vrijede sva svojstva: der., transl. ...)

Vratimo se malo na odnos $\mathcal{S}'(\mathbb{R}^d)$ i $\mathcal{D}'(\mathbb{R}^d)$.

$F \in \mathcal{S}'(\mathbb{R}^d)$, tada vrijedi:

$$\varphi_n \xrightarrow{\mathcal{S}(\mathbb{R}^d)} \varphi \Rightarrow \langle F, \varphi_n \rangle \to \langle F, \varphi \rangle$$

$F|_{\mathcal{D}(\mathbb{R}^d)} \in \mathcal{D}'(\mathbb{R}^d)$?

$$\left.\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R}^d)} \varphi \Rightarrow \varphi_n \xrightarrow{\mathcal{S}(\mathbb{R}^d)} \varphi \Rightarrow \langle F, \varphi_n \rangle \longrightarrow \langle F, \varphi \rangle \right\} \Rightarrow F|_{\mathcal{D}(\mathbb{R}^d)} \text{ je iz } \mathcal{D}'(\mathbb{R}^d)$$

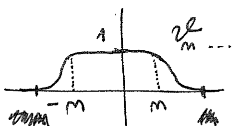
Ako sada uzmemo funkcional iz $\mathcal{D}'(\mathbb{R}^d)$, hoćemo li ga moći uvijek proširiti do funkcionala iz $\mathcal{S}'(\mathbb{R}^d)$?
NE! (slijedi primjer)

PRIMJER 4. $f(x) = e^{x^2}$

$$f \in L^1_{loc}(\mathbb{R}) \Rightarrow f \in \mathcal{D}'(\mathbb{R})$$

Pogledajmo je li $\int_{\mathbb{R}} e^{x^2} \varphi \, dx$ definirano za svaki $\varphi \in \mathcal{S}(\mathbb{R})$.

$\varphi(x) = e^{-\frac{1}{2}x^2} \in \mathcal{S}(\mathbb{R})$



χ_m ... rezuća f-ja iz C_c^∞

$$\int_{\mathbb{R}} e^{x^2}\varphi\,dx = \int_{\mathbb{R}} e^{\frac{1}{2}x^2}dx \to \text{ovaj integral ne postoji}$$

$\Rightarrow$ našli smo f-ju koja definira ~~temperiranu~~ distribuciju, ali ne i temperiranu distribuciju

$$\Rightarrow \quad \mathcal{S}'(\mathbb{R}^d) \subsetneq \mathcal{D}'(\mathbb{R}^d)$$

MALO PRECIZNIJE:

$\langle T,\varphi\rangle := \int_{\mathbb{R}} e^{x^2}\varphi(x)\,dx \;, \varphi\in\mathcal{D}(\mathbb{R})$

$\underset{\mathcal{D}(\mathbb{R})}{\varphi\varkappa_m} \xrightarrow{\mathcal{S}} \underset{\mathcal{S}(\mathbb{R})}{\varphi}$

Međutim, $\langle T,\varphi\varkappa_m\rangle = \int_{\mathbb{R}} e^{x^2}e^{-\frac{1}{2}x^2}\varkappa_m(x)\,dx \geq \int_{-m}^{m} e^{\frac{1}{2}x^2}dx \geq 2m \to \infty \Rightarrow\Leftarrow$

tj. $\langle T,\varphi\rangle$ nije def.

PRIMJER 5.

- $\delta_0 \in \mathcal{S}'(\mathbb{R})$

$$\langle\hat{\delta}_0,\varphi\rangle = \langle\overset{\delta_0}{\overset{\|}{\tilde{\delta}_0}},\hat{\varphi}\rangle = \hat{\varphi}(0) = \int_{-\infty}^{+\infty}\varphi(x)\,dx < \infty \,,\quad \varphi\in\mathcal{S}(\mathbb{R})$$

jer $\mathcal{S}\subseteq L^1$

$$\Rightarrow \hat{\delta}_0 = 1$$

$\widehat{f*\delta_0} = \hat{f}\cdot\hat{\delta}_0 = \hat{f} \;/^\vee \Rightarrow f*\delta_0 = f$ ... δ_0 je jedinica u algebri s konvolucijom

- $\hat{\delta}_a = ?$

$$\delta_a = \tau_a\delta_0 \;/^\wedge \Rightarrow \hat{\delta}_a = \widehat{\tau_a\delta_0} = e^{-2\pi i a\xi}\hat{\delta}_0 = e^{-2\pi i a\xi}$$

- $\widehat{\delta_0^{(k)}}(\xi) = (2\pi i\xi)^k$ $\to$ iz ovoga također vidimo da su polinomi iz $\mathcal{S}'$ jer $\mathcal{F}:\mathcal{S}'\to\mathcal{S}'$

$$\Rightarrow \check{x}^k = \frac{1}{(2\pi i)^k}\delta_0^{(k)} \quad \text{ili} \quad \hat{x}^k = \frac{1}{(-2\pi i)^k}\delta_0^{(k)}$$

Kako znati što je temperirana distribucija?

Na temp. distr. znamo računati F. pretvorbu i to je najveći prostor na kojem to možemo pa je zato bitan.

1) Po definiciji

$$T\in\mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \underbrace{\varphi_n \xrightarrow{\mathcal{S}} 0}_{\forall(\varphi_n)\subseteq\mathcal{S}(\mathbb{R}^d)} \Rightarrow \langle T,\varphi_n\rangle \to 0$$

2) Karakterizacija

$$T\in\mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \forall(\varphi_n)\subseteq\mathcal{D}(\mathbb{R}^d),\ \varphi_n \xrightarrow{\mathcal{S}} 0 \Rightarrow \langle T,\varphi_n\rangle \to 0$$

3) $\forall p\geq 1,\ L^p(\mathbb{R}^d)\subseteq\mathcal{S}'(\mathbb{R}^d)$ (slijedi iz $\mathcal{S}(\mathbb{R}^d)\subseteq L^p(\mathbb{R}^d),\ p\geq 1$)

4) $U\in\mathcal{S}'(\mathbb{R}^d)$ & $T=\hat{U} \Rightarrow T\in\mathcal{S}'(\mathbb{R}^d)$

5) $T \in \mathcal{S}'(\mathbb{R}^d)$, i) $\forall \alpha \in \mathbb{N}_0^d$, $x^\alpha T \in \mathcal{S}'(\mathbb{R}^d)$
ii) $\forall \alpha \in \mathbb{N}_0^d$, $\partial^\alpha T \in \mathcal{S}'(\mathbb{R}^d)$
iii) $\partial^\alpha G = T \Rightarrow G \in \mathcal{S}'(\mathbb{R}^d)$

KOMENTAR. Sada se možemo vratiti u ZAD.3: $f_1, f_2, f_6 \in \mathcal{S}' \Rightarrow \hat{f}_1, \hat{f}_2, \hat{f}_6 \in \mathcal{S}'$
$\Rightarrow$ definirano je $\hat{\hat{f}}_1, \hat{\hat{f}}_2, \hat{\hat{f}}_6$.
Također iz PRIMJER 1.
$\left(\frac{\sin x}{x}\right)^\wedge(\xi) = \pi \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi)$

PRIMJER 6.

$\ln|x| \in L^1_{loc}(\mathbb{R}) \Rightarrow \ln|x| \in \mathcal{D}'(\mathbb{R})$

Dokažimo pomoću karakterizacije da je i u $\mathcal{S}'(\mathbb{R})$.

