Soboljevljeva nejednakost

- želimo ispitati Soboljevljeva ulaganja u terminima polunormi na $W^{m,p}(\Omega)$, $p \in [1, \infty\rangle$:

$$|u|_{j,p} = \Big(\sum_{|\alpha|=j} \int_\Omega |\partial^\alpha u(x)|^p \, dx \Big)^{\frac{1}{p}}, \quad 0 \le j \le m$$

- uz neke pretpostavke (npr. Ω ograničen) $|\cdot|_{m,p}$ je norma na $W_0^{m,p}(\Omega)$, ekvivalentna s $\|\cdot\|_{m,p}$
- zasad, promatramo te polunorme kao funkcije na $C_c^\infty(\mathbb{R}^d)$
- dakle, pitamo se postoje li konstante $K < \infty$ i $q \ge 1$ takve da za $\varphi \in C_c^\infty(\mathbb{R}^d)$ vrijedi:

$$\int_{\mathbb{R}^d} |\varphi(x)|^q \, dx \le K^q \Big(\sum_{|\alpha|=m} \int_{\mathbb{R}^d} |\partial^\alpha \varphi(x)|^p \, dx \Big)^{\frac{q}{p}}$$

TEOREM.

$m \cdot p < d \Rightarrow (\exists K > 0)(\forall \varphi \in C_c^\infty(\mathbb{R}^d)) \; \|\varphi\|_q \le K \cdot |\varphi|_{m,p}$

ako i samo ako je $q = p^* = \frac{dp}{d - mp}$.

Dokaz.

$(\Rightarrow)$ $\varphi \longrightarrow \varphi_t(x) = \varphi(tx)$, $0 < t < \infty$

$$\longrightarrow \int_{\mathbb{R}^d} |\varphi(x)|^q dx \le K^q \cdot t^{d + mq - \frac{dq}{p}} \cdot \Big(\sum_{|\alpha|=m} \int_{\mathbb{R}^d} |\partial^\alpha \varphi(x)|^p dx \Big)^{\frac{q}{p}}$$

$$\Longrightarrow d + mq - \frac{dq}{p} = 0 \Longrightarrow q = \frac{dp}{d - mp} =: p^*.$$

$(\Leftarrow)$ $m=1$, i onda indukcija po m.

prvo dokazujemo za $m=1$, $p=1$:

$$\int_{\mathbb{R}^d} |\varphi(x)|^{\frac{d}{d-1}} dx \le K \cdot \Big(\sum_{j=1}^{d} \int_{\mathbb{R}^d} |\partial_j \varphi(x)|\, dx \Big)^{\frac{d}{d-1}} \qquad (*)$$

nakon toga, za $1<p<d$ i $p^* = \frac{dp}{d-p}$ primijenimo $(*)$ na $|\varphi(x)|^s$, $s = \frac{(d-1)p^*}{d}$:

$$\int_{\mathbb{R}^d} |\varphi(x)|^{p^*} dx \le K \cdot \Big(\sum_{j=1}^{d} \int_{\mathbb{R}^d} s \cdot |\varphi(x)|^{s-1} \cdot |\partial_j \varphi(x)|\, dx \Big)^{\frac{d}{d-1}}$$

$$\overset{\text{HÖLDER}}{\le} K_1 \Big(\sum_{j=1}^{d} \|\varphi\|^{s-1}_{(s-1)p'} \cdot \|\partial_j \varphi\|_p \Big)^{\frac{d}{d-1}}$$

$(s-1)p' = p^* \Rightarrow$ tvrdnja.

Preostaje pokazati $(*)$.

