$B^{s;p,q}(\Omega)$ ... Besovljevi prostori
$\shortparallel$
$(L^p(\Omega), W^{m,p}(\Omega))_{\frac{s}{m}, q; J}$ } dobiveni realnom metodom interpolacije

1

$m \in \mathbb{N}$, $p \in \langle 1, \infty \rangle$, Ω t.d. postoji op. proširenja

$$\{u|_{\partial\Omega} : u \in W^{m,p}(\Omega)\} = B^{m-\frac{1}{p}; p,p}(\partial\Omega)$$

[AF, 7.39] (poopćenje [AF, 7.43]) & [AF, 7.45(2)].

2 $m \in \mathbb{N}$,

$$B^{m;p,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow B^{m;p,2}(\Omega) \quad , 1 < p \le 2$$

$$B^{m;p,2}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow B^{m;p,p}(\Omega) \quad , 2 \le p < \infty$$

Dakle, samo za $p=2$ imamo

$$\boxed{W^{m,2}(\Omega) = B^{m;2,2}(\Omega)}$$

$$W^{s,p}(\Omega) = [L^p(\Omega), W^{m,p}(\Omega)]_{\frac{s}{m}} \quad (m > s) \qquad s > 0$$

Ukoliko postoji operator proširenja za $\partial\Omega$ tada se podudaraju sa standardnim Sob. prostorom za $s \in \mathbb{N}$.

$$W^{s,p}(\Omega) \hookrightarrow B^{s;p,p}(\Omega) \ , \ p \ge 2$$

$$B^{s;p,p}(\Omega) \hookrightarrow W^{s,p}(\Omega), \ p \le 2$$

$$\Rightarrow \quad W^{m,2}(\Omega)|_{\partial\Omega} = H^m(\Omega)|_{\partial\Omega} = B^{m-1/2;2,2}(\partial\Omega) = W^{m-1/2,2}(\partial\Omega) = H^{m-\frac{1}{2}}(\partial\Omega).$$

Ukoliko je $\Omega = \mathbb{R}^d$ tada na raspolaganju imamo Fourierovu pretvorbu

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} u(x)\, dx \quad , u \in C_c^\infty(\mathbb{R}^d)$$

$\mathcal{F}$ se proširuje do unitarnog operatora $\mathcal{F}: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

$$\boxed{W^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \langle \xi \rangle^s u \in L^2(\mathbb{R}^d)\}} \quad , \langle \xi \rangle = (1 + |2\pi\xi|^2)^{1/2}$$

$s \geq 0, s \in \mathbb{R}$

Za $p \neq 2$ se koriste Fourierovi množitelji.

TEOREM 1 (Apsolutna vrijednost Soboljevljeve funkcije)

Neka je $f \in W^{1,r}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$. Tada $|f| \in W^{1,r}(\Omega)$ pri čemu je

$$(*) \quad (\nabla |f|)(x) = \begin{cases} \frac{1}{|f(x)|} \underbrace{\left(\mathrm{Re} f(x) \, \nabla \mathrm{Re} f(x) + \mathrm{Im} f(x) \, \nabla \mathrm{Im} f(x) \right)}_{= \mathrm{Re}\left(\overline{f(x)} \nabla f(x)\right)} & , f(x) \neq 0 \\ 0 & , f(x) = 0 \end{cases}$$

Dz. Pokažimo da je $|f| \in W^{1,r}(\Omega)$ ukoliko $(*)$ vrijedi.

Kako $f \in L^r(\Omega) \Rightarrow |f| \in L^r(\Omega)$, dovoljno je pokazati da je

$\frac{1}{|f|} \left(\mathrm{Re} f \, \nabla \mathrm{Re} f + \mathrm{Im} f \, \nabla \mathrm{Im} f \right) \in L^r(\Omega)$.

$$\left| \frac{1}{|f|} \left(\mathrm{Re} f \, \nabla \mathrm{Re} f + \mathrm{Im} f \, \nabla \mathrm{Im} f \right) \right|^2 = \frac{1}{|f|^2} \Big((\mathrm{Re} f)^2 |\nabla \mathrm{Re} f|^2 + (\mathrm{Im} f)^2 |\nabla \mathrm{Im} f|^2 + 2 \mathrm{Re} f \, \mathrm{Im} f \, \nabla \mathrm{Re} f \cdot \nabla \mathrm{Im} f \Big)$$

$$\overset{\text{Cauchy-Schwarz-Bunjakovski}}{\leq} \frac{1}{|f|^2} \Big((\mathrm{Re} f)^2 |\nabla \mathrm{Re} f|^2 + (\mathrm{Im} f)^2 |\nabla \mathrm{Im} f|^2 + 2 |\mathrm{Re} f| |\mathrm{Im} f| |\nabla \mathrm{Re} f| |\nabla \mathrm{Im} f| \Big)$$

