TEOREM.

$C_c^\infty(\Omega)$ je gust u $W^{m,p}(\mathbb{R}^d)$ ako i samo ako je Ω^c (m,p')-polaran. $(1<p<\infty)$

Dz.

I) Pretpostavljamo: $C_c^\infty(\Omega)$ gust u $W^{m,p}(\mathbb{R}^d)$

$T \in W^{-m,p'}(\mathbb{R}^d)$, $\operatorname{supp} T \subseteq \Omega^c$

$u \in W^{m,p}(\mathbb{R}^d)$, $\varphi_j \in C_c^\infty(\Omega)$ t.d. $\|u-\varphi_j\|_{m,p} \to 0$

$$T(u) = \lim_j \underbrace{T(\varphi_j)}_{=0} = 0 \Rightarrow T=0.$$

II) Pretpostavljamo: Ω^c (m,p')-polaran

Pretpostavljamo suprotno: $C_c^\infty(\Omega)$ nije gust u $W^{m,p}(\mathbb{R}^d)$

$$\Rightarrow (\exists u \in W^{m,p}(\mathbb{R}^d))(\exists k>0)(\forall \varphi \in C_c^\infty(\Omega))$$

$$\|u-\varphi\|_{m,p,\mathbb{R}^d} \geq k$$

Hahn-Banach [AF, 1.13] $\Rightarrow$ $(\exists T \in W^{-m,p'}(\mathbb{R}^d))$ t.d.

$(\forall \varphi \in C_c^\infty(\Omega))\ T(\varphi)=0$ & $T(u)\neq 0$

$\Rightarrow \operatorname{supp} T \subseteq \Omega^c$, $T \neq 0$, kontradikcija s (m,p')-polarnošću skupa Ω^c

Q.E.D.

$f \in C^1(\langle a,b\rangle \times \langle c,d\rangle)$, $\frac{\partial f}{\partial x} \equiv 0$, $\frac{\partial f}{\partial y} \equiv 0 \Rightarrow f = C$

$\rightarrow$ proširenje na „manje glatke" funkcije (L^1_{loc})

Lema 1.

$B = \langle a_1, b_1\rangle \times \ldots \times \langle a_d, b_d\rangle \subseteq \mathbb{R}^d$, $\varphi \in \mathcal{D}(B)$

Ako je $\int_B \varphi(x)\,dx = 0$, onda je $\varphi(x) = \sum_{j=1}^{d} \varphi_j(x)$,

$$\varphi_j \in \mathcal{D}(B), \quad \int_{a_j}^{b_j} \varphi_j(x_1, \ldots, x_j, \ldots, x_d)\,dx_j = 0.$$

Korolar 1.

$T \in \mathcal{D}'(B)$ i $\partial_j T = 0$ za $1 \le j \le d$

$$\Rightarrow (\exists k \in \mathbb{C})(\forall \varphi \in \mathcal{D}(B)) \quad T(\varphi) = k \cdot \int_B \varphi(x)\,dx.$$

Lema 2.

$u \in L^1_{loc}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ otvoren, $\int_\Omega u(x)\varphi(x)\,dx = 0$

za svaki $\varphi \in \mathcal{D}(\Omega) \Rightarrow u(x) = 0$ (s.s. $x \in \Omega$).

Korolar 2.

$u \in L^1_{loc}(B)$, $\partial_j u = 0$ za $1 \le j \le d$

$$\Rightarrow (\exists k \in \mathbb{C}) \quad u(x) = k \quad (\text{s.s. } x \in B).$$

Dz.

$$\int_B u(x)\varphi(x)\,dx = T_u(\varphi) \overset{\text{Korolar 1.}}{=} k \cdot \int_B \varphi(x)\,dx \overset{\text{Lema 2.}}{\Longrightarrow} u(x) - k = 0 \quad (\text{s.s. } x \in B)$$

Teorem. Neka je $m \geq 1$.

(a) $W^{m,p}(\Omega) = W_0^{m,p}(\Omega) \Rightarrow \Omega^c$ je (m,p')-polaran.