$1 \le j \le d$, $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$

$$u_j(\hat{x}_j) = \Big(\int_{-\infty}^{\infty} |\partial_j \varphi(x)|\, dx_j \Big)^{\frac{1}{d-1}}$$

$$\varphi(x) = \int_{-\infty}^{x_1} \partial_1 \varphi(t, \hat{x}_1)\, dt \Rightarrow |\varphi(x)| \le (u_1(\hat{x}_1))^{d-1}$$

Općenito, $|\varphi(x)| \le (u_j(\hat{x}_j))^{d-1} \ \Big| \prod_{j=1}^{d} \Big| \int_{\mathbb{R}^d}$

$$\Rightarrow \int_{\mathbb{R}^d} |\varphi(x)|^{\frac{d}{d-1}} dx \le \int_{\mathbb{R}^d} \prod_{j=1}^{d} u_j(\hat{x}_j)\, dx$$

$$\underset{\text{SEMINAR!}}{\le} \Big(\prod_{j=1}^{d} \int_{\mathbb{R}^{d-1}} |u_j(\hat{x}_j)|^{d-1} d\hat{x}_j \Big)^{\frac{1}{d-1}} \le |\varphi|_{1,1}^{\frac{d}{d-1}}.$$

Varijante Soboljevljevih nejednakosti (2 seminara)

mješovite norme

$\vec{p} = p = (p_1, \dots, p_d)$

$0 < p_i \leq \infty$

$p_i \geq 1 \Rightarrow$ Banachov prostor

$$\|u\|_{\vec{p}} = \Big\| \dots \big\| \|u\|_{L^{p_1}(dx_1)} \big\|_{L^{p_2}(dx_2)} \dots \Big\|_{L^{p_d}(dx_d)}$$

$$p_i < \infty, \quad \|u\|_{\vec{p}} = \left[\int_{-\infty}^{\infty} \dots \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |u(x_1, \dots, x_d)|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} dx_2 \right]^{\frac{p_3}{p_2}} \dots dx_d \right]^{\frac{1}{p_d}}$$

a) anizotropne Soboljevljeve nejednakosti

$$\|\varphi\|_q \leq K \cdot \sum_{|\alpha| = m} \|\partial^\alpha \varphi\|_{p_\alpha}$$

Za multiindeks β, $|\beta| = m-1$, definiramo

$$\beta[j] = (\beta_1, \dots, \beta_{j-1}, \beta_j + 1, \beta_{j+1}, \dots, \beta_d)$$

$$\sum_{j=1}^{d} \frac{1}{p_{\beta[j]}} > m, \qquad \frac{1}{q} = \frac{1}{d^m} \sum_{|\alpha| = m} \binom{m}{\alpha} \frac{1}{p_\alpha} - \frac{m}{d}$$

b) reducirane Soboljevljeve nejednakosti

$$\|\varphi\|_q \leq K \cdot \sum_{\alpha \in \mathcal{M}} \|\partial^\alpha \varphi\|_p$$

$$m \cdot p < d, \quad q = p^*$$

$$\mathcal{M} = \mathcal{M}(d, m) = \{\alpha : |\alpha| = m, \ (\forall j \in 1..d) \ \alpha_j \in 0..1\}$$

$W^{m,p}(\Omega)$ – Banachova algebra?

$u, v \in W^{m,p} \Rightarrow u \cdot v \in W^{m,p}$?

Općenito, NE!

$m \cdot p > d$, Ω zadovoljava uvjet konusa $\Rightarrow$ DA!

Teorem.

Neka $\Omega \subseteq \mathbb{R}^d$ zadovoljava uvjet konusa. Ako je $m \cdot p > d$ ili $(p=1,\ m \geq d)$ postoji konstanta C^* t.d.

$$\|u \cdot v\|_{m,p} \leq C^* \|u\|_{m,p} \cdot \|v\|_{m,p}.$$

Posebno, za ekvivalentnu normu $\|u\|^*_{m,p} = C^* \|u\|_{m,p}$ imamo $\|u \cdot v\|^*_{m,p} \leq \|u\|^*_{m,p} \cdot \|v\|^*_{m,p}$

Dz.

$m \cdot p > d$; $(p=1,\ m \geq d)$ slično i jednostavnije

dokazujemo da za $|\alpha| \leq m$:

$$\int_\Omega |\partial^\alpha (u(x) v(x))|^p \leq C_\alpha \|u\|^p_{m,p} \cdot \|v\|^p_{m,p}$$

pretpostavimo prvo da je $u \in C^\infty(\Omega)$, Leibnizovo pravilo:

$$\partial^\alpha (u \cdot v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \underline{\underline{\partial^\alpha u\ \partial^{\alpha-\beta} v}}$$

Znamo (za $w \in W^{m,p}(\Omega)$):

a) $\int_\Omega |\partial^\beta w(x)|^r dx \leq C(\beta) \cdot \|w\|^r_{m,p}$ — $(m-|\beta|)p \leq d$, $p \leq r \leq \frac{dp}{d-(m-|\beta|)p}$

— $(m-|\beta|)p = d$, $p \leq r < \infty$

b) $|\partial^\beta w(x)| \leq C(\beta) \cdot \|w\|_{m,p}$ (ss $x \in \Omega$) — $(m-|\beta|)p > d$

Neka je k najveći cijeli broj takav da je $(m-k)p > d$, imamo $k \geq 0$.