$(\varphi_n) \subseteq \mathcal{D}(\mathbb{R})$, $\varphi_n \xrightarrow{\mathcal{S}} \varphi$

$$|\langle \ln|x|, \varphi_n\rangle| \leq \int_{\mathbb{R}} |\ln|x|| \, |\varphi_n(x)| \, dx \leq \underbrace{\|\varphi_n\|_{L^\infty(\mathbb{R})}}_{\to 0} \underbrace{\int_K |\ln|x|| \, dx}_{<\infty \text{ jer je } \ln|x| \in L^1_{loc}(\mathbb{R})}$$

$\Rightarrow \ln|x| \in \mathcal{S}'(\mathbb{R})$.

$(\ln|x|)' = pv\left(\frac{1}{x}\right) \Rightarrow pv\left(\frac{1}{x}\right) \in \mathcal{S}'(\mathbb{R})$.

Iz tog razloga ima smisla promatrati $\left(pv\left(\frac{1}{x}\right)\right)^\wedge$.

$$\left\langle \left(pv\left(\tfrac{1}{x}\right)\right)^\wedge, \varphi\right\rangle = \left\langle pv\left(\tfrac{1}{x}\right), \widetilde{\hat{\varphi}}\right\rangle = \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{\overline{\hat{\varphi}(x)}}{x} dx$$

$$= \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \int_{\mathbb{R}} e^{-2\pi i x\xi} \, \overline{\varphi(\xi)} \, \frac{1}{x} \, d\xi \, dx$$

$$\overset{\text{FUBINI}}{=} \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \overline{\varphi(\xi)} \int_{|x|\geq\varepsilon} e^{-2\pi i x\xi} \frac{1}{x} \, dx \, d\xi$$

$$\int_{|x|\geq\varepsilon} e^{-2\pi i x\xi} \frac{1}{x} dx = \underbrace{\int_{|x|\geq\varepsilon} \underbrace{\frac{\cos(2\pi x\xi)}{x}}_{\text{neparna f-ja}} dx}_{=0} - i \int_{|x|\geq\varepsilon} \underbrace{\frac{\sin(2\pi x\xi)}{x}}_{\text{parna f-ja}} dx$$

$$= -2i \int_{\varepsilon}^{+\infty} \frac{\sin(2\pi x\xi)}{x} dx = \begin{cases} -2i \displaystyle\int_{2\pi\xi\varepsilon}^{+\infty} \frac{\sin t}{t} dt, & \xi > 0 \\ +2i \displaystyle\int_{-2\pi\xi\varepsilon}^{+\infty} \frac{\sin t}{t} dt, & \xi < 0 \end{cases}$$

Budući da je $\lim_{\varepsilon\to 0}\int_{\varepsilon}^{+\infty}\frac{\sin t}{t}\,dt=\frac{\pi}{2}$, imamo:

$$\langle (pv(\tfrac{1}{x}))^{\wedge},\varphi\rangle = -i\pi\int_{\mathbb{R}} sign(\xi)\,\overline{\varphi(\xi)}\,d\xi$$

$$\Rightarrow (pv(\tfrac{1}{x}))^{\wedge} = -i\pi\, sign(x).$$

$$\Rightarrow (sign(x))^{\wedge}(\xi) = \frac{1}{i\pi}\, pv(\tfrac{1}{x})$$

ZAD. 6 $Pf(\frac{1}{x^n})^{\wedge}$

$$\frac{1}{x^n} = \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{1}{x}\right)^{(n-1)}$$

$$\Rightarrow \left(Pf(\tfrac{1}{x^n})\right)^{\wedge}(\xi) = \frac{(-1)^{n-1}}{(n-1)!}(2\pi i\xi)^{n-1}\left(pv(\tfrac{1}{x})\right)^{\wedge}$$

$$= \frac{(-2\pi i\xi)^{n-1}}{(n-1)!}(-\pi i)\,sign(\xi).$$

ZAD. 7.

$$\left(\frac{1}{x-a}\right)^{\wedge}(\xi) = \left(\tau_a \frac{1}{x}\right)^{\wedge}(\xi) = e^{-2\pi i a\xi}\left(-i\pi\, sign(\xi)\right).$$

ZAD. 8. $\mathcal{F}\left(\frac{1}{a^2+x^2}\right) = ?$, $a>0$.

Rj: Ovaj put pomoću formula.

$$\frac{1}{a^2+x^2} = \frac{1}{(x+ia)(x-ia)}$$ rastav na parcijalne razlomke

$$= \frac{A}{x+ia}+\frac{B}{x-ia}, \quad A=-\frac{1}{2ia}, \quad B=\frac{1}{2ia}$$

$h_1(x) = \frac{1}{a-ix}$

$\hat{h}_1(x) = \tilde{f}_2(\xi) = f_1(\xi)$

$\Rightarrow \widehat{h_1(\tfrac{x}{2\pi})} = 2\pi\, f_1(2\pi\xi)$

$$\left(\frac{A}{x+ia}\right)^{\wedge}(\xi) = \frac{1}{2a}\left(\frac{-1}{-a+ix}\right)^{\wedge}(\xi) = \frac{1}{2a}\widehat{h_1\left(\tfrac{\cdot}{2\pi}\right)}(\xi)$$

$$= \frac{\pi}{a}e^{-2\pi a\xi}H(2\pi\xi)$$

$$\left(\frac{B}{x-ia}\right)^{\wedge}(\xi) = \frac{1}{2a}\left(\frac{1}{a+ix}\right)^{\wedge}(\xi) = \frac{1}{2a}\widehat{h_2\left(\tfrac{\cdot}{2\pi}\right)}(\xi)$$

$$= \frac{\pi}{a}e^{2\pi a\xi}H(-2\pi\xi)$$

$$\Rightarrow \left(\frac{1}{a^2+x^2}\right)^{\wedge}(\xi) = \frac{\pi}{a}e^{-2\pi a|\xi|}$$

Dobili smo isti rezultat kao ranije, a ovdje smo koristili formule za F. pretvorbu f-ja $h_1(x) = \frac{-1}{-a+2\pi i x}$ i $h_2(x) = \frac{1}{a+2\pi i x}$.

~~ZAD. Isto, ali za $a<0$.~~ ~~DZ~~

$\frac{1}{x-a} = \tau_a \, pv\left(\frac{1}{x}\right)$, $a>0$

ali ako je $a \in \mathbb{C}$, $\operatorname{Im} a \neq 0$ tada je $\frac{1}{x-a}$ „dobra" f-ja kojoj smo našli F. pretvorbu (jedan od prethodnih zadataka), i ne vrijedi tada

$$\frac{1}{x-a} = \tau_a \, pv\left(\frac{1}{x}\right)$$

[VIDI: Howell: Principles of Fourier Analysis str. 685]

ZAD. 9. $\mathcal{F}\left(\frac{2x-1}{(x-2)(x+1)} * \delta_1\right) = ?$

Rj:

$$\mathcal{F}\left(\frac{2x-1}{(x-2)(x+1)} * \delta_1\right) = \left(\frac{2x-1}{(x-2)(x+1)}\right)^\wedge(\xi)\ \hat{\delta}_1(\xi)$$

- $\hat{\delta}_1(\xi) = e^{-2\pi i \xi}$

- $\frac{2x-1}{(x-2)(x+1)} = \frac{1}{x-2} + \frac{1}{x+1}$

$$\Rightarrow \left(\frac{2x-1}{(x-2)(x+1)}\right)^\wedge(\xi) = \left(\tau_2 \frac{1}{x}\right)^\wedge(\xi) + \left(\tau_{-1} \frac{1}{x}\right)^\wedge(\xi)$$

$$= e^{-2\pi i 2\xi}(-i\pi \operatorname{sign}(\xi)) + e^{-2\pi i(-1)\xi}(-i\pi \operatorname{sign}(\xi))$$

$$= -i\pi \operatorname{sign}(\xi)\left(e^{-4\pi i \xi} + e^{2\pi i \xi}\right)$$

$\Rightarrow$ rješenje je: $\underline{-i\pi \operatorname{sign}(\xi)\left(e^{-6\pi i \xi} + 1\right)}$.

