Prostori razlomljenog reda

$W^{s,p}(\Omega)$, $p \in \mathbb{R}$

$p=2$, $\Omega = \mathbb{R}^d \longrightarrow (1+|y|^2)^{\frac{s}{2}} \hat{u}(y) \in L^2(\mathbb{R}^d)$

Općenito : teorija interpolacije!

Tehnički detalji:

1) Bochnerov integral

$f : [a,b] \to X$, X Banachov

$\int_a^b f(t)\, d\mu(t) \longrightarrow d\mu(t) = w(t)\,dt$

$w > 0$ neprekinuta

$\mu(A) = \int_A w(t)\,dt$

$\searrow$ definiramo integral prvo za jednostavne funkcije, i onda preko njih definiramo pojmove izmjerivosti i sumabilnosti

$|f(t)| \longrightarrow \|f(t)\|_X$

2) $L^q(a,b;d\mu,X)$

3) $L^q_* = L^q(0,\infty;\frac{dt}{t},\mathbb{C})$

L^q_* je ekvivalentan s $L^q(\mathbb{R})$ preko transformacije $F(s) = f(e^s)$

$$\|f\|_{L^q_*} = \|F\|_{q;\mathbb{R}}$$

$$f * g(t) = \int_0^\infty f\left(\frac{t}{s}\right) g(s) \frac{ds}{s}$$

Realna metoda interpolacije

X_0, X_1 Banachovi, $X_0 \hookrightarrow \mathcal{X}$, $X_1 \hookrightarrow \mathcal{X}$ = Hausdorffov TVP

$X_0 \cap X_1 \neq \emptyset$

$$X_0 + X_1 = \{u = u_0 + u_1 : u_0 \in X_0,\ u_1 \in X_1\}$$

$$\|u\|_{X_0 \cap X_1} = \max\{\|u\|_{X_0}, \|u\|_{X_1}\}$$

$$\|u\|_{X_0 + X_1} = \inf\{\|u_0\|_{X_0} + \|u_1\|_{X_1} : u = u_0 + u_1,\ u_0 \in X_0,\ u_1 \in X_1\}$$

$$X_0 \cap X_1 \hookrightarrow \underset{\substack{\uparrow \\ \text{međuprostor između } X_0 \text{ i } X_1}}{X} \hookrightarrow X_0 + X_1$$

Želimo konstruirati familiju $X_{\theta,q}$ međuprostora sa svojstvom:

$$\left.\begin{aligned} T: X_0 \to Y_0 \text{ ograničen} \\ T: X_1 \to Y_1 \text{ ograničen} \end{aligned}\right\} \Rightarrow T: X_{\theta,q} \to Y_{\theta,q} \text{ ograničen}$$

Ideja: $\max\{\|u\|_{X_0}, t \cdot \|u\|_{X_1}\} = J(t;u)$

$$\inf\{\|u_0\|_{X_0} + t \cdot \|u_1\|_{X_1} : u = u_0 + u_1,\ u_0 \in X_0,\ u_1 \in X_1\} = K(t;u)$$

Za $t > 0$ dobivamo međusobno ekvivalentne norme na $X_0 \cap X_1$, odnosno $X_0 + X_1$

Npr. $\min\{1,t\}\|u\|_{X_0 \cap X_1} \leq J(t;u) \leq \max\{1,t\}\|u\|_{X_0 \cap X_1}$

$J(\cdot\,;u)$ je konveksna, a $K(\cdot\,,u)$ konkavna; neprekinute i monotono rastuće

$$u \in X_0 \cap X_1 \Rightarrow K(t;u) \leq \min\left\{1, \frac{t}{s}\right\} J(s;u)$$

K-metoda interpolacije

$0 \leq \theta \leq 1$, $1 \leq q \leq \infty$,

$$(X_0, X_1)_{\theta, q; K} = \{ u \in X_0 + X_1 : t^{-\theta} K(t; u) \in L^q_* \}$$

$$\|u\|_{\theta, q; K} = \| t^{-\theta} K(t; u) \|_{L^q_*}$$

$$q < \infty \Rightarrow (X_0, X_1)_{0, q; K} = (X_0, X_1)_{1, q; K} = \{0\}$$

$$q = \infty \Rightarrow \begin{cases} X_0 \hookrightarrow (X_0, X_1)_{0, \infty; K} \\ X_1 \hookrightarrow (X_0, X_1)_{1, \infty; K} \end{cases}$$