A) $|\beta| \leq k$:

$$\int_\Omega |\partial^\beta u(x)\, \partial^{\alpha-\beta} v(x)|^p \, dx \leq C(\beta)^p \|u\|_{m,p}^p \, \|\partial^{\alpha-\beta} v\|_p^p \quad \checkmark$$

B) $|\alpha-\beta| \leq k$:

$$\int_\Omega |\partial^\beta u(x)\, \partial^{\alpha-\beta} v(x)|^p \, dx \leq C(\alpha-\beta)^p \|v\|_{m,p}^p \cdot \|\partial^\alpha u\|_p^p \quad \checkmark$$

C) $|\beta| > k$ i $|\alpha-\beta| > k$

$(m-|\beta|)p \leq d$ i $(m-|\alpha-\beta|)p \leq d$

$$\Rightarrow 0 \leq \underbrace{\frac{d-(m-|\beta|)p}{d}}_{<\frac{1}{r}} + \underbrace{\frac{d-(m-|\alpha-\beta|)p}{d}}_{<\frac{1}{r'}} = 2 - \frac{(2m-|\alpha|)p}{d} < 2 - \frac{mp}{d} < 1$$

$$\frac{1}{r} + \frac{1}{r'} = 1$$

$$\Rightarrow p \leq rp < \frac{dp}{d-(m-|\beta|)p}, \quad p \leq r'p < \frac{dp}{d-(m-|\alpha-\beta|)p}$$

Hölder

$$\Rightarrow \int_\Omega |\partial^\alpha u(x)\, \partial^{\alpha-\beta} v(x)|^p \, dx \leq \left(\int_\Omega |\partial^\alpha u(x)|^{r \cdot p} dx\right)^{\frac{1}{r}} \cdot \left(\int_\Omega |\partial^{\alpha-\beta} v(x)|^{r' \cdot p} dx\right)^{\frac{1}{r'}}$$

$$\leq C(\beta)^{\frac{1}{r}} C(\alpha-\beta)^{\frac{1}{r'}} \cdot \|u\|_{m,p}^p \cdot \|v\|_{m,p}^p$$

$u \in W^{m,p}$, postoji $u_j \in C^\infty(\Omega)$ t.d. $\|u-u_j\|_{m,p} \to 0$

$$\|u_j v - u_k v\|_{m,p} = \|(u_j-u_k) v\|_{m,p} \leq C^* \|u_j - u_k\|_{m,p} \cdot \|v\|_{m,p}$$

$$\Rightarrow (\exists w \in W^{m,p}) \ \|u_j v - w\|_{m,p} \to 0$$

Također, $m \cdot p > d \Rightarrow u, v \in C_b$.

$$\|w - uv\|_p \leq \|w - u_j v\|_p + \|u_j v - uv\|_p$$

$$\leq \underbrace{\|w - u_j v\|_p}_{\to 0} + \underbrace{\|v\|_\infty}_{<\infty} \cdot \underbrace{\|u_j - u\|_p}_{\to 0} \quad \Rightarrow w = u \cdot v$$

$$\Rightarrow \|u \cdot v\|_{m,p} = \|w\|_{m,p} \leq \limsup_j \|u_j v\|_{m,p} \leq C^* \|u\|_{m,p} \|v\|_{m,p}$$

<u>Optimalnost ulaganja</u>

- pokazujemo da nije moguće „poboljšati" teorem o Soboljevljevim ulaganjima, čak niti za „dobre" domene

Primjer 1.

