Interpolacija u redu glatkoće

za $0 \le j \le m$ želimo $|\cdot|_{j,p}$ ocijeniti pomoću $|\cdot|_{m,p}$ i $\|\cdot\|_p$

Teorem.
Neka $\Omega \subseteq \mathbb{R}^d$ zadovoljava uvjet konusa. Za svaki $\varepsilon_0 > 0$ postoje konstante K, K' t.d. za $0 < \varepsilon \le \varepsilon_0$, $0 \le j \le m$, $u \in W^{m,p}(\Omega)$ vrijedi:

(1) $$|u|_{j,p} \le K\left(\varepsilon |u|_{m,p} + \varepsilon^{-j/(m-j)} \|u\|_p\right),$$

(2) $$\|u\|_{j,p} \le K'\left(\varepsilon \|u\|_{m,p} + \varepsilon^{-j/(m-j)} \|u\|_p\right),$$

(3) $$\|u\|_{j,p} \le 2K' \|u\|_{m,p}^{j/m} \cdot \|u\|_p^{(m-j)/m}.$$

Dz.
Dovoljno je dokazati (1).
(2) slijedi iz (1) jer $k \le j \Rightarrow \frac{k}{m-k} \le \frac{j}{m-j}$
(3) slijedi iz (2) izborom $\varepsilon \|u\|_{m,p} = \varepsilon^{-j/(m-j)} \|u\|_p$
(1) dokazujemo uz pomoć tri leme.

Lema 1. (varijanta za $m=2$, $j=1$, $d=1$) vrijedi
Za $\rho > 0$, $1 \le p < \infty$, $K_p = 2^{p-1} 9^p$ i $g \in C^2([0,\rho])$ vrijedi

$$|g'(0)|^p \le \frac{K_p}{\rho}\left(\rho^p \int_0^\rho |g''(t)|^p \, dt + \rho^{-p} \int_0^\rho |g(t)|^p \, dt\right).$$

Dz.
Dovoljno je uzeti $f \in C^2([0,1])$, općenito $f(t) = g(\rho t)$.

$$x \in [0, \tfrac{1}{3}],\ y \in [\tfrac{2}{3}, 1] \Rightarrow (\exists z \in \langle x, y \rangle)\ |f'(z)| = \left|\frac{f(y)-f(x)}{y-x}\right| \le 3(|f(x)| + |f(y)|)$$

$$|f'(0)| = \left|f'(z) - \int_0^z f''(t)\, dt\right| \le 3|f(x)| + 3|f(y)| + \int_0^1 |f''(t)|\, dt \ \Big/ \int_0^{1/3} dx \int_{2/3}^1 dy$$

$\Rightarrow$ tvrdnja za $p=1$.

Sada potenciramo na p-tu i iskoristimo:

$$(a+b)^p \le 2^{p-1}(a^p+b^p),$$

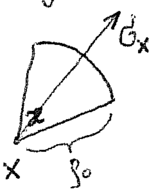
$$\left(\int_0^1 |h(t)|\,dt\right)^p \le \left(\int_0^1 1^{p'}\,dt\right)^{\frac{p}{p'}} \cdot \int_0^1 |h(t)|^p\,dt.$$

Lema 2. ($m=2$, $j=1$, $\varepsilon_0 \underset{\uparrow}{=} \rho_0 =$ visina konusa iz uvjeta konusa)

BSO jer općenito $\varepsilon \longrightarrow \varepsilon \cdot \frac{\varepsilon_0}{\rho_0}$

Dz.

$\Sigma \subseteq \mathbb{R}^d$ jedinična sfera, $K_0 = |\Sigma| = \int_\Sigma d\sigma$



$|\sigma_x| = 1$

$$\Sigma_x = \left\{ \sigma \in \Sigma : \measuredangle(\sigma, \sigma_x) \le \frac{\varkappa}{2} \right\}$$

$u \in C^\infty(\Omega)$, primijenimo Lemu 1 na $g(t) = u(x+t\sigma)$, $x \in \Omega$, $\sigma \in \Sigma_x$

$$\Rightarrow \underbrace{|\sigma \cdot \nabla u(x)|^p}_{\int_\Sigma\ \geq\ K_1 \cdot |\nabla u(x)|^p} \le \frac{K_p}{\rho} \cdot \underbrace{\left(\rho^p \int_0^\rho \left| \frac{d^2}{dt^2} u(x+t\sigma) \right|^p dt + \rho^{-p} \int_0^\rho |u(x+t\sigma)|^p\,dt \right)}_{=:\, I(\rho, p, u, x, \sigma)} \quad \Big/ \int_\Sigma \int_\Omega$$

$$\Rightarrow \int_\Omega |\nabla u(x)|^p\,dx \le \frac{K_p}{K_1 \rho} \underbrace{\int_\Sigma \int_\Omega I(\rho, p, u, x, \sigma)\,dx\,d\sigma}_{?}$$