NAPOMENA. Drugi način računanja $(\operatorname{sign}(x))^\wedge$.

Uočimo $f_a(x) := \operatorname{sign}(x)\, e^{-a|x|} \xrightarrow[a\to 0]{\mathcal{S}'} \operatorname{sign}(x)$.

Budući da je $\mathcal{F}$ neprekinuta na $\mathcal{S}'$ imamo:

$$(\operatorname{sign}(x))^\wedge(\xi) = \lim_{a\to 0} \hat{f}_a(\xi) = \lim_{a\to 0} \frac{-4\pi i \xi}{a^2+4\pi^2\xi^2} = -\frac{i}{\pi}\frac{1}{\xi} = \frac{1}{i\pi}\frac{1}{\xi}$$

ZAD. 10. Izračunajte $\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx$

Rj.

Plancherelova f.

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} \, \frac{\sin x}{x} \, dx = \int_{-\infty}^{+\infty} \left(\left(\frac{\sin x}{x}\right)^{\wedge}(\xi)\right)^2 d\xi$$

$$= \pi^2 \int_{-\infty}^{+\infty} \chi_{\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]}(\xi)\, d\xi$$

$$= \pi^2 \frac{1}{\pi}$$

$$= \pi$$

ELEMENTARNA RJEŠENJA OSNOVNIH PDJ

1. JEDNADŽBA PROVOĐENJA

$$P = \frac{d}{dt} - \nu \Delta_x \quad , \nu > 0 .$$

$$\Phi_t - \nu \Delta \Phi = \delta_0(t,x) \quad , \quad \Phi \ldots \text{elem. rj.}$$

Djelujemo F. pretvorbom samo po x varijablama:

$$\partial_t \hat{\Phi} + 4\pi^2 \nu |\xi|^2 \hat{\Phi} = 1(\xi)\, \delta_0(t) \quad , \quad \mathcal{F}_x(\delta_0(t,x))(\xi) = \mathcal{F}_x(\delta_0(t)\,\delta_0(x))(\xi)$$

$$= \delta_0(t)\, \mathcal{F}_x(\delta_0(x))(\xi)$$

$$= \delta_0(t)\, 1(\xi)$$

Tražimo elem. rješenje ~~...~~ običnog dif. op.:

$$\frac{d}{dt} + \underbrace{4\pi^2 \nu |\xi|^2}_{}$$ ← ne ovisi o t

Prije (str. 20.): $\hat{\Phi}_\xi = H(t)\, f(t)$,

$$\begin{cases} f' + (4\pi^2 \nu |\xi|^2) f = 0 \\ f(0) = 1 \end{cases}$$

$$\Rightarrow f(t) = e^{-4\pi^2 \nu |\xi|^2 t}$$

$$\Rightarrow \hat{\Phi}(t,\xi) = H(t)\, e^{-4\pi^2 \nu |\xi|^2 t} \quad \Big/ \ \mathcal{F}_\xi^{-1} = \overline{\mathcal{F}}_\xi$$

$$\Rightarrow \boxed{\Phi(t,x) = H(t) \frac{1}{(4\pi\nu t)^{d/2}}\, e^{-\frac{|x|^2}{4\nu t}}}$$

$$u_t - \nu \Delta u = f$$ ← vidjet ćemo poslije kako bismo uveli i poč. uvjet, tj. rješavali na $\mathbb{R}^+$, a ne na $\mathbb{R}$ po t

$$u = (\Phi * f) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} \frac{1}{(4\pi\nu(t-\tau))^{d/2}}\, e^{-\frac{|x-\xi|^2}{4\nu(t-\tau)}}\, f(\tau,\xi)\, d\xi\, d\tau$$

↑ $H(t-\tau) \Rightarrow \tau \le t$

② LAPLACEOVA JEDNADŽBA

$$P = -\Delta$$

$-\Delta\Phi = \delta_0 \,/\,\hat{}$; Φ ... elem. rj.

$$4\pi^2|\xi|^2\hat{\Phi}(\xi) = 1$$

a) $\underline{d=1}$

$$4\pi^2\xi^2\hat{\Phi}(\xi) = 1 \Rightarrow \xi^2\left(4\pi^2\hat{\Phi}(\xi) - Pf\left(\tfrac{1}{\xi^2}\right)\right) = 0$$

$$\vdots$$

$$\Rightarrow \hat{\Phi}(\xi) = \frac{1}{4\pi^2}Pf\left(\frac{1}{\xi^2}\right) + C_1\delta_0 + C_2\delta_0'$$

$$\Rightarrow \Phi(x) = \frac{1}{2}x\,\mathrm{sign}(-x) + \underbrace{Ax+B}_{\text{harmonička f-ja}}$$

$$\left(Pf\left(\tfrac{1}{\xi^2}\right)\right)^{\vee}(x) = 2\pi^2 x\,\mathrm{sign}(-x)$$

harmonička f-ja (nisu to jedine harmoničke, već jedine u $\mathcal{S}'$)

$$\Rightarrow \boxed{\Phi(x) = \frac{1}{2}x\,\mathrm{sign}(-x)}$$ ← možemo uzeti

b) $\underline{d=2}$ ($\frac{1}{|\xi|^\alpha}$ je u $L^1_{loc}(\mathbb{R}^d)$ za $0<\alpha<d$ pa $\frac{1}{|\xi|^2} \notin L^1_{loc}(\mathbb{R}^2)$)

Definiramo distribuciju:

$$\left\langle Pf\left(\frac{1}{|\xi|^2}\right), \varphi\right\rangle := \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1}\frac{\overline{\varphi}(x)-\overline{\varphi}(0)}{|x|^2}dx + \int_{|x|\geq 1}\frac{\overline{\varphi}(x)}{|x|^2}dx$$

Pokaže se $Pf\frac{1}{|\xi|^2} \in \mathcal{S}'(\mathbb{R}^2)$ & → $|\varphi(x)-\varphi(0)| \leq C|x|$, a $\frac{1}{|x|}$ je lokalno int.

$$\left(Pf\frac{1}{|\xi|^2}\right)^{\wedge}(x) = -2\pi\ln|x| - 2\pi C, \quad C \text{ ... konst.}$$

$$\left\langle 4\pi^2|\xi|^2\left(\frac{1}{4\pi^2}Pf\left(\frac{1}{|\xi|^2}\right)\right), \varphi\right\rangle = \left\langle Pf\frac{1}{|\xi|^2}, |\xi|^2\varphi\right\rangle$$

$$= \lim_{\varepsilon\to 0}\int_{\varepsilon<|\xi|<1}\frac{\overline{\varphi}(\xi)|\xi|^2 - \left(\overline{\varphi}(\xi)|\xi|^2\right)\big|_{\xi=0}}{|\xi|^2} + \int_{|\xi|\geq 1}\frac{|\xi|^2\overline{\varphi}(\xi)}{|\xi|^2}d\xi$$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon < |\xi| < 1} \bar{\varphi}(\xi)\, d\xi + \int_{|\xi| \geq 1} \bar{\varphi}(\xi)\, d\xi$$

$$= \int_{\mathbb{R}^2} \bar{\varphi}(\xi)\, d\xi$$

$$= \langle \mathbb{1}, \varphi \rangle \quad , \varphi \in \mathcal{S} .$$

$\Rightarrow$ uzet ćemo $\hat{\Phi}(\xi) = \frac{1}{4\pi^2} \mathrm{Pf} \frac{1}{|\xi|^2}$

