Marcinkiewiczev interpolacijski teorem

$u : \Omega \to \mathbb{C}$ izmjeriva,

funkcija distribucije

$$\delta_u(t) = \lambda_d(\{x \in \Omega : |u(x)| > t\}$$
$$= |\{x \in \Omega : |u(x)| > t\}$$

Čebiševljeva nejednakost:

$$u \in L^p,\ 0 < p < \infty \Rightarrow \delta_u(t) \leq t^{-p} \cdot \|u\|_p^p$$

slaba L^p norma:

$$[u]_p = [u]_{p,\Omega} = \left(\sup_{t>0} t^p \delta_u(t)\right)^{\frac{1}{p}}$$

operator slabog tipa (p,q), $p \in [1,\infty]$, $q \in [1,\infty\rangle$:

$$[F(u)]_{q,\Omega'} \leq C \cdot \|u\|_{p,\Omega}$$

sublinearan operator:

$$|F(u+v)| \leq |F(u)| + |F(v)|, \quad |F(cu)| = |c| \cdot |F(u)|$$

TEOREM (Marcinkiewicz)

$1 \leq p_1 \leq q_1 < \infty$, $1 \leq p_2 \leq q_2 \leq \infty$, $q_1 < q_2$

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad 0 < \theta < 1$$

$$\left.\begin{array}{l} F \text{ slabog tipa } (p_1,q_1) \\ F \text{ slabog tipa } (p_2,q_2) \end{array}\right\} \Rightarrow F : L^p(\Omega) \to L^q(\Omega') \text{ neprekidan.}$$

Lema 1.

$\Omega \subseteq \mathbb{R}^d$ zadovoljava uvjet konusa,

$u \in C^\infty(\Omega)$, $x \in \Omega$, $0 < r \leq \rho$

$$\Rightarrow |u(x)| \leq C \cdot \Big(\sum_{|\alpha| \leq m-1} r^{|\alpha|-d} \cdot \int_{C_{x,r}} |\partial^\alpha u(y)| dy \qquad (*)$$

$$+ \sum_{|\alpha|=m} \int_{C_{x,r}} |\partial^\alpha u(y)| \cdot |x-y|^{m-d} dy \Big)$$

$r = \min\{1, \rho\}$, $C_{x,r} \subseteq K[x,1]$, $\chi_1 = \chi_{K[0,1]}$, $\omega_m(x) = |x|^{m-d}$

$$(*) \Rightarrow |u(x)| \leq C \cdot \Big(\sum_{|\alpha| \leq m-1} \underbrace{\chi_1 * |\partial^\alpha u|(x)}_{\text{Lema } 2,3} + \sum_{|\alpha|=m} \underbrace{\chi_1 \omega_m * |\partial^\alpha u|(x)}_{\text{Lema } 2,3} \Big) \qquad (**)$$

Dz. Taylorova formula s integralnim ostatkom:

$$f(1) = \sum_{j=0}^{m-1} \frac{1}{j!} f^{(j)}(0) + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(t)\, dt$$

$$\uparrow$$

$$f(t) = u(tx + (1-t)y), \quad x \in \Omega, y \in C_{x,r}$$

$$f^{(j)}(t) = \sum_{|\alpha|=j} \frac{j}{\alpha!} \partial^\alpha u(tx + (1-t)y) \cdot (x-y)^\alpha$$

$$\alpha! = \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_d!$$

$$(x-y)^\alpha = (x_1-y_1)^{\alpha_1} (x_2-y_2)^{\alpha_2} \cdot \ldots \cdot (x_d-y_d)^{\alpha_d}$$

$$\Rightarrow |u(x)| \leqslant \sum_{|\alpha|\leqslant m-1} \frac{1}{\alpha!} |\partial^\alpha u(y)| \cdot |x-y|^{|\alpha|}$$

$$+ \sum_{|\alpha|=m} \frac{m}{\alpha!} |x-y|^m \int_0^1 (1-t)^{m-1} |\partial^\alpha u(tx+(1-t)y)| \, dt$$

Integriramo po y na skupu $C_{x,r}$ $\left(|C_{x,r}| = c \cdot r^d \right)$:

$$c r^d |u(x)| \leqslant \sum_{|\alpha|\leqslant m-1} \frac{r^{|\alpha|}}{\alpha!} \int_{C_{x,r}} |\partial^\alpha u(y)| \, dy$$

$$+ \sum_{|\alpha|=m} \frac{m}{\alpha!} \underbrace{\int_{C_{x,r}} |x-y|^m \int_0^1 (1-t)^{m-1} |\partial^\alpha u(tx+(1-t)y)| \, dt \, dy}$$