BSO $\sigma = e_d = (0, \dots, 0, 1)$, u suprotnom: linearna kombinacija sličnih ocjena

$$\int_\Omega I(\rho, p, u, x, e_d) = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_0^\rho \left(\rho^p |\partial_d^2 u(x', x_d+t)|^p + \rho^{-p} |u(x', x_d+t)|^p \right) dt\,dx_d\,dx'$$

$$= \int_{\mathbb{R}^{d-1}} \int_0^\rho \int_{-\infty}^{\infty} \left(\rho^p |\partial_d^2 u(x)|^p + \rho^{-p} |u(x)|^p \right) dx_d\,dt\,dx'$$

$$\underset{\swarrow \le}{=} \rho \cdot \int_\Omega \left(\rho^p |\partial_d^2 u(x)|^p + \rho^{-p} |u(x)|^p \right) dx$$

$$\Rightarrow |u|_{1,p}^p \leq d \cdot \int_\Omega |\nabla u(x)|^p dx \leq \frac{dK_pK_0}{K_1}\left(\rho^p |u|_{2,p}^p + \rho^{-p} \|u\|_p^p\right)$$

$| ^{\frac{1}{p}}$, $\rho = \varepsilon$, $C^\infty(\Omega)$ gust u $W^{2,p}(\Omega)$

Lema 3. (tvrdnja teorema)

$m \geq 2$, ako vrijedi

$$|u|_{1,p} \leq K(\delta |u|_{2,p} + \delta^{-1} \|u\|_p), \quad u \in W^{2,p}(\Omega), \quad 0 < \delta \leq \delta_0, \quad (*)$$

onda vrijedi (za $0 \leq j \leq m-1$)

$$|u|_{j,p} \leq K'(\varepsilon |u|_{m,p} + \varepsilon^{\frac{-j}{m-j}} \|u\|_p), \quad u \in W^{m,p}(\Omega), \quad 0 < \varepsilon \leq \varepsilon_0,$$

gdje je $\varepsilon_0 = \min\{\delta_0, \delta_0^2, \dots, \delta_0^{m-1}\}$.

Dz.

BSO $1 \leq j \leq m-1$

Prvo dokazujemo za $j = m-1$ indukcijom po m.

Pretpostavka ove leme (Lema 2) je baza indukcije ($m=2$).

Pretpostavimo dakle

$$|u|_{k-1,p} \leq K_1(\delta |u|_{k,p} + \delta^{-(k-1)} \|u\|_p), \quad \underset{(0<\delta\leq\delta_0)}{u \in W^{k,p}(\Omega)}, \text{ za neki } 2 \leq k \leq m-1. \quad (**)$$

$u \in W^{k+1,p}(\Omega)$,

$$|u|_{k,p} \leq K_2 \cdot \sum_{|\alpha|=k-1} |\partial^\alpha u|_{1,p} \overset{(*)}{\leq} K_3(\delta |u|_{k+1,p} + \delta^{-1} |u|_{k-1,p})$$

$$\overset{(**)}{\leq} K_3(\delta |u|_{k+1,p} + \underbrace{K_1 \delta^{-1} \eta |u|_{k,p}}_{?} + K_1 \delta^{-1} \eta^{1-k} \|u\|_p), \quad 0 < \eta \leq \delta_0.$$

$\eta = \frac{\delta}{2K_1K_3}$, $2K_1K_3 \geq 1$ ✓

Dokazali smo:

$$|u|_{m-1,p} \leq \tilde{K}(\delta |u|_{m,p} + \delta^{1-m} \|u\|_p), \quad 0<\delta\leq\delta_0.$$

Indukcijom (unatrag) po j dokazujemo:

$$|u|_{j,p} \leq (\delta^{m-j} |u|_{m,p} + \delta^{-j} \|u\|_p), \quad 1\leq j\leq m-1, \quad 0<\delta\leq\delta_0. \checkmark$$

$$\varepsilon = \delta^{\frac{1}{m-j}}, \quad \delta\leq\delta_0 \Rightarrow \varepsilon \Rightarrow \varepsilon_0.$$

<u>Primjedba.</u>

$\delta_0 = \infty$ (npr. $\Omega = \mathbb{R}^d_+$) $\Rightarrow$ ocjene iz teorema vrijedi za svaki $\epsilon>0$, te su konstante K i K' neovisne o ϵ