$\boxed{\begin{array}{l} 2ab \leq a^2 + b^2 \\ a = |\mathrm{Im} f| |\nabla \mathrm{Re} f| \\ b = |\mathrm{Re} f| |\nabla \mathrm{Im} f| \end{array}}$

$$\leq \frac{1}{|f|^2} \Big((\mathrm{Re} f)^2 |\nabla \mathrm{Re} f|^2 + (\mathrm{Im} f)^2 |\nabla \mathrm{Im} f|^2 + (\mathrm{Im} f)^2 |\nabla \mathrm{Re} f|^2 + (\mathrm{Re} f)^2 |\nabla \mathrm{Im} f|^2 \Big)$$

$$\overset{|f|^2 = (\mathrm{Re} f)^2 + (\mathrm{Im} f)^2}{=} |\nabla \mathrm{Re} f|^2 + |\nabla \mathrm{Im} f|^2 = |\nabla f|^2$$

$$\Rightarrow \left| \frac{1}{|f|} \left(\mathrm{Re} f \, \nabla \mathrm{Re} f + \mathrm{Im} f \, \nabla \mathrm{Im} f \right) \right| \leq |\nabla f|$$

$$\Rightarrow \frac{1}{|f|} \left(\mathrm{Re} \, \nabla \mathrm{Re} f + \mathrm{Im} f \, \nabla \mathrm{Im} f \right) \in L^r(\Omega).$$

Pokažimo sada $(*)$.

$G_\varepsilon(s_1, s_2) := \sqrt{\varepsilon^2 + s_1^2 + s_2^2} - \varepsilon$. Vrijedi:

- $G_\varepsilon(0,0) = 0$
- $\left| \frac{\partial G_\varepsilon}{\partial s_i}(s_1, s_2) \right| = \left| \frac{s_i}{\sqrt{\varepsilon^2 + s_1^2 + s_2^2}} \right| \leq 1$

Koristimo sljedeću lemu :

LEMA ([LL, 6.16])

$$\left.\begin{array}{l} G \in C^1_b(\mathbb{R}^N; \mathbb{C}) \ \& \ G(0) = 0 \\ \vec{u} = (u_1, \dots, u_N) \in W^{1,p}(\Omega; \mathbb{R}^N) \end{array}\right\} \Rightarrow G(\vec{u}) \in W^{1,p}(\Omega)$$

$$\& \quad \partial_j(G(\vec{u})) = \sum_{k=1}^{N} \frac{\partial G}{\partial s_k}(u)\, \partial_j u_k$$

$$\Rightarrow \quad K_\varepsilon := G_\varepsilon(\mathrm{Re} f, \mathrm{Im} f) \in W^{1,p}(\Omega)$$

$$\& \quad \nabla K_\varepsilon = \frac{\mathrm{Re} f\, \nabla \mathrm{Re} f + \mathrm{Im} f\, \nabla \mathrm{Im} f}{\sqrt{\varepsilon^2 + (\mathrm{Re} f)^2 + (\mathrm{Im} f)^2}} = \frac{\mathrm{Re} f\, \nabla \mathrm{Re} f + \mathrm{Im} f\, \nabla \mathrm{Im} f}{\sqrt{\varepsilon^2 + |f|^2}}$$

Za $\varphi \in C_c^\infty(\Omega)$ imamo

$$\int_\Omega \nabla\varphi(x)\, K_\varepsilon(x)\, dx = -\int_\Omega \varphi(x)\, \nabla K_\varepsilon(x)\, dx$$