$mp < d$, $q > p^* = \frac{kp}{d-mp}$

Konstruiramo: $u \in W^{m,p}(\Omega)$, $u \notin L^q(\Omega_k)$

BSO $0 \in \Omega$, $\Omega_k = \{x \in \Omega : x_{k+1} = \ldots = x_d = 0\}$

$R > 0$, $K_R = K(0,R)$

Fiksiramo R t.d. $\overline{K_{2R}} \subset \Omega$; $v(x) \in |x|^\mu$

$$u \in C^\infty(\mathbb{R}^d \setminus \{0\}), \ u(x) = \begin{cases} v(x), & x \in K_R \\ 0, & x \notin K_{2R} \end{cases}$$

$$u \in W^{m,p}(\Omega) \Leftrightarrow v \in W^{m,p}(K_R)$$

$$\partial^\alpha v(x) = P_\alpha(x)\cdot |x|^{\mu-2|\alpha|}, \quad P_\alpha \text{ homogeni polinom stupnja } |\alpha|$$

$$|\partial^\alpha v(x)| \leq C_\alpha \cdot |x|^{\mu-|\alpha|} =: C_\alpha \cdot \rho^{\mu-|\alpha|}$$

$$\int_{K_R} |\partial^\alpha v(x)|^p dx \leq C \cdot \int_0^R \rho^{(\mu-|\alpha|)p+d-1} d\rho$$

$$u \in W^{m,p} \iff (\mu-|\alpha|)p+d-1 > -1 \iff \mu > |\alpha| - \frac{d}{p}$$
$$\iff \mu > m - \frac{d}{p}$$

S druge strane, $\tilde{x}_k = (x_1, \dots, x_k)$, $r = |\tilde{x}_k|$:

$$\int_{\Omega_k} |u(x)|^q d\tilde{x}_k \geqslant \int |v(x)|^q d\tilde{x}_k = C_k \int_0^R r^{\mu q + k-1} dr$$

$$u \notin L^q(\Omega_k) \iff \mu q + k - 1 \leqslant -1 \iff \mu \leq \frac{-k}{q}$$

Postoji li μ t.d. je $m - \frac{d}{p} < \mu \leq \frac{-k}{q}$? Slijedi iz $q > p^*$.

Primjer 2.

$mp > d > (m-1)p, \quad \lambda > m - \frac{d}{p}$

Koristimo funkciju u iz Primjera 1.

Znamo $u \in W^{m,p} \iff m - \frac{d}{p} < \mu < \lambda$

S druge strane, za $|x| < R$:

$$\frac{|u(x) - u(0)|}{|x-0|^\lambda} = |x|^{\mu-\lambda} \to \infty \text{ kad } |x| \to 0 \Rightarrow u \notin C^{0,\lambda}(\bar{\Omega})$$

Primjer 3.

$mp = d$, $p > 1$

konstruiramo: $u \in W^{m,p}(\Omega)$, $u \in L^\infty(\Omega)$

$$v(x) = \log\left(\log \frac{4R}{|x|}\right)$$

ostatak konstrukcije isti kao u Primjeru 1

očito : $u \in L^{\infty}(\Omega)$

$$\partial^{\alpha} v(x) = \sum_{j=1}^{|\alpha|} P_{\alpha,j}(x) \cdot |x|^{-2|\alpha|} \left(\log \frac{4R}{|x|}\right)^{-j},$$

$P_{\alpha,j}$ homogeni polinomi stupnja $|\alpha|$

$$\rightarrow \int_{K_R} |\partial^{\alpha} v(x)|^p \, dx \leq C \cdot \sum_{j=1}^{|\alpha|} \int_0^R \left(\log \frac{4R}{\rho}\right)^{-jp} \cdot \rho^{-\frac{|\alpha| \cdot d}{m} + d - 1} \, d\rho$$

jedini netrivijalni slučaj je $|\alpha| = m \rightarrow \sigma = \log \frac{4R}{\rho} \rightarrow p > 1$

Primjer 4.