J-metoda interpolacije

$0 \leq \theta \leq 1$, $1 \leq q \leq \infty$,

$$(X_0, X_1)_{\theta, q; J} = \{ u \in X_0 + X_1 : u = \int_0^\infty f(t) \frac{dt}{t}, \ f \in L^1(0, \infty; \tfrac{dt}{t}, X_0 + X_1), \ t^{-\theta} J(t; f) \in L^q_* \}$$

$$S(u) = \{ f \in L^1(0, \infty; \tfrac{dt}{t}, X_0 + X_1) : u = \int_0^\infty f(t) \frac{dt}{t} \},$$

$$\|u\|_{\theta, q; J} = \inf_{f \in S(u)} \| t^{-\theta} J(t; f(t)) \|_{L^q_*}$$

$$q > 1 \Rightarrow (X_0, X_1)_{0, q; J} = (X_0, X_1)_{1, q; J} = \{0\}$$

$$q = 1 \Rightarrow \begin{cases} (X_0, X_1)_{0, 1; J} \hookrightarrow X_0 \\ (X_0, X_1)_{1, 1; J} \hookrightarrow X_1 \end{cases}$$

$$0<\theta<1 \Rightarrow \begin{cases} (X_0,X_1)_{\theta,q;J} \hookrightarrow (X_0,X_1)_{\theta,q;K} \\ (X_0,X_1)_{\theta,q;K} \hookrightarrow (X_0,X_1)_{\theta,q;J} \\ \text{isti prostori, s ekvivalentnim normama} \end{cases}$$

Općenite klase međuprostora X između X_0 i X_1:

I) $X \in \mathcal{K}(\theta; X_0, X_1)$: $K(t;u) \le C_1 t^{\theta} \|u\|_X \iff X \hookrightarrow (X_0,X_1)_{\theta,\infty;K}$

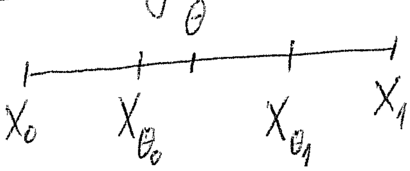
II) $X \in \mathcal{J}(\theta; X_0, X_1)$: $\|u\|_X \le C_2 t^{-\theta} J(t;u) \iff (X_0,X_1)_{\theta,1;J} \hookrightarrow X$

III) $\mathcal{K}(\theta; X_0, X_1) \cap \mathcal{J}(\theta; X_0, X_1) =: \mathcal{H}(\theta; X_0, X_1)$

<u>Posljedica:</u> $(X_0,X_1)_{\theta,q;J} = (X_0,X_1)_{\theta,q;K} \in \mathcal{H}(\theta; X_0, X_1)$, $0<\theta<1$

$$X_0 \in \mathcal{H}(0; X_0, X_1)$$
$$X_1 \in \mathcal{H}(1; X_0, X_1)$$

<u>Reiteracijski teorem:</u>



$$\theta = (1-\lambda)\theta_0 + \lambda\theta_1$$

$$X_{\theta_0}, X_{\theta_1} \in \mathcal{H}(\underset{\theta_1}{\theta_0}; X_0, X_1), \quad 0<\lambda<1$$

$$\Rightarrow (X_{\theta_0}, X_{\theta_1})_{\lambda,q;J} = (X_{\theta_0}, X_{\theta_1})_{\lambda,q;K}$$
$$= (X_0,X_1)_{\theta,q;K} = (X_0,X_1)_{\theta,q;J}$$

Interpolacijski teorem

$$\|Tu\|_{Y_1} \le M_1 \|u\|_{X_1}, \quad \|Tu\|_{Y_0} \le M_0 \|u\|_{X_0}$$

$$\Rightarrow \|Tu\|_{(Y_0,Y_1)_{\theta,q;K}} \le M_0^{1-\theta} M_1^{\theta} \|u\|_{(X_0,X_1)_{\theta,q;K}} \quad \begin{pmatrix} 0<\theta<1 \\ \text{ili } q=\infty \end{pmatrix},$$

$$\|Tu\|_{(Y_0,Y_1)_{\theta,q;J}} \le M_0^{1-\theta} M_1^{\theta} \|u\|_{(X_0,X_1)_{\theta,q;J}} \quad \begin{pmatrix} 0<\theta<1 \\ \text{ili } q=1 \end{pmatrix}$$

Primjer.

$$(L^1(\Omega), L^\infty(\Omega))_{1-\frac{1}{p},q;K} =: L^{p,q}(\Omega), \quad 1<p<\infty, \quad 1\le q\le\infty$$