- $K_\varepsilon(x) = \sqrt{\varepsilon^2 + |f(x)|^2} - \varepsilon$

$$\Rightarrow \quad K_\varepsilon(x) \xrightarrow{\varepsilon \to 0} |f(x)| \text{ po točkama}$$

$$\& \quad |K_\varepsilon(x)| \leq \varepsilon + |f(x)| - \varepsilon = |f(x)|$$

$$\Rightarrow \int_\Omega \nabla\varphi(x) K_\varepsilon(x)\, dx \xrightarrow{\varepsilon \to 0} \int_\Omega \nabla\varphi(x)\, |f(x)|\, dx$$

- $|\nabla K_\varepsilon(x)| \leq \frac{1}{|f|} \left| \mathrm{Re} f\, \nabla \mathrm{Re} f + \mathrm{Im} f\, \nabla \mathrm{Im} f \right|$

$$\leq |\nabla f(x)|$$

$$\& \quad \nabla K_\varepsilon(x) \longrightarrow \underbrace{\frac{\mathrm{Re} f(x)\, \nabla \mathrm{Re} f(x) + \mathrm{Im} f(x)\, \nabla \mathrm{Im} f(x)}{|f(x)|}}$$

$$\Rightarrow \int_\Omega \varphi(x) \nabla K_\varepsilon(x)\, dx \xrightarrow{\varepsilon \to 0} \int_\Omega \varphi(x) \cdot dx$$

$$\Rightarrow \int_\Omega \nabla\varphi(x)\, |f|(x)\, dx = -\int_\Omega \varphi(x) \frac{\mathrm{Re} f(x)\, \nabla \mathrm{Re} f(x) + \mathrm{Im} f(x)\, \nabla \mathrm{Im} f(x)}{|f|(x)}\, dx$$

$$\Rightarrow \quad |f| \in W^{1,p}(\Omega) \ \& \text{ vrijedi } (*).$$

■

NAPOMENA

1) Pokazali smo zapravo da za $f \in W^{1,p}(\Omega)$ vrijedi
$$|\nabla|f|| \leq |\nabla f| \quad \text{s.s. u } \Omega .$$
Štoviše, iz analize ocjene u dokazu imamo
$$|\nabla|f|| = |\nabla f| \text{ s.s. u } \Omega \iff (\exists C \in \mathbb{R})\ C\,\mathrm{Re} f = \mathrm{Im} f .$$
Posebno, za $C=0$ imamo da jednakost vrijedi za realne f-je, što također vrijedi iz $(*)$ jer tada
$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & , \ f(x) > 0 \\ -\nabla f(x) & , \ f(x) < 0 \\ 0 & , \ f(x) = 0 \end{cases} .$$

2) Iz prethodnog teorema direktno slijedi da za $f \in W^{1,p}(\Omega)$ imamo $f_+ := \max\{f, 0\}$, $f_- := \min\{f, 0\} \in W^{1,p}(\Omega)$, odnosno općenitije da za $f, g \in W^{1,p}(\Omega)$ imamo $\max\{f,g\}, \min\{f,g\} \in W^{1,p}(\Omega)$ (vidi [LL, 6.18]).

Umjesto Laplaceovog operatora Δ kojim se opisuje gibanje čestice (elektrona) na kvantnoj razini, promotrimo <u>magnetski Laplaceov operator</u> ili <u>Laplaceov operator u magnetskom polju</u>
$$(\nabla + iA)^2 = \underbrace{\nabla \cdot \nabla}_{=\Delta} - i \operatorname{div} A + iA \cdot \nabla + A^2 ,$$
pri čemu je veza vektorskog potencijala $A \in L^2_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ i magnetskog polja $B : \mathbb{R}^3 \to \mathbb{R}^3$ dana s
$$\operatorname{rot} A = B .$$
Uočimo da A nije jedinstveno određen (A i $A + \nabla\varphi$ odgovaraju istom magnetskom polju).

Ovaj operator se koristi pri opisivanju gibanja čestice (elektrona) pod djelovanjem magnetskog polja.