$$\Rightarrow \quad \Phi(x) = \frac{1}{4\pi^2} \left(\mathrm{Pf} \frac{1}{|\xi|^2}\right)^{\vee}(x)$$

$$= \frac{1}{4\pi^2} \left(-2\pi \ln|x| - 2\pi C\right)$$

$$= -\frac{1}{2\pi} \ln|x| - \underbrace{\frac{\mathbb{C}}{2\pi}}$$

harmonička f-ja, tj. rješenje homogene pa za elem. rj. možemo uzeti:

$$\boxed{\Phi(x) = -\frac{1}{2\pi} \ln|x|}$$

c) $\underline{d=3}$

$\frac{1}{4\pi^2 |\xi|^2}$ je lokalno integrabilna

$$\Rightarrow \quad \hat{\Phi}(\xi) = \frac{1}{4\pi^2 |\xi|^2}$$

$$\Rightarrow \quad \Phi(x) = \frac{1}{4\pi |x|}$$

[Vladimirov: Eq. of Math. Phy., (9.44) str. 156
&
DZ 2, 2013, ZAD 7.

$f(x) := |x|^{-\alpha}$, $0 < \mathrm{Re}(\alpha) < d$

$f : \mathbb{R}^d \to \mathbb{R}$

$\hat{f}(\xi) = C_\alpha |\xi|^{-(d-\alpha)}$

($d=3 \Rightarrow C_\alpha = 1$)

d) $\underline{d \geq 3}$

Osim prethodnom metodom može i na sljedeći način

$$\Phi(x) = \int_{-\infty}^{+\infty} \Phi_H(t, x)\, dt \quad , \text{ gdje je } \Phi_H \text{ elementarno rj. jednadžbe provođenja}$$

Laplaceova jednadžba

$$-\Delta u = f \quad \text{na } \mathbb{R}^d$$

Ako postoji rješenje nije jedinstveno (uvijek možemo dodati netrivijalnu harmoničku f-ju).

Pretpostavimo da postoji Φ t. d. $-\Delta\Phi = \delta_0$. Tada imamo:

$$-\Delta(\Phi * f) = (-\Delta\Phi) * f = \delta_0 * f = f$$

svojstvo diferenciranja konvolucije; neutralni element

$\Rightarrow \Phi * f$ je rješenje. ← USPOREDITI S PDJ 1

(gornji postupak ima smisla za $\Phi \in \mathcal{S}'$ i $f \in \mathcal{S}$).

Φ... fundamentalno rješenje

$$\left\{\begin{array}{l} f(x) := |x|^{-\alpha}, \quad 0 < \mathrm{Re}(\alpha) < d \\ f: \mathbb{R}^d \to \mathbb{R} \\ \hat{f}(\xi) = c_\alpha |\xi|^{-(d-\alpha)} \end{array}\right. \qquad (d=3,\ c_\alpha = 1)$$

<u>d=3</u>

$$-\Delta\Phi = \delta_0 \quad /\,\hat{}$$

$$+4\pi^2|\xi|^2\hat{\Phi}(\xi) = 1$$

$$(*)\quad \hat{\Phi}(\xi) = +\frac{1}{4\pi^2}\frac{1}{|\xi|^2} \implies \Phi(x) = +\frac{1}{4\pi^2|x|}$$

PROBLEM: Rekli smo da rješenje nije jedinstveno, a dobili smo samo jedno rješenje.

RAZLOG: (*) nije korektan jer ~~se~~ se ne radi o množenju f-ja, nego o množenju f-je i distribucije, npr. u 1D:

$$\xi^2\hat{\Phi}(\xi) = 1 \implies \xi^2\left(\hat{\Phi}(\xi) - \mathrm{Pf}\left(\tfrac{1}{\xi^2}\right)\right) = 0$$

$$\vdots$$

$$\not\Rightarrow \hat{\Phi}(\xi) = \frac{1}{\xi^2} \qquad \hat{\Phi}(\xi) = \mathrm{Pf}\left(\tfrac{1}{\xi^2}\right) + C_1\delta_0 + C_2\delta_0'$$

$(\mathcal{P}f(\frac{1}{\xi^2}))^\wedge(x) = 2\pi^2 x\,\mathrm{sign}(-x)$

$\Rightarrow$ (pomnožimo s $\frac{1}{4\pi^2}$) jedno posebno rj. je $\frac{1}{2} x\,\mathrm{sign}(-x)$

$$\Phi(\xi) = (\mathcal{P}f(\tfrac{1}{\xi^2}))^\wedge + \underbrace{(C_1\delta_0 + C_2\delta_0')^\wedge(\xi)}_{= A\xi + B \ \dots \text{ polinomijalna harmonička f-ja}}$$

Općenito je to istina da samo dodavanjem polinomijalnih harmoničkih f-ja dobivamo sva rješenja. Naglasak je na polinomijalne jer nepolinomijalne harmoničke f-je brzo rastu pa nisu temperirane distribucije, a time ih nismo ni mogli dobiti jer nije definirana njihova F. pretvorba.

Neka je $\Omega \subseteq \mathbb{R}^d$ otvoren i ograničen, $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$.

$$\begin{cases} -\Delta u = f & \text{u } \Omega \\ u = g & \text{na } \partial\Omega \end{cases}$$

Fourierova pretvorba je definirana za f-je koje su definirane na cijelom $\mathbb{R}^d$ pa moramo proširiti f-ju f (najlakše ju je samo proširiti nulom). Označimo proširenje s F.

$$-\Delta(\Phi * F) = F \quad \& \quad F|_\Omega = f$$

$$\Rightarrow \quad v := \Phi * F|_\Omega \text{ zadovoljava}$$

$$-\Delta v = f \quad \text{u } \Omega$$

$w := u - v$

w mora zadovoljavati : $\begin{cases} -\Delta w = 0 & \text{u } \Omega \\ w = g - h & \text{na } \partial\Omega, \end{cases}$

gdje je $h = \Phi * F|_{\partial\Omega}$.

Može se pokazati da je h neprekinuta f-ja pa je klasičan Dirichletov problem : naći harmoničku f-ju na Ω s definiranom vrijednosti na $\partial\Omega$.

Tada je $u = w + v$ rješenje.

Ovdje uočavamo da Fourierova pretvorba nije pogodna za rubne probleme, tj. glavni problem kod rješavanja ostaje.

Schrödingerova jednadžba

$u_t = i\Delta u \quad , \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d$

$\hookrightarrow$ jednadžba gibanja čestice bez vanjske sile

$$P(\text{čestica u } E \text{ u vremenu } t) = \int_E |u(t,x)|^2 dx$$

$$P(\text{moment čestice je u } E \text{ u vremenu } t) = \int_E |\hat{u}(t,\xi)|^2 d\xi$$

Može se pokazati da $\int_E |u(t,x)|^2 dx$ ne ovisi o t pa zbog gornje interpretacije smo uzeli $\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = 1$

(sad po Plancharelovoj formuli slijedi i $\|\hat{u}(t,\cdot)\|_{L^2(\mathbb{R}^d)} = 1$).