Fubini,
$z = tx + (1-t)y$
$z - x = (1-t)(y-x)$
$dz = (1-t)^d \cdot dy$

$$\int_0^1 (1-t)^{(m-1)-m-d} \int_{C_{x,(1-t)\cdot r}} |z-x|^m |\partial^\alpha u(z)| \, dz \, dt$$

Fubini

$|z-x| \leqslant (1-t)\cdot r$
$\Rightarrow t \leqslant 1 - \frac{|z-x|}{r}$

$$\int_{C_{x,r}} |z-x|^m |\partial^\alpha u(z)| \int_0^{1-\frac{|z-x|}{r}} (1-t)^{-d-1} \, dt \, dz$$

1. dio, A)

$m \cdot p > d$ ili $(m=d,\ p=1)$

$$W^{m,p} \hookrightarrow C^0_b$$
$$\hookrightarrow L^q(\Omega_k),\quad p \leq q \leq \infty$$

$u \in W^{m,p} \cap C^\infty$, $x \in \Omega$

tvrdimo: $|u(x)| \leq C \cdot \|u\|_{m,p}$

Lema 1 daje tvrdnju za $p=1$, $m=d$.

$p>1 \Rightarrow$ Hölder na $(*)$:

$$|u(x)| \leq C \cdot \Big(\sum_{|\alpha|\leq m-1} c^{1/p'} \cdot r^{|\alpha|-d+\frac{d}{p'}} \|\partial^\alpha u\|_p$$
$$+ \sum_{|\alpha|=m} \|\partial^\alpha u\|_p \cdot \Big[\int_{C_{x,r}} |x-y|^{(m-d)p'} dy \Big]^{1/p'} \Big)$$

$$(m-d)p' > -d$$
$$\Leftrightarrow (m-d) \cdot \frac{p}{p-1} > -d$$
$$\Leftrightarrow m \cdot p - d \cdot p > -dp + d$$
$$\Leftrightarrow m \cdot p > d$$

$u \in W^{m,p}$, $u_k \in W^{m,p} \cap C^\infty$, $\|u_k - u\|_{m,p} \to 0$

$|u_k(x) - u_\ell(x)| \leq C \cdot \|u_k - u_\ell\|_{m,p}$

$\Rightarrow u_k \rightrightarrows u'$, $u' = u$ (ss)

Napomena.

Pokazali smo i više:

$$|u(x)| \leq C \cdot \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{p, C_{x,r}} \qquad (***)$$

$\Omega_k = \Omega \cap H$, $\Omega_{k,\rho} = \{x \in \mathbb{R}^d : d(x, \Omega_k) < \rho\}$, $u = \tilde{u}$

$$\int_{\Omega_k} |u(x)|^p \, dx \overset{(***)}{\leq} C \cdot \sum_{|\alpha| \leq m} \int_{\Omega_k} \int_{K[x,\rho]} |\partial^\alpha u(y)|^p \, dy \, dx$$

$$\overset{\text{Fubini}}{=} C \cdot \sum_{|\alpha| \leq m} \int_{\Omega_{k,\rho}} |\partial^\alpha u(y)|^p \int_{H \cap K[y,\rho]} dx \, dy$$

$$\left.\begin{aligned} &\Rightarrow W^{m,p} \hookrightarrow L^p(\Omega_k) \\ &(***) \Rightarrow W^{m,p} \hookrightarrow L^\infty(\Omega_k) \end{aligned}\right\} \xRightarrow{[AF,\ 2.11]} W^{m,p} \hookrightarrow L^q(\Omega_k), \; p \leq q \leq \infty$$

Q.E.D.

<u>Lema 2.</u>

$p \geq 1$, $1 \leq k \leq d$, $d - mp < k$, $m \leq d$

$v \in L^p(\mathbb{R}^d) \Rightarrow \| \chi_r \omega_m * |v| \|_{p,H} \leq C \cdot r^{m - \frac{d-k}{p}} \cdot \|v\|_p$

Posebno, $\| \chi_1 * |v| \|_{p,H} \underset{\substack{\uparrow \\ m \leq d}}{\leq} \| \chi_1 \omega_m * |v| \|_{p,H} \leq C \cdot \|v\|_p$

<u>Dz.</u> $p > 1$, Hölder:

$$\chi_r \omega_m * |v|(x) = \int_{K[x,r]} |v(y)| \, |x-y|^{m-d-s+s} \, dy$$

$$\leq \left(\int_{K[x,r]} |v(y)|^p \cdot |x-y|^{-sp} \, dy \right)^{\frac{1}{p}} \left(\int_{K[x,r]} |x-y|^{\overbrace{(s+m-d)p'}^{>-d}} \, dy \right)^{\frac{1}{p'}}$$

$$= C \cdot r^{\overbrace{s+m-\frac{d}{p}}^{>0}} \left(\int_{K[x,r]} |v(y)|^p \cdot |x-y|^{-sp} \, dy \right)^{\frac{1}{p}}$$