$(m-1)p = d, \ p > 1$

konstruiramo : $u \in W^{m,p}(\Omega), \ u \notin C^{0,1}(\overline{\Omega})$

$$v(x) = |x| \cdot \log\left(\log \frac{4R}{|x|}\right)$$

da je $v \in W^{m,p}(K_R) \ (u \in W^{m,p}(\Omega))$: na isti način

$$\frac{|v(x) - v(0)|}{|x - 0|} = \log\left(\log \frac{4R}{|x|}\right) \rightarrow \infty \text{ kad } |x| \rightarrow 0 \Rightarrow u \notin C^{0,1}(\overline{\Omega})$$

Nepravilne domene

Soboljevljeva ulaganja ne vrijede ako :

a) domena je neomeđena i preuska u beskonačnosti

$$\lim_{\substack{|x| \to \infty \\ x \in \Omega}} d(x, \partial\Omega) = 0$$

b) rub domene ima eksponencijalni šiljak

Teorem 1.

$\Omega \subseteq \mathbb{R}^d$ neomeđena, $|\Omega| < \infty$, $q > p$.

Tada $W^{m,p}$ nije uložen u L^q.

Dz.

$\rho = |x|$, $u(x) = u(\rho)$

BSO $|\Omega| = 1$

$A(\rho) = |\Omega \cap S(0,\rho)|$, $\int_0^\infty A(\rho)\,d\rho = 1$

$r_0 = 0$, definiramo r_k $(k \in \mathbb{N})$ t.d. $\int_{r_{k-1}}^{r_k} A(\rho)\,d\rho = \frac{1}{2^k} = \int_{r_k}^{\infty} A(\rho)\,d\rho$

$r_k \nearrow \infty$

$\Delta r_k = r_{k+1} - r_k$, $0 < \varepsilon < \frac{1}{mp} - \frac{1}{mq}$

Kontradikcijom: postoji rastući niz (k_j) t.d. $\Delta r_{k_j} \geq 2^{-\varepsilon k_j}$

BSO $k_1 \geq 1 \Rightarrow k_j \geq j$

$a_0 = 0$, $a_j = r_{k_j+1}$, $b_j = r_{k_j}$ $\Rightarrow a_{j-1} \leq b_j < a_j$

$a_j - b_j = \Delta r_{k_j} \geq 2^{-\varepsilon k_j}$

$f \in C^\infty(\mathbb{R})$, $0 \leq f \leq 1$

$f(t) = 0$ za $t \leq 0$, $f(t) = 1$ za $t \geq 1$

$|f^{(\varkappa)}(t)| \leq M$ za $1 \leq \varkappa \leq m$

$$u(x) = \begin{cases} 2^{\frac{k_{j-1}}{q}} & , a_{j-1} \leq \rho \leq b_j \\ 2^{\frac{k_{j-1}}{q}} + \left(2^{\frac{k_j}{q}} - 2^{\frac{k_{j-1}}{q}}\right) f\left(\frac{\rho - b_j}{a_j - b_j}\right) & , b_j \leq \rho \leq a_j \end{cases}$$

$\Omega_j = \{x \in \Omega : a_{j-1} \leq \rho < a_j\}$

$$\int_{\Omega_j} |u(x)|^p \, dx = \left(\int_{a_{j-1}}^{b_j} + \int_{b_j}^{a_j} \right) u(x)^p A(\rho) \, d\rho \leq \frac{1}{2^{(j-1)(1-p/q)}}$$

$p < q \to$ geometrijski red $\to u \in L^p(\Omega)$

slično vrijedi i za derivacije $\left(\frac{d^\alpha u}{d\rho^\alpha}\right) \to u \in W^{m,p}(\Omega)$

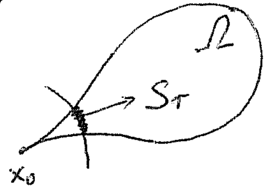
$$\int_{\Omega_j} |u(x)|^q \, dx \geq 2^{k_j-1} \int_{a_{j-1}}^{a_j} A(\rho) \, d\rho$$

$$= 2^{k_j-1}\left(\frac{1}{2^{k_j-1+1}} - \frac{1}{2^{k_j+1}} \right) \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \to u \notin L^q$$

„eksponencijalni šiljak"