Prirodno se ovdje pojavljuje prostor grafa operatora $\nabla + iA$, odnosno <u>magnetski Soboljevljev prostor</u> dan s

$$H^1_A(\mathbb{R}^3) := \{ u \in L^2(\mathbb{R}^3) : (\nabla + iA) u \in L^2(\mathbb{R}^3) \}$$

opskrbljen normom $\|u\|_{H^1_A} := \sqrt{\|u\|_2^2 + \|(\nabla+iA)u\|_2^2}$. Lako se pokaže da je $H^1_A(\mathbb{R}^3)$ Banachov prostor. Štoviše, $H^1_A(\mathbb{R}^3)$ je Hilbertov prostor uz skalarni produkt

$$\langle \cdot | \cdot \rangle_{H^1_A} := \langle \cdot | \cdot \rangle + \langle (\nabla + iA) \cdot | (\nabla + iA) \cdot \rangle .$$

Općenito,

$$u \in H^1_A(\mathbb{R}^3) \not\Rightarrow u \in H^1(\mathbb{R}^3) .$$

Međutim,

$$u \in H^1_A(\mathbb{R}^3) \Longrightarrow |u| \in H^1(\mathbb{R}^3) .$$

<u>TEOREM 2</u> (Diamagnetic inequality)

$A \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d)$, $u \in H^1_A(\mathbb{R}^d)$.

Tada $|u| \in H^1(\mathbb{R}^d)$, te vrijedi

$$|\nabla |u| (x)| \leq |(\nabla + iA) u(x)| \quad \text{s.s. } x \in \mathbb{R}^d .$$

<u>Dz.</u> $u \in H^1_A(\mathbb{R}^d) \Rightarrow \nabla u + \underbrace{iAu}_{\in L^1_{loc}(\mathbb{R}^d)} \in L^2(\mathbb{R}^d) \subseteq L^1_{loc}(\mathbb{R}^d)$

$$\Rightarrow \nabla u \in L^1_{loc}(\mathbb{R}^d)$$

$$\Rightarrow u \in W^{1,1}_{loc}(\mathbb{R}^d)$$

$$\Rightarrow |u| \in W^{1,1}_{loc}(\mathbb{R}^d) \quad \& \quad (\nabla |u|)(x) = \begin{cases} \mathrm{Re}\left(\frac{\overline{u}}{|u|} \nabla u\right)(x), & u(x) \neq 0 \\ 0, & u(x) = 0 \end{cases}$$

(analogno kao Teorem 1)

Kako je

$$\mathrm{Re}\left(\frac{\bar{u}}{|u|}\, i u A\right) = \mathrm{Re}(i|u|A) = 0 ,$$

imamo

$$(\nabla|u|)(x) = \begin{cases} \mathrm{Re}\left(\frac{\bar{u}}{|u|}(\nabla + iA)u\right)(x) , & u(x) \neq 0 \\ 0 , & u(x) = 0 \end{cases} .$$

$$\left|\mathrm{Re}\left(\frac{\bar{u}}{|u|}(\nabla + iA)u\right)(x)\right| \leq \left|\left(\frac{\bar{u}}{|u|}(\nabla + iA)u\right)(x)\right| = |(\nabla + iA)u(x)|$$

$$\Rightarrow \quad |\nabla|u|(x)| \leq |(\nabla + iA)u(x)|$$

$$\Rightarrow \quad \nabla|u| \in L^2(\mathbb{R}^d) .$$

<u>TEOREM 3.</u> Neka je $A \in L^2_{loc}(\mathbb{R}^3;\mathbb{R}^3)$.

a) Za svaki $q \in [2,6]$ postoji $C > 0$ (neovisna o A) t.d.

$$\|u\|_q \leq C\|u\|_{H^1_A(\mathbb{R}^3)} , \quad u \in H^1_A(\mathbb{R}^3) ,$$

tj. $H^1_A(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$.

b) Ukoliko je dodatno $A \in L^r(\mathbb{R}^3;\mathbb{R}^3)$ za neki $r \in [3,\infty]$, tada

$$H^1(\mathbb{R}^3) \hookrightarrow H^1_A(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3) ,$$

tj. $H^1_A(\mathbb{R}^3) = H^1(\mathbb{R}^3)$.

<u>Dz.</u>

a) Za $u \in H^1_A(\mathbb{R}^3)$ po Teoremu 2 imamo $|u| \in H^1(\mathbb{R}^3)$ pa koristeći Soboljevljeva ulaganja i Teorem 2 imamo

$$\|u\|_q = \||u|\|_q \leq C\||u|\|_{1,2} \underset{T-2}{\leq} C\|u\|_{H^1_A(\mathbb{R}^3)} .$$