$p=1$, dovoljno je $s+m-d \geq 0$ (bez Höldera)

Potenciramo na p-tu i integriramo na H:

$$\| \chi_r \omega_m * |v| \|_{p,H}^p \leq C \cdot r^{(s+m)p-d} \int_H \int_{K[x,r]} |v(y)|^p \cdot |x-y|^{\overset{>-k}{\widetilde{-sp}}} \, dy \, dx$$

$$\underset{\text{Fubini}}{\leq} C \cdot r^{(s+m)p-d} \cdot r^{k-sp} \cdot \|v\|_p^p$$

$$= C \cdot r^{mp-(d-k)} \cdot \|v\|_p^p$$

$$\left.\begin{array}{l} k-sp>0 \longrightarrow s<\frac{k}{p} \\ s+m-\frac{d}{p}>0 \longrightarrow s>\frac{d}{p}-m \end{array}\right\} \frac{k}{p}>\frac{d}{p}-m \; / \cdot p$$

$$\iff \underline{k>d-mp.}$$

Q.E.D.

<u>Lema 3.</u>

$p>1$, $m \cdot p<d$, $d-mp<k \leq d$, $p^* = \frac{kp}{d-mp}$

$$\frac{d}{kp} - \frac{1}{p^*} = \frac{m}{k} \longleftrightarrow p$$

$v \in L^p(\mathbb{R}^d)$

$$\Rightarrow \| \chi_1 * |v| \|_{p^*,H} \underset{m \leq d}{\leq} \| \chi_1 \omega_m * |v| \|_{p^*,H} \leq \| \omega_m * |v| \|_{p^*,H} \underset{\text{dokazujemo ovo!}}{\leq} C \cdot \|v\|_p$$

Dz. (Marcinkiewicz)

Dokazat ćemo da je $v \mapsto (\omega_m * |v|)|_H$ slabog tipa (p, p^*).

$$\int_{\mathbb{R}^d \setminus K[x,r]} |v(y)||x-y|^{m-d} \, dy \leq \|v\|_p \cdot \Big(\int_{\mathbb{R}^d \setminus K[x,r]} |x-y|^{(m-d)p'} \Big)^{\frac{1}{p'}}$$

$$= C_1 \cdot \|v\|_p \cdot \Big(\int_r^\infty t^{(m-d)p'+d-1} \, dt \Big)^{\frac{1}{p'}}$$

$(m-d)p' + d < 0$

$\Leftrightarrow m \cdot p < d$

$$= C_1 \cdot r^{m-\frac{d}{p}} \cdot \|v\|_p$$

$t > 0$, biramo r t.d. $C_1 \cdot r^{m-\frac{d}{p}} \cdot \|v\|_p = \frac{t}{2}$

pretpostavimo $(\omega_m * |v|)(x) = \int_{\mathbb{R}^d} |v(y)||x-y|^{m-d} \, dy > t$

$$\Rightarrow (\chi_r \omega_m * |v|)(x) = \int_{K[x,r]} |v(y)| \cdot |x-y|^{m-d} \, dy > \frac{t}{2}$$

$$\Rightarrow \big| \{ x \in H : \omega_m * |v|(x) > t \} \big| \leq \Big| \Big\{ x \in H : \chi_r \omega_m * |v|(x) > \frac{t}{2} \Big\} \Big|$$

$$\overset{\text{Čebišev}}{\leq} \Big(\frac{2}{t} \Big)^p \cdot \| \chi_r \omega_m * |v| \|_{p,H}^p$$

$$\overset{\text{Lema 2}}{\leq} \Big(\frac{r^{\frac{d}{p}-m}}{C_1 \|v\|_p} \Big)^p \cdot C \cdot r^{mp-d+k} \cdot \|v\|_p^p$$

$$= C_2 \cdot r^k$$

$$= C_2 \cdot \Big(\frac{2C_1 \|v\|_p}{t} \Big)^{p^*}$$

$$\left.\begin{array}{l} mp < d \Rightarrow p < \frac{d}{m} \\ d - mp < k \Rightarrow p > \frac{d}{m} - \frac{k}{m} \end{array}\right\} p \in \left\langle \frac{d}{m} - \frac{k}{m}, \frac{d}{m} \right\rangle$$

$$p_1 < p < p_2, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad 0 < \theta < 1$$

$$\frac{1}{p^*} = \frac{d/k}{p} - \frac{m}{k} = \frac{1-\theta}{p_1^*} + \frac{\theta}{p_2^*}$$

$p_1 \leq p_1^*$ ✓

$p_2 \leq p_2^*$ ✓ $\rightsquigarrow$ Marcinkiewicz!