Interpolacija u redu sumabilnosti

$\Omega \subseteq \mathbb{R}^d$ zadovoljava (slabi) uvjet konusa

<u>Teorem 1.</u>

Neka je ($m\cdot p > d$, $p\leq q\leq\infty$) ili ($m\cdot p = d$, $p\leq q<\infty$) ili ($m\cdot p<d$, $p\leq q\leq p^* = \frac{d\cdot p}{d-m\cdot p}$). Tada

$$(\exists K>0)(\forall u\in W^{m,p}) \quad \|u\|_q \leq K\cdot \|u\|_{m,p}^{\theta}\cdot\|u\|_p^{1-\theta}, \text{ gdje je}$$

$$\theta = \frac{d}{mp} - \frac{d}{mq}.$$

Dz.

$$(m\cdot p<d,\ p\leq q\leq p^*) \Rightarrow \|u\|_q \leq \|u\|_{p^*}^{\theta} \|u\|_p^{1-\theta} \leq K\cdot\|u\|_{m,p}^{\theta}\|u\|_p^{1-\theta}$$

interpolacijska nejednakost $\left(\frac{1}{q} = \frac{\theta}{p^*} + \frac{1-\theta}{p}\right)$ (AF, Teorem 2.11) Soboljevljeva ulaganja

U ostalim slučajevima,

$$|u(x)| \leq K_1\cdot\Big(\sum_{|\alpha|\leq m-1} r^{|\alpha|-d}\chi_r * |\partial^\alpha u|(x) + \sum_{|\alpha|=m}(\chi_r\omega_m)*|\partial^\alpha u|(x)\Big),$$

$u\in C^\infty(\Omega)$

$\chi_r = \chi_{K[0,r]}$,

$\omega_m(x) = |x|^{m-d}$,

$0<r\leq\rho =$ visina konusa.

$$\|\chi_r * |\partial^\alpha u|\|_q \leq \|\chi_r\|_s \, \|\partial^\alpha u\|_p = K_2 \cdot r^{d-\frac{d}{p}+\frac{d}{q}} \|\partial^\alpha u\|_p$$

$$\|(\chi_r \omega_m) * |\partial^\alpha u|\|_q \leq \|\chi_r \omega_m\|_s \, \|\partial^\alpha u\|_p = K_3 \cdot r^{m-\frac{d}{p}+\frac{d}{q}} \|\partial^\alpha u\|_p$$

Youngova nejednakost $\left(\frac{1}{p}+\frac{1}{s} = 1+\frac{1}{q}\right)$

$$\Rightarrow \|u\|_q \leq K_4 \Big(\sum_{j=0}^{m-1} r^{j-\frac{d}{p}+\frac{d}{q}} \underbrace{|u|_{j,p}}_{\leq K_5 \cdot (r^{m-j} |u|_{m,p} + r^{-j}\|u\|_p)} + r^{m-\frac{d}{p}+\frac{d}{q}} |u|_{m,p} \Big)$$

ε iz prethodnog teorema

$$\leq K_6 \left(r^{m-\frac{d}{p}+\frac{d}{q}} \|u\|_{m,p} + r^{-\frac{d}{p}+\frac{d}{q}} \|u\|_p \right), \quad r \leq 1$$

povećanjem konstanti ako je potrebno

r biramo tako da su dva člana na desnoj strani jednaka

$\Rightarrow$ tvrdnja.

Posebno, $mp > d \Rightarrow \|u\|_\infty \leq K \cdot \|u\|_{m,p}^{\frac{d}{mp}} \cdot \|u\|_p^{1-\frac{d}{mp}}$.

želimo ovdje dobiti općenitiju normu $(\|u\|_q)$

<u>Teorem 2.</u>

Neka je $m \cdot p > d$, $p > 1$, te ili $1 \leq q \leq p$ ili $(q > p,\ m \cdot p - p < d)$.

$(\exists K > 0)(\forall u \in W^{m,p})\ \|u\|_\infty \leq K \cdot \|u\|_{m,p}^{\theta} \|u\|_q^{1-\theta}$, gdje je

$$\theta = \frac{dp}{dp + (mp-d)q}.$$

Dz.

$$1 \leq q \leq p \Rightarrow \|u\|_\infty \underset{\text{Teorem 1.}}{\leq} K \cdot \|u\|_{m,p}^{\frac{d}{mp}} \|u\|_p^{1-\frac{d}{mp}}$$

$$\|u\|_p \leq \|u\|_q^{q/p} \cdot \|u\|_\infty^{1-q/p}$$

Za $q>p$ dokazujemo prvo slučaj $m=1$ $(p>d)$.