Iz gornjeg vidimo da je f-ja $x \mapsto |u(t,x)|^2$ gustoća slučajne varijable pronalaska čestice, a analogno je $\xi \mapsto |\hat{u}(t,\xi)|^2$ gustoća za moment pa uvodimo i sljedeće pojmove iz vjerojatnosti:

$$x^{av} := \int_{\mathbb{R}^d} x |u(t,x)|^2 dx \quad \dots \text{ očekivani položaj u vremenu } t$$

$$\xi^{av} := \int_{\mathbb{R}^d} \xi |\hat{u}(t,\xi)|^2 d\xi \quad \dots \text{ očekivani moment } \text{—— '' ——}$$

$$\left.\begin{aligned} (\delta x_j) &:= \int_{\mathbb{R}^d} (x_j - x_j^{av}) |u|^2 dx \dots \\ (\delta \xi_j) &:= \int_{\mathbb{R}^d} (\xi_j - \xi_j^{av}) |\hat{u}|^2 d\xi \dots \end{aligned}\right\} \text{varijanca ili disperzija}$$

$$\frac{\begin{cases} \partial_t u = i\Delta u & / \mathcal{F}_x \\ u(0,\cdot) = f & / \mathcal{F}_x \end{cases}}{\begin{aligned} \partial_t \hat{u} &= -4\pi^2 i |\xi|^2 \hat{u} \\ \hat{u}(0,\cdot) &= \hat{f} \end{aligned}}$$

$$\Rightarrow \hat{u}(t,\xi) = e^{-4\pi^2 i t |\xi|^2} \hat{f}(\xi)$$

$$\Rightarrow u(t,x) = \bar{\mathcal{F}}\left(e^{-4\pi^2 i t |\xi|^2} \hat{f}(\xi)\right)$$

3. SCHRÖDINGEROVA JEDNADŽBA

$$P = \frac{d}{dt} - i\nu\Delta_x, \quad \nu > 0.$$

$$\frac{d}{dt}\Phi - i\nu\Delta\Phi = \delta_0(t,x) \;/\; \mathcal{F}_x$$

$$\frac{d}{dt}\hat{\Phi} + 4\pi^2\nu|\xi|^2 i\,\hat{\Phi} = \delta_0(t)\,\mathbb{1}(\xi)$$

Kao za j. provođenja $\Rightarrow$

$$\Rightarrow \hat{\Phi}(t,\xi) = H(t)\,e^{-4\pi^2\nu|\xi|^2 t i}$$

PROBLEM: Naći F. pretvorbu f-je $x \mapsto e^{-a|x|^2}$, $a \in \mathbb{C}$, $\mathrm{Re}\, a \geq 0$.

1. $u \in \mathcal{D}'(\mathbb{R})$ rješava $\frac{du}{dx} + \lambda x u = 0$, $\lambda \in \mathbb{C} \setminus \{0\}$

ako je $u(x) = C e^{-\lambda \frac{x^2}{2}}$.

Posebno, za $\mathrm{Re}\lambda \geq 0$ je $u \in \mathcal{S}'$.

2. $v := \hat{u}$ rješava sljedeću diferencijalnu jednadžbu:

$$\lambda v' + 4\pi^2 \xi v = 0$$

(jer je $\widehat{u'} = 2\pi i \xi \hat{u}$, a $\widehat{-2\pi i x u} = \hat{u}'$),

te je stoga

$$v(\xi) = K e^{-2\pi^2 \xi^2/\lambda}$$

(uzimamo $\mathrm{Re}\lambda \geq 0$, $\lambda \neq 0$).

3. $u(x) = e^{-x^2}$ znači: $C = 1$, $\lambda = 2$; pa je

$$v(\xi) = K e^{-\pi^2 \xi^2}.$$

K određujemo iz uvjeta da je $K = v(0) = \int_{-\infty}^{\infty} u(x)dx = \sqrt{\pi}$.

4. $u(x) = e^{-4\pi^2 t x^2 i}$, $C = 1$, $\lambda = +8\pi^2 t i$ $\quad(\mathrm{Re}\lambda = 0)$

$$v(\xi) = K e^{-2\pi^2 \xi^2/\lambda} = K e^{+i\xi^2/4t}$$

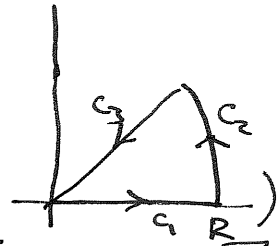
[Posebno, $\frac{1}{4t} = \pi$ daje $u(x) = e^{-i\pi x^2}$, $v(\xi) = K e^{i\pi \xi^2}$

pa je $v = K\bar{u}$, $u = \bar{\mathcal{F}} v = K \bar{\mathcal{F}} \bar{u} = K \overline{\hat{u}} = K \bar{v} = |K|^2 u$;

odakle je $|K|^2 = 1$. Niti u ovom slučaju ne možemo dobiti točan izraz za K.]

Pokušajmo kao u 3:

$$K = v(0) = \int_{-\infty}^{\infty} u(x)\,dx = \int_{-\infty}^{\infty} e^{-4\pi^2 t i x^2}\,dx$$



Poznato je (integracija po kompleksnom području) da je $2\int_0^{\infty} e^{ix^2}dx = (1+i)\sqrt{\frac{\pi}{2}}$, odnosno $2\int_0^{\infty} \overline{e^{ix^2}}\,dx = (1-i)\sqrt{\frac{\pi}{2}}$

$$\int_{-\infty}^{\infty} e^{-(4\pi^2 t) i x^2}dx = \frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{-iy^2}dy = \frac{1}{2\pi\sqrt{t}}(1-i)\sqrt{\frac{\pi}{2}} = \frac{1-i}{2\sqrt{2\pi t}}$$

$y = 2\pi\sqrt{t}\,x,\ dy = 2\pi\sqrt{t}\,dx$

Dakle, $\hat{u}(\xi) = \dfrac{1-i}{2\sqrt{2\pi t}}\, e^{i\xi^2/4t}$.

Napomena: $$\oint_{C_1\cup C_2\cup C_3} e^{iz^2}dz = 0.$$

$\oint_{C_1}$ ... traženi

$\oint_{C_2}$ ... Jordanova lema ... $\to 0$ za $R \to \infty$

$\oint_{C_3}$ ... $z(t) = te^{i\frac{\pi}{4}} = \dfrac{1+i}{\sqrt{2}}t$

$dz = \dfrac{1+i}{\sqrt{2}}dt$

$$-\oint_{C_3} e^{iz^2}dz = \ldots = \frac{1+i}{\sqrt{2}}\underbrace{\int_0^{\infty} e^{-t^2}dt}_{=\sqrt{\pi}/2}$$

47

Fresnel Integrals

David Sirajuddin

Itcanbeshown.com
May 29, 2008

The following integral

$$\int_0^\infty \cos x^2 dx \tag{1}$$

can be evaluated by way of complex calculus (see *An Engineer's Guide to Complex Integration*). At first, the problem can prove difficult due to the odd choice of complex contour one must use in order to obtain a solution. This integral is known as a *Fresnel integral* and arises in the field of optics in the description of near field Fresnel diffraction. While the function is transcendental when evaluating the integral over definite limits, a solution can be found when the bounds are treated as semi-infinite. In fact, the convergence of the real integral over semi-infinite bounds is suggested when looking at a trace of the function (Figure 1).

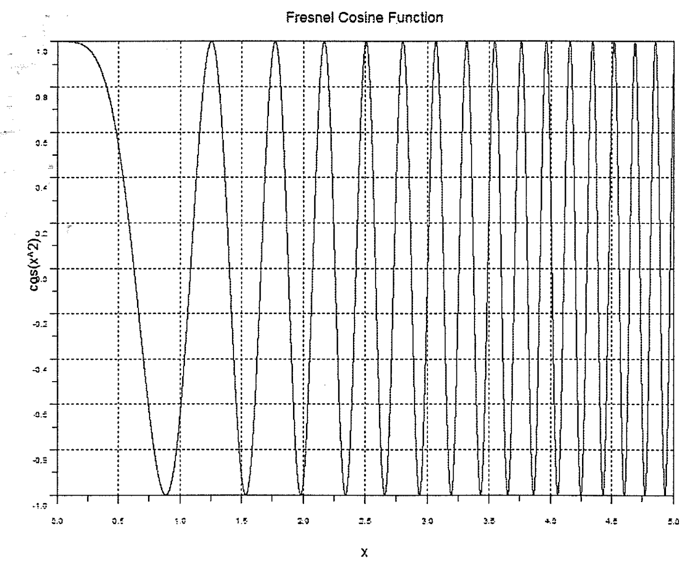


Figure 1 - A plot is shown of the Fresnel cosine function. The frequency increases with x.