$p_1^* < p_2^*$ ✓

1\. dio, C), $p > 1$

$$m \cdot p < d \quad i \quad d - mp < k \leq d$$

$$\Rightarrow W^{m,p} \hookrightarrow L^q(\Omega_k), \quad p \leq q \leq p^*$$

$u \in C^\infty(\Omega)$, $u = \tilde{u}$

$$(**) \rightarrow |u(x)| \leq C \cdot \left(\sum_{|\alpha| \leq m-1} \chi_1 * |\partial^\alpha u|(x) + \sum_{|\alpha| = m} \chi_1 \omega_m * |\partial^\alpha u|( ) \right)$$

$$p \leq q \leq p^*, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p^*}, \quad \theta \in [0,1]$$

$$\Rightarrow \|u\|_{q,\Omega_k} \leq \|u\|_{p,H}^{1-\theta} \cdot \|u\|_{p^*,H}^{\theta}$$

$$\leq C \cdot \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p \right)^{1-\theta} \cdot \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p \right)^{\theta}$$

Lema 2,3

$$\leq C \cdot \|u\|_{m,p,\Omega}.$$

1. dio, B), $p>1$

$$m\cdot p = d \Rightarrow W^{m,p} \hookrightarrow L^q(\Omega_k), \quad p \leq q < \infty$$

$1 < p_1 < p < p_2$ t.d. $\left.\begin{array}{l} mp_1 < d \\ d - mp_1 < k \end{array}\right\}$ C)

$mp_2 > d \longrightarrow$ A)

$$\frac{1}{q} = \frac{\theta}{p_1} \longrightarrow \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

(**), Marcinkiewicz $\rightsquigarrow$ Q.E.D.

1. dio $\longrightarrow$ preostaje pokazati B) i C) za $p=1$

Ključni korak : $W^{1,1} \hookrightarrow L^p, \quad 1 \leq p \leq \frac{d}{d-1}$

B) $m = d \Rightarrow W^{m,1} \hookrightarrow L^q(\Omega_k), \quad 1 \leq q < \infty$ (dokazano pod A))

C) $m < d$ i $d - m \underset{\uparrow}{\leq} k \Rightarrow W^{m,1} \hookrightarrow L^q(\Omega_k), \quad 1 \leq q \leq 1^* = \frac{k}{d-m}$

dokazat ćemo za strogu nejednakost

$$W^{m,1} \hookrightarrow W^{m-1,p}, \quad 1 \leq p \leq \frac{d}{d-1}$$

$$\hookrightarrow L^q(\Omega_k), \quad p \leq q \leq p^*$$

$$p^* = \frac{kp}{d-(m-1)p} = \frac{k \cdot \frac{d}{d-1}}{d-(m-1)\cdot\frac{d}{d-1}} = \frac{k}{d-m}$$

2. dio

$\Omega \subseteq \mathbb{R}^d$ zadovoljava strogi lokalno Lipschitzov uvjet:
- lokalno konačan otvoren pokrivač $\{U_j\}$ ruba $\partial\Omega$

$(\exists \delta, M > 0)(\exists$ funkcije f_j od $d-1$ varijabli$)$

(i) $(\exists R \in \mathbb{N})$ svaka kolekcija $R+1$ skupova U_j ima prazan presjek

(ii) $(\forall x, y \in \Omega_\delta)\ |x-y| < \delta \Rightarrow (\exists j)\ x, y \in V_j := \{z \in U_j : d(z, \partial U_j) > \delta\}$

(iii) $|f_j(\xi) - f_j(\eta)| \leq M \cdot |\xi - \eta|$

(iv) postoji koordinatni sustav na U_j t.d.

$\Omega \cap U_j \ \dots \ y_d < f_j(y_1, \dots, y_{d-1})$

Tvrdimo:

$$W^{m,p} \hookrightarrow C^{0,\lambda}(\overline{\Omega}), \text{ gdje je } m \cdot p > d \text{ i}$$

(a) $0 < \lambda \leq m - \frac{d}{p}$ za $d > (m-1)p$, ili

(b) $0 < \lambda < 1$ za $d = (m-1)p$ i $p > 1$, ili

(c) $0 < \lambda \leq 1$ za $d = m-1$ i $p = 1$.