Lorentzovi prostori

Karakterizacija Lorentzovih prostora:

$$\delta_u(t) = |\{x\in\Omega : |u(x)| > t\}|$$

$$u^*(s) = \inf\{t : \delta_u(t) \le s\}$$

u^* nerastuća i $\delta_{u^*}(t) = \delta_u(t)$ – preslagivanje funkcije u

$$u^{**}(t) = \frac{1}{t}\int_0^t u^*(s)\,ds$$

$$\|u\|_{L^{p,q}(\Omega)} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} u^{**}(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1\le q<\infty \\ \sup_{t>0} t^{\frac{1}{p}} u^{**}(t), & q=\infty \end{cases} \quad (1\le p\le\infty)$$

$1<p<\infty \Rightarrow u^{**}$ možemo zamijeniti s u^*

$1<p<\infty,\ p=q \Rightarrow L^{p,p}(\Omega) = L^p(\Omega)$

$L^{p,\infty}(\Omega)$ = slabi $L^p(\Omega)$

Besovljevi prostori

$X_0 = L^p(\Omega)$, $X_1 = W^{m,p}(\Omega)$

Teorem.
$\Omega \subseteq \mathbb{R}^d$ zadovoljava strogi lokalno Lipschitzov uvjet,
$0 < k < m$, $1 \leq p < \infty$

$$\Rightarrow W^{k,p}(\Omega) \in \mathcal{H}\left(\frac{k}{m}; L^p(\Omega), W^{m,p}(\Omega)\right).$$

Dokaz.

Trebamo dokazati: $\|u\|_{k,p} \leq C t^{-\frac{k}{m}} J(t;u) \longrightarrow$ klasa $\mathcal{J}$ (*)

$K(t;u) \leq C t^{\frac{k}{m}} \|u\|_{k,p} \longrightarrow$ klasa $\mathcal{K}$ (**)

Teorem 5.2 [AF] $\Rightarrow \|u\|_{k,p} \leq \underline{C \|u\|_p^{1-\frac{k}{m}} \|u\|_{m,p}^{\frac{k}{m}}}$ $\quad (u \in W^{m,p}(\Omega))$

$$\overset{?}{\leq} \; t^{-\frac{k}{m}} J(t;u) = \max_{t>0} \left\{ t^{-\frac{k}{m}} \|u\|_p, \; t^{1-\frac{k}{m}} \|u\|_{m,p} \right\}$$

max se postiže za $t = \dfrac{\|u\|_p}{\|u\|_{m,p}} \Rightarrow$ (*)

Pretpostavimo da ~~vrijedi (**) i da~~ je $u \in W^{k,p}(\Omega)$.

$$K(t;u) \leq \|u\|_p + t \cdot \|0\|_{m,p} = \|u\|_p \leq \underline{\|u\|_{k,p}}$$

$$\overset{?}{\leq} \; C t^{\frac{k}{m}} \|u\|_{k,p} \qquad t \geq 1 \checkmark$$

$0 < t \leq 1$, biramo rastav $u = \underset{L^p}{\underset{\cap}{u_0}} + \underset{W^{m,p}}{\underset{\cap}{u_1}}$ t.d. $\|u_0\|_p + t \cdot \|u_1\|_{m,p} \leq 2K(t;u)$

$\Rightarrow \|u-u_1\|_p = \|u_0\|_p \leq 2Ct^{\frac{k}{m}} \|u\|_{k,p}$

i $\|u_1\|_{m,p} \leq 2Ct^{\frac{k}{m}-1} \|u\|_{k,p}$, a to je (za $t=\varepsilon^m$, $u_\varepsilon = u_1$)

aproksimacijski problem iz 5.31.

Obratno, ako aproksimacijski problem ima rješenje, tj. za $\varepsilon \leq 1$ postoji $u_\varepsilon \in W^{m,p}(\Omega)$ t.d.

$$\|u-u_\varepsilon\|_p \leq C\varepsilon^k \|u\|_{k,p} \quad \text{i} \quad \|u_\varepsilon\|_{m,p} \leq C\varepsilon^{k-m} \|u\|_{k,p},$$

onda za $\varepsilon = t^{\frac{1}{m}}$ vrijedi:

$$t^{-\frac{k}{m}} K(t;u) \leq t^{-\frac{k}{m}} \left(\|u-u_\varepsilon\|_p + t \cdot \|u_\varepsilon\|_{m,p} \right)$$

$$\leq C \cdot \|u\|_{k,p}, \text{ tj. vrijedi } (**)$$

$0 < s < \infty$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, m najmanji cijeli broj $> s$.

$$B^{s;p,q}(\Omega) = \left(L^p(\Omega), W^{m,p}(\Omega) \right)_{\frac{s}{m}, q; J}$$