As evident from above, the frequency of the function increases with x, until the wavelength of the function tends to zero as $x \to \infty$. The function oscillates above and below the x-axis, suggesting that it is possible in the limit for large x, that sufficient contributions from the area sweeped out by the function will be cancelled out by its negative and positive portions, leading to a finite result. This is precisely the case, and this finite value of the integral is found via complex integration.

The real-valued function $f(x) = \cos x^2$ is transposed into the complex domain as a complex exponential $f(z) = \exp(iz^2)$. The complex function f is identified to hold no singularities; however, this only suggests that the complex closed contour integral of this function about any domain is zero, not that all path integrals making up the closed contour are themselves zero. Thus, a solution could still be obtained in this manner. In order to choose a proper contour, begin by examining the behavior of the complex exponential function in the complex plane, this is qualitatively shown in the figure below:

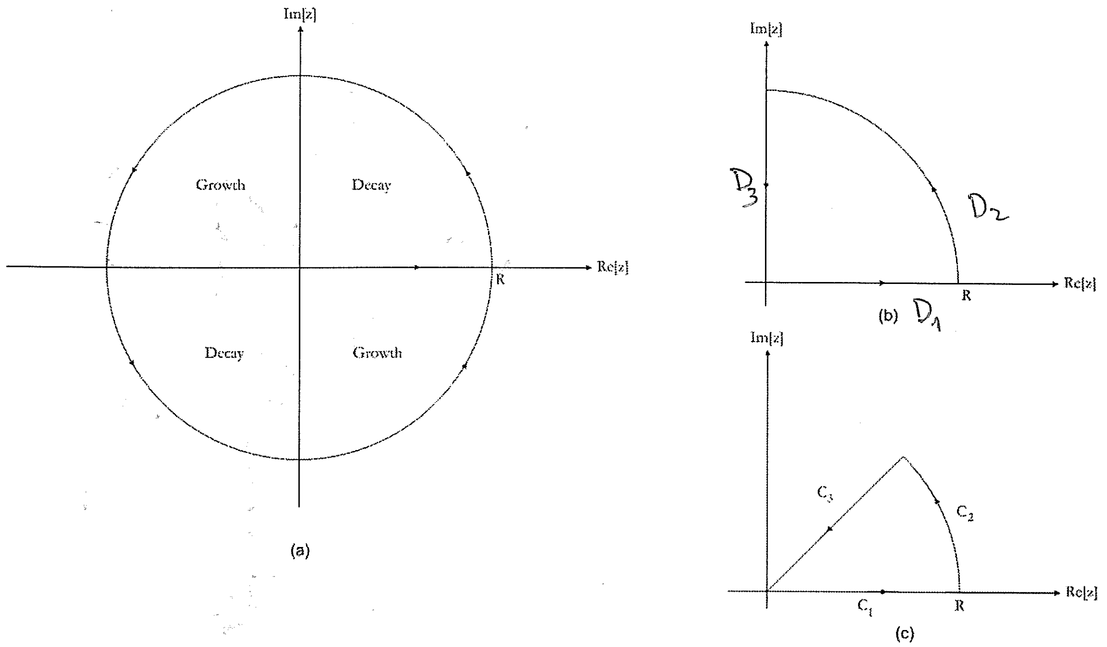


Figure 2 - The behavior of the function e^{iz^2} is examined. In (a) the function is described along a positively oriented closed circular contour. The regions where the function decays and increases are labelled. In (b), a proposed quarter circle contour is shown that evades the increasing regions of the function. And, in (c) an eighth circle is given, which is a contour that allows for the integration to yield a proper, finite value.

It is seen by inspection that, when examined along a positively oriented closed circular contour, the complex exponential function $\exp(iz^2)$ increases and decays in different quadrants. The integral of the function in a region of growth is not capable of admitting a finite result since the limits of integration (and hence the radius of any circular path) must eventually be extended to infinity in accordance with the original problem. However, it would seem that it is still possible to integrate, so long as the increasing regions of the function are avoided. Thus, a first choice of contour could be the quarter-circle illustrated in Figure 2(b). However, when choosing this contour, applying residue theorem, and taking limits, the solution to the problem does not yield a value, but rather an identity:

Integral po D_2 je nula kao što je i po C_2 (vidi poslije)

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx \qquad (2)$$

$\int_{D_1} e^{iz^2} dz \rightarrow \int_0^\infty e^{ix^2} dx$

This is certainly nice to know, but a value of both these functions integrated over the stipulated limits is still yet to be found. In order to integrate the function over the desired bounds so as to obtain a finite result, the 1/8th circle, shown in Figure 5(c), is used.

Applying residue theorem to the system provides the following statement

$\int_{D_2} e^{iz^2} dz \underset{\substack{z=iy \\ dz=idy}}{=} i\int_R^0 e^{-iy^2} dy \rightarrow -\int_0^\infty e^{i(-y^2+\frac{\pi}{2})} dy$

$\Rightarrow$ analogija (samo gledam Re)

$$\oint_C e^{iz^2}\,dz = 0$$ ← moramo, ovaj integral apsolutno ne konvergira na $\langle 0, +\infty\rangle$, ali on sam jest

where C is used to denote the closed contour shown in Figure 2(c). Notice how since there are no residues contained within the contour, the contour integral of the function is equal to zero. That is to say, residue theorem is reduced to Cauchy's integral theorem in this instance. The integral can further be broken up into a summation of path integrals, where the sum of the paths is equivalent to the contour C.

$$\underbrace{\int_{C_1} e^{iz^2}\,dz}_{\mathrm{I}} + \underbrace{\int_{C_2} e^{iz^2}\,dz}_{\mathrm{II}} - \underbrace{\int_{C_3} e^{iz^2}\,dz}_{\mathrm{III}} = 0 \tag{3}$$

The integrals have been labelled as I, II, and III for convenient referencing, and each path C_1, C_2, and C_3 are as shown in Figure 5(c). Each integral is evaluated below.

Integral I

The only work that needs to be done on this integral is to parameterize it along the real line, and take limits. Thus,

$$\int_{C_1} e^{iz^2}\,dz = \int_0^R e^{ix^2}\,dx$$

since $f(z) = f(x,y)$, and $y = 0$ for all points on the real line. Letting $R \to \infty$, Integral I becomes

$$\int_0^\infty e^{ix^2}\,dx. \tag{4}$$

where it is noted that the real part of this integral can be taken to make this equation of the same form as the original integral in the problem statement (Eqn. (4)).