Već znamo da je $W^{m,p} \hookrightarrow C_B^0(\Omega)$ pa je dovoljno dokazati:

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\lambda} \leq C \cdot \|u\|_{m,p}.$$

$m \cdot p > d \geq (m-1) \cdot p$

$\uparrow$ možemo primijeniti B) i C) na $W^{\overset{d}{1}+(m-1),p}$

$\Rightarrow W^{m,p} \hookrightarrow W^{1,r}$

$\Rightarrow$ dovoljno je tvrdnju pokazati za $m = 1$

$\underline{m=1}$

$$W^{1,p} \hookrightarrow C^{0,\lambda}(\overline{\Omega}), \quad p>d, \quad 0<\lambda \leq 1-\frac{d}{p}$$

$$\longrightarrow \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda} \leq C \cdot \|u\|_{1,p}$$

Dz.

1) Ω = kocka jediničnog brida

Q_t = kocka brida duljine t

Već znamo da je tvrdnju dovoljno dokazati za $u \in C^\infty(\Omega)$

$x,y\in\Omega, \quad |x-y| = \sigma < 1$

$\Rightarrow$ postoji Q_σ t.d. $x,y\in Q_\sigma$

$$z\in Q_\sigma, \quad u(z)-u(x) = \int_0^1 \frac{d}{dt} u(x+t(z-x))\,dt$$

$$\Longrightarrow |u(x)-u(z)| \leq \sigma\sqrt{d} \cdot \int_0^1 |\nabla u(x+t(z-x))|\,dt$$

+ Fubini

$$\Longrightarrow \left| u(x) - \frac{1}{\sigma^d}\int_{Q_\sigma} u(z)\,dz \right| = \left| \frac{1}{\sigma^d}\int_{Q_\sigma} (u(x)-u(z))\,dz \right| \leq$$

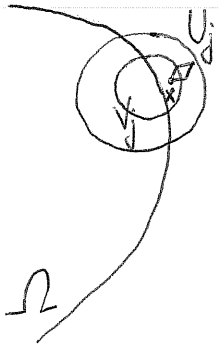
$$\leq C\cdot \sigma^{1-\frac{d}{p}} \|\nabla u\|_p$$

$$\Longrightarrow |u(x)-u(y)| \leq 2C\cdot |x-y|^{1-\frac{d}{p}} \cdot \|\nabla u\|_p$$

2) Ω = paralelepiped

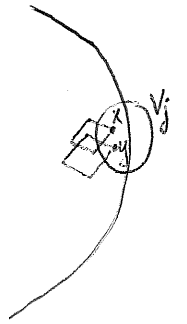
- nesingularna linearna transformacija
- poseban slučaj m-glatke transformacije

3) Ω = proizvoljna domena koja zadovoljava strogi lokalno Lipschitzov uvjet



($\exists$ paralelepiped P dijametra δ)

$$(\forall j)(\exists P_j \cong P)(\forall x \in V_j \cap \Omega) \ x + P_j \subseteq \Omega$$



$$(\exists \delta_0, \delta_1 > 0, \ \delta_0 \leq \delta)(\forall x, y \in V_j \cap \Omega)$$

$$|x-y| < \delta_0 \Rightarrow (\exists z \in (x+P_j) \cap (y+P_j)),$$

$$|x-z| + |y-z| \leq \delta_1 \cdot |x-y|$$

$$\left.\begin{aligned} |u(x)-u(y)| &\leq |u(x)-u(z)| + |u(y)-u(z)| \\ &\leq C\,|x-z|^\lambda \|u\|_{1,p} + C\,|y-z|^\lambda \|u\|_{1,p} \\ &\leq C_1 \cdot |x-y|^\lambda \|u\|_{1,p} \end{aligned}\right\} (+)$$

Konačno, $x, y \in \Omega$ proizvoljne.

$$|x-y| < \delta_0 \leq \delta, \quad x, y \in \Omega_\delta \overset{(ii)}{\Longrightarrow} (\exists j)\ x, y \in V_j \text{ pa vrijedi } (+)$$

$|x-y| < \delta_0$, $x \in \Omega_\delta$, $\underline{y \in \Omega \setminus \Omega_\delta}$

$|x-y| < \delta_0$, $\underline{x, y \in \Omega \setminus \Omega_\delta}$ nije problem

$|x-y| \geq \delta_0$

$$\Rightarrow |u(x)-u(y)| \leq |u(x)| + |u(y)| \overset{1.\text{dio}}{\leq} C \cdot \|u\|_{1,p}$$

$$\leq C \cdot (\delta_0^{-1} |x-y|)^\lambda \cdot \|u\|_{1,p}$$