Reiteracijski teorem

$$\Rightarrow \begin{cases} B^{s;p,q}(\Omega) = \left(L^p(\Omega), B^{s_1;p,q_1}(\Omega) \right)_{\frac{s}{s_1}, q; J} & (s_1 > s,\ 1 \leq q_1 < \infty) \\ B^{s;p,q}(\Omega) = \left(W^{k,p}(\Omega), W^{m,p}(\Omega) \right)_{\theta, q; J} & (0 \leq k < s < m,\ s = (1-\theta)k + \theta m) \\ B^{s;p,q}(\Omega) = \left(B^{s_1;p,q_1}(\Omega), B^{s_2;p,q_2}(\Omega) \right)_{\theta, q; J} & (0 < s_1 < s < s_2,\ s = (1-\theta)s_1 + \theta s_2,\ 1 \leq q_1, q_2 \leq \infty) \end{cases}$$

Karakterizacija traga

$1<p<\infty$, sljedeća dva uvjeta na izmjerivu funkciju u na $\mathbb{R}^{d-1}$ su ekvivalentna:

(i) Postoji funkcija $U \in W^{m,p}(\mathbb{R}^d)$ takva da je u njezin trag na $\mathbb{R}^{d-1}$;

(ii) $u \in B^{m-\frac{1}{p};p,p}(\mathbb{R}^{d-1})$.

Nažalost,

$$B^{m;p,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow B^{m;p,2}(\Omega), \quad 1 < p \leq 2$$

$$B^{m;p,2}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow B^{m;p,p}(\Omega), \quad 2 \leq p < \infty$$

$$p=2: \ W^{m,2}(\Omega) = B^{m;2,2}(\Omega)$$

Kompleksna interpolacija

$X_0, X_1 \hookrightarrow \mathcal{X}, \quad X_0 + X_1$

$\mathcal{F} = \mathcal{F}(X_0, X_1) =$ prostor funkcija kompleksne varijable $\zeta = \theta + i\tau$ s vrijednostima u $X_0 + X_1$ sa sljedećim svojstvima:

(a) f je neprekinuta i ograničena na pruzi $0 \leq \theta \leq 1$

(b) f je analitička (f' postoji) na pruzi $0 < \theta < 1$

(c) f je neprekinuta na $\theta = 0$ (u X_0) i
$$\|f(i\tau)\|_{X_0} \to 0 \text{ kad } |\tau| \to \infty$$

(d) f je neprekinuta na $\theta = 1$ (u X_1) i
$$\|f(1+i\tau)\|_{X_1} \to 0 \text{ kad } |\tau| \to \infty$$

$$\|f\|_{\mathcal{F}} = \max\left\{ \sup_{\tau} \|f(i\tau)\|_{X_0},\ \sup_{\tau} \|f(1+i\tau)\|_{X_1} \right\}$$

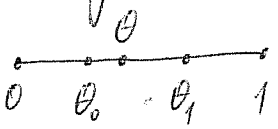
Banachov prostor

$$\theta \in \langle 0,1 \rangle,\ X_\theta = [X_0, X_1]_\theta = \{ u \in X_0 + X_1 : u = f(\theta) \text{ za neki } f \in \mathcal{F} \}$$

$\uparrow$ kompleksni interpolacijski prostor između X_0 i X_1

$$\|u\|_{X_\theta} = \|u\|_{[X_0, X_1]_\theta} = \inf\left\{ \|f\|_{\mathcal{F}} : f(\theta) = u \right\}$$

- interpolacijski teorem
- reiteracijski teorem



$$\theta = (1-\lambda)\theta_0 + \lambda\theta_1, \quad 0 < \lambda < 1$$

$$\left[[X_0, X_1]_{\theta_0},\ [X_0, X_1]_{\theta_1} \right]_\lambda \underset{\uparrow}{=} [X_0, X_1]_\theta$$

s ekvivalentnim normama

<u>Primjer.</u>

$1 \le p_1, p_2 \le \infty$, $p_1 \neq p_2$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ $\quad (0 < \theta < 1)$

$$\Rightarrow \left[L^{p_0}(\Omega), L^{p_1}(\Omega) \right]_\theta = L^p(\Omega).$$

<u>Definicija.</u>

$$W^{s,p}(\Omega) = \left[L^p(\Omega), W^{m,p}(\Omega) \right]_{\frac{s}{m}} \quad (m > s)$$

$s = m \in \mathbb{N}$ – imamo li ekvivalenciju s prijašnjom definicijom?

$\Omega = \mathbb{R}^d$ DA!

ako vrijede teoremi proširenja $\longrightarrow$ DA!