Integral II

Since integral II has a complex-valued, nonconstant path, it would be convenient to prove this integral tends to zero. This integral does – indeed – turn out to be identical to zero, by way of Jordan's lemma. This is shown by initially factoring the function in the integrand in the following way

$$\int_{C_2} e^{iz^2}\,dz = \int_{C_2} e^{i(z^2-z)}e^{iz}\,dz = \int_{C_2} g(z)e^{iz}\,dz.$$

The above equation is now in the form of the statement in Jordan's lemma, where $g(z) = e^{iz(z-1)}$. Since the function g is *entire*, that is, it is analytic for all points in the complex plane, it meets the requirements for Jordan's lemma. Thus, it can be said that

$$\int_{C_2} e^{iz^2}\,dz = 0 \tag{5}$$

$z = Re^{i\theta}$

$g(z) = e^{R^2 e^{i(2\theta+\frac{\pi}{2})} - R e^{i(\theta+\frac{\pi}{2})}} \Rightarrow |g(z)| = e^{\overbrace{R^2\cos(2\theta+\frac{\pi}{2})}^{<0} - R\cos(\theta+\frac{\pi}{2})} \searrow 0, \ R\to\infty$

nalazimo se na C_2 pa je $\theta \in [0, \frac{\pi}{4}] \Rightarrow$
$\Rightarrow 2\theta + \frac{\pi}{2} \in [\frac{\pi}{2}, \pi]$

Integral III

For the third integral, the following parametrization is introduced

$$z(t) = te^{i(\pi/4)} = \frac{1+i}{\sqrt{2}}t \quad (0 \leq t \leq t_0)$$

it then follows that

$$dz = \frac{1+i}{\sqrt{2}}dt$$

for an appropriate value of t_0, such that $\text{dist}|0, z(t_0)| = R$. Inputting these substitutions into integral III in Eqn. (3)

$$\begin{aligned}
\int_{C_3} e^{iz^2}dz &= \frac{1+i}{\sqrt{2}} \int_0^{t_0} \exp\left[i\left(e^{i\frac{\pi}{4}t}\right)^2\right] dt \\
&= \frac{1+i}{\sqrt{2}} \int_0^{t_0} \exp\left[i\left(e^{i\frac{\pi}{4}t^2}\right)^2\right] dt \\
&= \frac{1+i}{\sqrt{2}} \int_0^{t_0} \exp(i^2t^2)dt \\
&= \frac{1+i}{\sqrt{2}} \int_0^{t_0} e^{-t^2}dt
\end{aligned}$$

nema minusa jer je u (3) uzet integral s minusom

Furthermore, if t_0 is allowed to extend to infinity then

$$\int_{C_3} e^{iz^2}dz = \frac{1+i}{\sqrt{2}} \int_0^{\infty} e^{-t^2}dt$$

The parametrization has rendered the original integral into a *Gaussian integral.* The above integral can be evaluated in a number of ways. Such methods include Feynmann's so-called parametric integration, or by integrating the square of the integral in polar coordinates, or by using the gamma function $\Gamma(t)$. It is identified that the integral above is equivalent to $\Gamma(3/2) = \sqrt{\pi}/2$. Thus, the solution to the above equation can be written as

$$\frac{1+i}{\sqrt{2}} \int_0^{\infty} e^{-t^2}dt = \left(\frac{1+i}{\sqrt{2}}\right)\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}} \tag{6}$$

Inserting Eqns. (4), (5), and (6) into (3), and solving for the real integral reveals

$$\int_0^{\infty} e^{ix^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}$$

Taking the real part of the above equation, and combining this result with the identity shown in Eqn. (2), the solution to integral (1) is found to be

$$\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{7}$$

Općenito, e^{-ax^2}, $a \in \mathbb{C}$, $\mathrm{Re}\, a \geq 0$.

Radimo kao u prethodnom računu, samo kut za C_3 uzimamo t.d. taj integral postane realan, tj.

$$\left.\begin{aligned} a &= |a| e^{i\theta_0} \\ z(t) &= t e^{i\theta} \end{aligned}\right\} \Rightarrow \; -az^2 = -|a| t^2 \underbrace{e^{i(2\theta+\theta_0)}}_{\text{želimo da ovo bude } = 1}$$

$$\Rightarrow \boxed{\theta = -\frac{\theta_0}{2}}$$

(ovo isto razmatranje može biti uzeto u prethodnom računu za $a = -i$ pa je $\theta_0 = -\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$)

$$\Rightarrow \int_{C_3} e^{-az^2} dz = e^{-i\frac{\theta_0}{2}} \int_0^{t_0} e^{-|a|t^2} dt \xrightarrow[t_0 \to \infty]{} \frac{1}{2}\sqrt{\frac{\pi}{|a|}}\, e^{-i\frac{\theta_0}{2}}$$

($\mathrm{Re}\, a \geq 0$ nam daje da g u C_2 opet zadovoljava Jordanovu lemu)

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{|a|}}\, e^{-i\frac{\theta_0}{2}}, \qquad a = |a| e^{i\theta_0}, \quad \theta_0 = \mathrm{arctg}\left(\frac{\mathrm{Im}\, a}{\mathrm{Re}\, a}\right) \ldots \text{ istog je predznaka kao i } \mathrm{Im}\, a.$$

Konačno:
$$\boxed{\begin{aligned} u(x) &= e^{-ax^2}, \quad a \in \mathbb{C},\ \mathrm{Re}\, a \geq 0 \\ \hat{u}(\xi) &= \sqrt{\frac{\pi}{|a|}}\, e^{-i\frac{\theta_0}{2}}\, e^{-\frac{\pi^2 \xi^2}{a}} \end{aligned}}$$

2. pristup

[Kenneth B. Howell: Principels of Fourier Analysis; § 23.3, § 34, ZAD 34.48]

↳ tu se može naći i alternativni oblik kompleksnog int. dobiven trig. relacijama:

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \frac{1}{|a|}\sqrt{\frac{\pi}{2}} \left(\sqrt{|a| + \mathrm{Re}\, a} - i\, \mathrm{sign}(\mathrm{Im}\, a) \cdot \sqrt{|a| - \mathrm{Re}\, a}\right)$$

za $\mathrm{Re}\, a > 0$

3. pristup

[NA: Kvantizacija; Lema 2]

↳ uzme se skalarna matrica B.

Izvedimo formulu i za $\mathbb{R}^d$.

$$\boxed{\begin{aligned} u(x) &= e^{-a|x|^2}, \quad a\in\mathbb{C},\ \mathrm{Re}\,a\geq 0 \\ \hat{u}(\xi) &= \left(\frac{\pi}{|a|}\right)^{d/2} e^{-i\frac{d\theta_0}{2}} e^{-\frac{\pi^2|\xi|^2}{a}} \end{aligned}} \quad (*)$$

ZADATAK JE NA IDUĆOJ STRANICI

Vratimo se sad zadatku.

$$u(t,x) = \left(\frac{\pi}{a}\right)^{d/2} \mathcal{F}\left(e^{-\left(\frac{\pi^2}{a}+4\pi^2 t i\right)|\xi|^2}\right)$$

$$b := \frac{\pi^2}{a} + 4\pi^2 t i \implies |b| = \pi^2\sqrt{\frac{1}{a^2}+16t^2} \quad \ldots \quad b = |b|e^{i\theta_0}$$

$$\theta_0 = \mathrm{arctg}\,(4at)$$

$$\bullet\ e^{-\frac{\pi^2|\xi|^2}{b}} = e^{-\frac{\bar{b}\pi^2|\xi|^2}{|b|^2}} = e^{-\frac{\frac{\pi^2}{a}\cancel{\pi^2}|\xi|^2}{\frac{\pi^4}{a^2}+16\pi^4 t^2}} \cdot e^{\frac{4\cancel{\pi^2} t i\, \cancel{\pi^2}|\xi|^2}{\pi^4\left(\frac{1}{a^2}+16t^2\right)}}$$

$$= e^{-\frac{a|\xi|^2}{1+16a^2t^2}}\, e^{\frac{i\,4a^2t\,|\xi|^2}{1+16a^2t^2}}$$

$$\implies u(t,x) = \left(\frac{\pi}{a}\right)^{d/2} \cdot \left(\frac{1}{\pi\sqrt{\frac{1}{a^2}+16t^2}}\right)^{d/2} e^{-i\frac{d\theta_0}{2}}\, e^{-\frac{a|\xi|^2}{1+16a^2t^2}}\, e^{\frac{i4a^2t\,|\xi|^2}{1+16a^2t^2}}$$