$$q \in [2, 2^*]$$

$$2^* = \frac{dp}{d - mp} = \frac{3\cdot 2}{3 - 1\cdot 2} = 6$$

b) $u \in H^1(\mathbb{R}^3)$,

$$\|u\|_{H^1_A(\mathbb{R}^3)} = \sqrt{\|u\|_2^2 + \|(\nabla + iA)u\|_2^2}$$

$$\leq \sqrt{\|u\|_2^2 + 2\|\nabla u\|_2^2 + 2\|Au\|_2^2}$$

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{r'} \geq \frac{1}{r'} \Rightarrow r' \geq 2$$

$$\leq \frac{1}{3} + \frac{1}{r'} \Rightarrow r' \leq 6$$

$$\Downarrow$$

$$r' \in [2,6]$$

$$\leq \sqrt{\|u\|_2^2 + 2\|\nabla u\|_2^2 + 2\|A\|_r^2 \|u\|_{r'}^2}$$

Sob. ulag.

$$\leq \sqrt{\underbrace{\|u\|_2^2 + 2\|\nabla u\|_2^2}_{\leq 2\|u\|_{1,2}^2} + 2\|A\|_r^2 \|u\|_{1,2}^2}$$

$$\leq \sqrt{2 + 2\|A\|_r^2}\, \|u\|_{1,2} \leq \sqrt{2}\,(1 + \|A\|_r)\, \|u\|_{1,2}$$

$$\Rightarrow H^1(\mathbb{R}^3) \hookrightarrow H^1_A(\mathbb{R}^3)$$

$u \in H^1_A(\mathbb{R}^3)$,

$$\|u\|_{H^1(\mathbb{R}^3)} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2}$$

$$= \sqrt{\|u\|_2^2 + \|(\nabla + iA)u - iAu\|_2^2}$$

$$\leq \sqrt{\underbrace{\|u\|_2^2 + 2\|(\nabla + iA)u\|_2^2}_{\leq 2\|u\|_{H^1_A(\mathbb{R}^3)}^2} + 2\underbrace{\|Au\|_2^2}_{\leq \|A\|_r^2 \|u\|_{r'}^2 \overset{(a)}{\leq} \|A\|_r^2 \|u\|_{H^1_A(\mathbb{R}^3)}^2}}$$

$$\leq \sqrt{2}\,(1 + \|A\|_r)\, \|u\|_{H^1_A(\mathbb{R}^3)}$$

■

Iako za $A \in L^r(\mathbb{R}^3; \mathbb{R}^3)$, $r \geq 3$, zapravo imamo $H^1_A(\mathbb{R}^3) = H^1(\mathbb{R}^3)$, u evolucijskim zadaćama kada A ovisi o vremenu analiza ipak postaje netrivijalna.

UKRATKO O APSTRAKTNOM CAUCHYJEVOM PROBLEMU

$X \ldots$ Banachov prostor

Promatramo
$$(\text{ACP}) \begin{cases} u'(t) = Au(t) = f(t) \\ u(0) = u_0 \end{cases},$$

gdje je $A : \operatorname{dom} A \subseteq X \to X$ gusto definiran linearan operator, $u_0 \in X$, $f : \langle 0, T\rangle \to X$, $T > 0$.

Kažemo da je u <u>klasično rješenje</u> (ACP) na $[0,T\rangle$ ako:

- $u \in C^1(\langle 0,T\rangle; X) \cap C([0,T]; X)$
- $u(t) \in \operatorname{dom} A$, $t \in \langle 0, T\rangle$
- u zadovoljava (ACP).

Za općenite rezultate nam je bitno da je A <u>infinitezimalni generator jako neprekinute polugrupe</u>.

$$\underbrace{\text{jako neprekinute polugrupe}}_{C_0\text{-polugrupe}}$$

Nećemo davati definicije već ćemo samo istaknuti da je

$$A = i\Delta : \operatorname{dom} A \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$

uz $\operatorname{dom} A = \{u \in L^2(\mathbb{R}^d) : i\Delta u \in L^2(\mathbb{R}^d)\} = H^2(\mathbb{R}^d)$,

infinitezimalni generator C_0-polugrupe.