$\Omega \subseteq \mathbb{R}^d$, $x_0 \in \partial\Omega$, $K_r = K(x_0, r)$

$\Omega_r = \Omega \cap K_r$, $S_r = \Omega \cap \partial K_r$



$A(r, \Omega) = |S_r|$

$(\forall k \in \mathbb{R}) \lim_{r \searrow 0} \frac{A(r,\Omega)}{r^k} = 0 \longrightarrow \Omega$ ima eksponencijalni šiljak u x_0

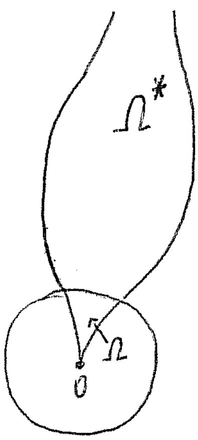
Teorem 2.

$\Omega \subseteq \mathbb{R}^d$ ima eksponencijalni šiljak u $x_0 \in \partial\Omega$, $q > p$.
Tada $W^{m,p}$ nije uložen u L^q.

Dz.

BSO $x_0 = 0$, $r = |x|$

$\Omega^* = \left\{ \frac{x}{|x|^2} : x \in \Omega, |x| < 1 \right\}$



$A(r, \Omega^*) = A(\frac{1}{r}, \Omega)\, r^{2(d-1)} \overset{EŠ}{\Longrightarrow} |\Omega^*| < \infty$

Neka je $p < t < q$.

Prema Teoremu 1. (i njegovom dokazu) postoji $\tilde{v} \in C^m(0,\infty)$ t.d.

(i) $\tilde{v}(r) = 0$ za $r \in \langle 0, 1]$, $\longrightarrow \Omega^* \cap K_1 = \emptyset$

(ii) $\int_1^\infty |\tilde{v}^{(j)}|^t A(r,\Omega^*)\,dr < \infty$, za $0 \le j \le m$ $\longrightarrow \tilde{v}(|y|) \in W^{m,t}(\Omega^*)$

(iii) $\int_1^\infty |\tilde{v}(r)|^q A(r,\Omega^*)\,dr = \infty$. $\longrightarrow \underbrace{\tilde{v}(|y|)}_{v(y)} \notin L^q(\Omega^*)$

$x = \frac{y}{|y|^2}$, $\rho = |x| = \frac{1}{|y|} = \frac{1}{r}$

$$u(x) = \tilde{u}(\rho) = r^\lambda \tilde{v}(r) = |y|^\lambda v(y), \quad \lambda = \frac{2d}{q}$$

$$\int_\Omega |u(x)|^q dx = \int_0^1 |\tilde{u}(\rho)|^q A(\rho,\Omega)\,d\rho = \int_1^\infty |\tilde{v}(r)|^q A(r,\Omega^*)\,dr = \infty \Rightarrow u \notin L^q(\Omega)$$

$$|\alpha| = j \le m, \quad |\partial^\alpha u(x)| \le |\tilde{u}^{(j)}(\rho)| \le \sum_{i=0}^{j} c_{ij}\, r^{\lambda+j+i}\, \tilde{v}^{(i)}(r)$$

$$\Rightarrow \int_\Omega |\partial^\alpha u(x)|^p dx \le C \cdot \sum_{i=0}^{j} \int_1^\infty |\tilde{v}^{(i)}(r)|^p\, r^{(\lambda+j+i)p-2d} A(r,\Omega^*)\,dr$$

Ako je $(\lambda + 2m)p \le 2d$, onda $p < t$, $|\Omega^*| < \infty \Rightarrow u \in W^{m,p}(\Omega)$.

U suprotnom,

$$k := ((\lambda+2m)p - 2d)\frac{t}{t-p} + 2d,$$

$$\int_1^\infty |\tilde{v}^{(i)}(r)|^p\, r^{(\lambda+j+i)p-2d} A(r,\Omega^*)\,dr \le \int_1^\infty \left[|\tilde{v}^{(i)}(r)|^p A(r,\Omega^*)^{\frac{p}{t}}\right] \cdot \left[r^{(k-2d)\cdot\frac{t-p}{t}} \cdot A(r,\Omega^*)^{\frac{t-p}{t}}\right] dr$$

$\frac{t}{p}$ (Hölder) $\frac{t}{t-p}$

$< \infty$ zbog (ii) $\qquad < \infty$ zbog EŠ