Koristimo ocjenu kao i malo prije:

$$|u(x)| \leq K_1 \cdot \left(r^{-d} \chi_r * |u|(x) + \sum_{|\alpha|=1} (\chi_r \omega_1) * |\partial^\alpha u|(x)\right), \quad 0<r\leq \rho.$$

$$\chi_r * |u|(x) \leq K_2 \cdot r^{d-\frac{d}{q}} \|u\|_q \leq K_4 \cdot r^{d-\frac{d}{q}} \|u\|_{1,p}$$

$$(\chi_r \omega_1) * |\partial^\alpha u|(x) \leq K_3 \cdot r^{1-\frac{d}{p}} \|\partial^\alpha u\|_p$$

Hölderova nejednakost

- izborom konstanti možemo postići da gornje ocjene vrijede za dovoljno veliki r
- r biramo tako da su dva sumanda na desnoj strani jednaka

Općenito $(m>1)$: $W^{m,p} \hookrightarrow W^{1,r}$, $r = \frac{dp}{d-(m-1)p}$

$$|u(x)| \leq K \cdot \|u\|_{1,r}^{\theta} \|u\|_q^{1-\theta} \leq K' \cdot \|u\|_{m,p}^{\theta} \|u\|_q^{1-\theta}$$

Teorem 3. (važan za teoreme kompaktnosti nekih ulaganja) $m, k \in \mathbb{N}$, $p>1$. Neka je $m \cdot p < d$ i $d - m\cdot p < k \leq d$, te ν najveći cijeli broj manji od $m \cdot p$ takav da je $d-\nu \leq k$. Tada

$$(\exists K>0)(\forall u \in W^{m,p}) \ \|u\|_{\frac{kq}{d}, \Omega_k} \leq K \cdot \|u\|_q^{1-\theta} \cdot \|u\|_{m,p}^{\theta}, \text{ gdje je}$$

$$q = p^* = \frac{dp}{d-mp}, \quad \theta = \frac{\nu p}{\nu p + (mp-\nu) q}.$$

Primjedba.

$k=d \Rightarrow \|u\|_q \leq K' \cdot \|u\|_{m,p}$

$\Rightarrow \|u\|_{\frac{kq}{d}, \Omega_k} = \|u\|_{r, \Omega_k} \leq K'' \cdot \|u\|_{m,p}$, $r = \frac{kp}{d-mp}$

- alternativni dokaz jednog dijela Soboljevljevih ulaganja

Interpolacija s kompaktnim poddomenama

dokazali smo:

$$|u|_{j,p} \leq K\left(\varepsilon |u|_{m,p} + \varepsilon^{\frac{-j}{m-j}} \|u\|_p\right), \quad 0 < \varepsilon \leq \varepsilon_0$$

možemo li ovdje dobiti $\|u\|_{p,\Omega_\varepsilon}$, gdje je $\Omega_\varepsilon \Subset \Omega$?

Da, ako je $\Omega \subseteq \mathbb{R}^d$ omeđen i zadovaljava ili uvjet konusa ili uvjet segmenta. (TEOREM!)

Lema 1. (jednodimenzionalna nejednakost) Tada postoji konstanta K i za svaki $1 \leq p < \infty$, $0 < l_1 < l_2 < \infty$, $\varepsilon > 0$ konstanta δ $(0 < 2\delta < l_1)$, takve da za $g \in C^1(a,b)$ $(l_1 \leq b-a \leq l_2)$ vrijedi:

$$\int_a^b |g(t)|^p\,dt \leq K\varepsilon \int_a^b |g'(t)|^p\,dt + K \cdot \int_{a+\delta}^{b-\delta} |g(t)|^p\,dt.$$

Dz.

$f \in C^1(0,1)$, $s \in \langle 0,1\rangle$, $\tau \in \langle \frac{1}{3}, \frac{2}{3}\rangle$:

$$|f(s)| = \left|f(\tau) + \int_\tau^s f'(\xi)\,d\xi\right| \leq |f(\tau)| + \int_0^1 |f'(\xi)|\,d\xi$$

$\int_{\frac{1}{3}}^{\frac{2}{3}} d\tau$, $|\;|^p$ + Hölder, $\int_0^1 ds$

$$\Rightarrow \int_0^1 |f(s)|^p\,ds \leq K_p \int_{\frac{1}{3}}^{2/3} |f(s)|^p\,ds + K_p \int_0^1 |f'(s)|^p\,ds, \quad K_p = 3 \cdot 2^{p-1}$$