Promotrimo sada polulinearnu apstraktnu zadaću

$$(\text{pACP}) \begin{cases} u'(t) - Au(t) = f(t, u(t)) \\ u(0) = u_0 \end{cases}.$$

$Y := (\operatorname{dom} A, \|\cdot\|_A)$ ... Banachov prostor

domena opskrbljena graf normom

$$(\|u\|_A = \|u\| + \|Au\|)$$

TEOREM [Pazy, Teorem 6.1.7] Neka je $A : \operatorname{dom} A \subseteq X \to X$ generator C_0-polugrupe. Neka je $f : [0,T] \times Y \to Y$ jednoliko (po t) lokalno Lipschitzova u Y, te neka je $\underbrace{t \mapsto f(t,y)}_{\text{za svaki } y \in Y}$ neprekinuta s $[0,T]$ u Y.

Ako je $u_0 \in \operatorname{dom} A$, tada (pACP) ima jedinstveno klasično rješenje na intervalu $[0, T_{max}\rangle$, te ukoliko je $T_{max} < T$ tada

$$\lim_{t \nearrow T_{max}} (\|u(t)\| + \|Au(t)\|) = \infty.$$

Primijenimo prethodni teorem na sljedećoj zadaći

$$\begin{cases} \partial_t u - i\Delta u + ik|u|^2 u = 0 & \text{u } \langle 0, \infty\rangle \times \mathbb{R}^2 \\ u(0,\cdot) = u_0 & \text{u } \mathbb{R}^2 \end{cases}, \quad \underline{k \in \mathbb{R}}.$$

$A := i\Delta$

$\operatorname{dom} A = \{u \in L^2(\mathbb{R}^2) : i\Delta u \in L^2(\mathbb{R}^2)\} = H^2(\mathbb{R}^2)$ (ovo je jednakost skupova)

$A : H^2(\mathbb{R}^2) \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ je infinitezimalni generator C_0-polugrupe.

$Y := (\operatorname{dom} A, \|\cdot\|_A) = H^2(\mathbb{R}^2)$

Samo trebamo pokazati da je preslikavanje

$$H^2(\mathbb{R}^2) = Y \ni u \xmapsto{F} -ik|u|^2 u \in Y = H^2(\mathbb{R}^2).$$

dobro definirano i lokalno Lip.

<u>LEMA</u> $u, v \in H^2(\mathbb{R}^2)$,

$$\|F(u)\|_{2,2} \lesssim \|u\|_{2,2}^3$$

$$\|F(u) - F(v)\|_{2,2} \lesssim (\|u\|_{2,2}^2 + \|v\|_{2,2}^2)\|u-v\|_{2,2}$$

$\Rightarrow$ F je lok. Lip. u Y.

<u>Dz.</u> Pokažimo tvrdnju za $F(u) = u^3$, u realna, a tvrdnja analogno slijedi i za gornji F.

$$\partial_j(u^3) = 3u^2 \partial_j u$$

$$\partial_{jj}(u^3) = 6u(\partial_j u)^2 + 3u^2 \partial_{jj} u,$$

pa imamo

$$\boxed{H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)}$$

$$\|F(u)\|_{2,2}^2 \lesssim \|u^3\|_2^2 + \|\Delta(u^3)\|_2^2$$

$$\lesssim \|u\|_{0,\infty}^4 \|u\|_{2,2}^2 + \|u\|_{0,\infty}^2 \|u\|_{1,4}^4 + \|u\|_{0,\infty}^4 \|u\|_{2,2}^2$$

$$\boxed{H^2(\mathbb{R}^2) \hookrightarrow W^{1,4}(\mathbb{R}^2)}$$

$$\lesssim \|u\|_{2,2}^6 + \|u\|_{2,2}^6 + \|u\|_{2,2}^6$$

$$\lesssim \|u\|_{2,2}^6 \quad \checkmark$$

Analogno za $u, v, z \in H^2(\mathbb{R}^2)$ imamo

$$\cancel{\|u^2 v\|_{2,2} \lesssim \|u\|_{2,2}^2 \|v\|_{2,2}} \quad \|uvz\|_{2,2} \lesssim \|u\|_{2,2}\|v\|_{2,2}\|z\|_{2,2}$$

$$\|u^3 - v^3\|_{2,2} \lesssim \|u^2(u-v)\|_{2,2} + \|v(u+v)(u-v)\|_{2,2}$$

$$\lesssim \|u\|_{2,2}^2 \|u-v\|_{2,2} + \|v\|_{2,2}\|u+v\|_{2,2}\|u-v\|_{2,2}$$

$$\lesssim (\|u\|_{2,2}^2 + \|v\|_{2,2}^2)\|u-v\|_{2,2}.$$

∎