Ako uzmemo da kod kompleksnog korjenovanja uzimamo rezultat s realnim dijelom ≥ 0, onda se formula (*) može zapisati:

$$\hat{u}(\xi) = \left(\frac{\pi}{a}\right)^{d/2} e^{-\frac{\pi^2|\xi|^2}{a}},$$

pa dobivamo malo „ljepšu" formulu:

$$u(t,x) = \left(\frac{\pi}{a}\right)^{d/2}\left(\frac{1}{\frac{\pi}{a}+4\pi t i}\right)^{d/2} e^{-\frac{a|\xi|^2}{1+16a^2t^2}}\, e^{\frac{i4a^2t|\xi|^2}{1+16a^2t^2}}$$

$$= \underbrace{\left(\frac{1}{a}\right)^{d/2}\left(\frac{1}{\frac{1}{a}+4ti}\right)^{d/2}}_{=(1+4ati)^{-d/2}} e^{-\frac{a|\xi|^2}{1+16a^2t^2}}\, e^{\frac{i4a^2t|\xi|^2}{1+16a^2t^2}}$$

$\rightarrow$ rješenje se poklapa s Rauchom str. 102. za $a=\frac{a}{2}$.

ZAD. $\partial_t u = i \Delta u$

$u(0,\cdot) = f \quad , \quad f(x) = e^{-a|x|^2} \quad , \quad a > 0$

Rj.

$$u(t,x) = \overline{\mathcal{F}}\left(e^{-4\pi^2 i t |\xi|^2} \hat{f}(\xi)\right)$$

u 1D:

$f_1(x) = e^{-ax^2} \quad , \quad a > 0$

$\hat{f}_1(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2 \xi^2}{a}}$

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x)\, dx = \int_{\mathbb{R}^d} e^{-2\pi i \sum_{i=1}^{d} \xi_i x_i}\, e^{-a \sum_{k=1}^{d} x_k^2}\, dx$$

$$= \prod_{k=1}^{d} \left(\int_{\mathbb{R}} e^{-2\pi i \xi_k x} e^{-ax^2}\, dx \right) = \prod_{k=1}^{d} \hat{f}_1(\xi_k)$$

$$= \left(\frac{\pi}{a}\right)^{d/2} e^{-\frac{\pi^2 |\xi|^2}{a}}$$

$$\Rightarrow \quad u(t,x) = \overline{\mathcal{F}}\left(e^{-4\pi^2 i t |\xi|^2} \left(\frac{\pi}{a}\right)^{d/2} e^{-\frac{\pi^2 |\xi|^2}{a}} \right)$$

$$= \left(\frac{\pi}{a}\right)^{d/2} \overline{\mathcal{F}}\left(e^{-\left(\frac{\pi^2}{a} + 4\pi^2 t i\right)|\xi|^2} \right)$$

$$= \left(\frac{\pi}{a}\right)^{d/2} \mathcal{F}\left(e^{-\left(\frac{\pi^2}{a} + 4\pi^2 t i\right)|\xi|^2} \right)$$ (jer je funkcija parna)

Problem smo sveli na računanje F. pretvorbe $e^{-a|\xi|^2}$ za $a \in \mathbb{C}$ i $\operatorname{Re} a \geq 0$.

ZAD. $\begin{cases} \partial_t u = +i\Delta u \\ u(0,\cdot) = f, \quad f(x) = \delta_0 \end{cases}$

Rj:

$$u(t,x) = \bar{\mathcal{F}}\left(e^{-4\pi^2 i t|\xi|^2}\,\hat{\delta}_0\right) = \bar{\mathcal{F}}\left(e^{-4\pi^2 i t|\xi|^2}\right)$$

$$a = 4\pi^2 t i$$

$$u(t,x) = \left(\frac{1-i}{2\sqrt{2\pi t}}\right)^d e^{\frac{i|x|^2}{4t}}$$

ili

$$u(t,x) = \left(\frac{1}{4\pi t i}\right)^{d/2} e^{\frac{i|x|^2}{4t}}$$

Jednadžba provođenja

$$\begin{cases} u_t = \nu\Delta u \\ u(0,\cdot) = f \end{cases}, \quad \nu > 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

Izvod kao za Schrödingerovu j.:

$$\begin{cases} \hat{u}_t = -\nu 4\pi^2 |\xi|^2 \hat{u} \\ \hat{u}(0,\cdot) = \hat{f} \end{cases}$$

$$\Rightarrow \hat{u} = e^{-\nu 4\pi^2 t|\xi|^2} \hat{f}(\xi)$$

$$\Rightarrow u(t,x) = \bar{\mathcal{F}}\left(\underbrace{e^{-\nu 4\pi^2 t|\xi|^2}}\,\hat{f}(\xi)\right)$$

ova f-ja je parna pa kad je $\hat{f}$ parna možemo umjesto $\bar{\mathcal{F}}$ računati $\mathcal{F}$ jer će i $\mathcal{F}$. pretvorba biti parna.

ZAD.

$$\begin{cases} \partial_t u = \nu \Delta u \\ u(0,\cdot) = f, \quad f = e^{-a|x|^2}, \quad a>0 \end{cases}$$

Rj.

$$u(t,x) = \left(\frac{\pi}{a}\right)^{d/2} \bar{\mathcal{F}}\left(e^{-\nu 4\pi^2 t|\xi|^2} e^{-\frac{\pi^2|\xi|^2}{a}}\right)$$ ← pazi

$$= \left(\frac{\pi}{a}\right)^{d/2} \bar{\mathcal{F}}\left(e^{-\underbrace{(\nu 4\pi^2 t + \frac{\pi^2}{a})}_{>0}|\xi|^2}\right)$$

$$= \left(\frac{\pi}{a}\right)^{d/2} \left(\frac{\cancel{\pi}}{\nu 4\pi^{\cancel{2}} t + \frac{\pi^{\cancel{2}}}{a}}\right)^{d/2} e^{-\frac{\pi^2|x|^2}{\nu 4\pi^2 t + \frac{\pi^2}{a}}}$$

$$= (\nu 4at+1)^{-d/2} e^{-\frac{a|x|^2}{\nu 4at+1}}$$

ZAD.

$$\begin{cases} \partial_t u = \nu \Delta u \\ u(0,\cdot) = \delta_0 \end{cases}$$

Rj.

$$u(t,x) = \bar{\mathcal{F}}\left(e^{-\nu 4\pi^2 t|\xi|^2}\right)$$

$$= \mathcal{F}\left(e^{-\nu 4\pi^2 t|\xi|^2}\right)$$

$$= \left(\frac{\pi}{\nu 4\pi^2 t}\right)^{d/2} e^{-\frac{\pi^2|x|^2}{\nu 4\pi^2 t}}$$

$$= (\nu 4\pi t)^{-d/2} e^{-\frac{|x|^2}{\nu 4t}}$$

... uočavamo da je za za svaki t (pa i t jako mali) $u(t,x) > 0$ za svaki $x \in \mathbb{R}^d$, tj. poremećaj u točki 0 u vremenu $t=0$ je već za vrijeme $0+\varepsilon$ proširen na cijeli $\mathbb{R}^d$ pa se tu vidi da jednadžba provođenja ima beskonačnu brzinu širenja