$f(s) = g(a + s(b-a)) = g(t)$

$$\Rightarrow \int_a^b |g(t)|^p\,dt \leq K_p \cdot (b-a)^p \int_a^b |g'(t)|^p\,dt + K_p \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} |g(t)|^p\,dt$$

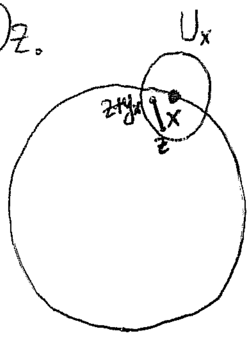
– da bi dobili ocjenu za svaki $\varepsilon > 0$ biramo $k \in \mathbb{N}$ t.d. je $k^{-p} \leq \varepsilon$, dijelimo interval $\langle a, b \rangle$ na k jednakih dijelova ($a_j = a + (b-a)\frac{j}{k}$, $j = 0, 1, \dots, k$), ocjenjujemo

$$\int_a^b |g(t)|^p \, dt = \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} |g(t)|^p \, dt,$$

te dobijemo traženu tvrdnju za $\delta \leq \frac{b-a}{3k}$.

Lema 2. ($m = 1$, $j = 0$)

Dz.



Ω ograničen $\Rightarrow \partial\Omega$ kompaktan

$\Rightarrow$ Konačan potpokrivač $\{U_1, U_2, \dots\}$

pripadni vektori $\{y_1, y_2, \dots\}$

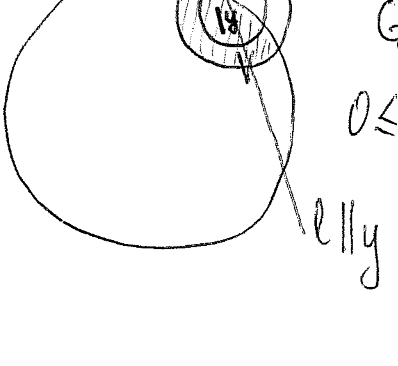
$\exists \, V_j \Subset U_j$ t.d. $\partial\Omega \subseteq \bigcup_j V_j$

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\} \subseteq \bigcup_j V_j$$

$$\Omega = \bigcup_j (V_j \cap \Omega) \cup \tilde{\Omega}, \quad \tilde{\Omega} \Subset \Omega$$

$\Rightarrow$ dovoljno je pokazati

$$|u|^p_{0,p,V_j \cap \Omega} \leq K_1 \varepsilon^p |u|^p_{1,p,\Omega} + K_1 \|u\|^p_{p,\Omega_{\varepsilon,j}}$$



$$Q = \{x + ty : x \in U \cap \Omega, \ 0 < t < 1\} \subseteq \Omega$$

$$0 \leq \eta < 1, \quad Q_\eta = \{x + ty : x \in V \cap \Omega, \ \eta < t < 1\} \Subset Q$$

Lema 1. $\Rightarrow (\exists \eta>0)(\exists K_1>0)(\forall u \in C^\infty(\Omega))$

$$\int_{\ell\cap Q_0} |u(x)|^p\, ds \le K_1 \varepsilon^p \int_{\ell\cap Q_0} |\partial_y u(x)|^p\, ds + K_1 \int_{\ell\cap Q_\eta} |u(x)|^p\, ds$$

– integriramo po projekciji od Q_0 na hiperravninu okomitu na y

$$\|u\|^p_{p,V\cap\Omega} \le \|u\|^p_{p,Q_0} \le K_1 \varepsilon^p |u|^p_{1,p,Q_0} + K_1 \|u\|^p_{p,\underbrace{Q_\eta}_{\Subset\Omega}}$$

Završetak dokaza teorema.

– primijenimo Lemu 2. na $\partial^\beta u$, $|\beta| = m-1$:

$$|u|_{m-1,p,\Omega} \le K_1 \varepsilon \cdot |u|_{m,p,\Omega} + K_1 \underbrace{|u|_{m-1,p,\Omega_\varepsilon}}_{\text{želimo primijeniti}}$$ prvi teorem od danas

– zamijenimo Ω_ε

s $\Omega' = \bigcup_{x\in\overline{\Omega}_\varepsilon} K(x,\delta)$, $\delta < d(\overline{\Omega}_\varepsilon, \partial\Omega)$

$\Omega' \Subset \Omega$ i očito zadovoljava uvjet konusa

$$\Rightarrow |u|_{m-1,p,\Omega'} \le K_2 \varepsilon |u|_{m,p,\Omega'} + K_2 \varepsilon^{-(m-1)} \|u\|_{p,\Omega'},$$

čime smo dobili tvrdnju u slučaju $j = m-1$.

– opći slučaj ide indukcijom (unatrag) po j.