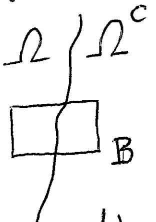
(b) Ako je Ω^c $(1,p)$-polaran i (m,p')-polaran, onda je

$$W^{m,p}(\Omega) = W_0^{m,p}(\Omega).$$

Dz.

(a) Dokazujemo prvo da je $\lambda_d(\Omega^c) = 0$.

Pretpostavimo suprotno:



$\lambda_d(B \cap \Omega) > 0$

$\lambda_d(B \cap \Omega^c) > 0$

$U \in C_c^\infty(\mathbb{R}^d)$, $U \equiv 1$ na $B \cap \Omega$

$u = U|_\Omega$, $u \in W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$

$\tilde{u}$ proširenje nulom $\Rightarrow \tilde{u} \in W^{m,p}(\mathbb{R}^d)$ i $\partial_j \tilde{u} = \widetilde{\partial_j u}$

$\widetilde{\partial_j u} = 0$ na $B \underset{\substack{\uparrow \\ \text{Korolar 2.}}}{\Longrightarrow} u(x) = k$ (s.s. $x \in B$), kontradikcija!

Sada ćemo dokazati da je $C_c^\infty(\Omega)$ gust u $W^{m,p}(\mathbb{R}^d)$ pa će po prethodnom teoremu slijediti tvrdnja.

$v \in W^{m,p}(\mathbb{R}^d)$, $u = v|_\Omega$

$u \in W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$

$\Rightarrow \tilde{u} \in W^{m,p}(\mathbb{R}^d)$ i može se aproksimirati funkcijama $\varphi_j \in C_c^\infty(\Omega)$

$\Rightarrow$ isto vrijedi i za v jer je

$v(x) = \tilde{u}(x)$ (s.s $x \in \mathbb{R}^d$)

(b) Neka je Ω^c $(1,p)$-polaran i (m,p')-polaran i neka je $u \in W^{m,p}(\Omega)$.

$\tilde{u} \in L^p(\mathbb{R}^d)$, $T_{\partial_j \tilde{u}} \in W^{-1,p}(\mathbb{R}^d)$

$\widetilde{\partial_j u} \in L^p(\mathbb{R}^d) \subseteq W^{-1,p}(\mathbb{R}^d)$

$$\left.\begin{array}{l} \Rightarrow T_{\partial_j \tilde{u}} - \widetilde{\partial_j u} \in W^{-1,p}(\mathbb{R}^d) \\ \quad \operatorname{supp}(T_{\partial_j \tilde{u}} - \widetilde{\partial_j u}) \subseteq \Omega^c \\ \quad \Omega^c \ (1,p)\text{-polaran} \end{array}\right\} \Rightarrow \partial_j \tilde{u} = \widetilde{\partial_j u}$$

$$\left.\begin{array}{l} \Rightarrow \partial_j \tilde{u} \in L^p(\mathbb{R}^d), \quad \tilde{u} \in W^{m,p}(\mathbb{R}^d) \\ (m,p')\text{-polarnost} \Rightarrow C_c^\infty(\Omega) \text{ je gust u } W^{m,p}(\mathbb{R}^d)) \end{array}\right\} u \in W_0^{m,p}(\Omega)$$

Q.E.D.

<u>Napomena.</u>

$p=2 \Rightarrow (m,p')$-polarnost povlači $(1,p)$-polarnost

Vrijedi i više:

<u>Teorem.</u>

$m \geq 1$ i $p \geq 2 \Rightarrow W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$ ako **i** samo ako je Ω^c (m,p')-polaran

<u>Dz.</u>

$$(m,p')\text{-polarnost} \underset{\substack{\uparrow \\ p' \leq p \\ \uparrow \\ F \cap K \text{ polaran za svaki kompakt } K}}{\Rightarrow} (m,p)\text{-polarnost} \Rightarrow (1,p)\text{-polarnost}$$

SOBOLJEVLJEV TEOREM ULAGANJA

$$W^{m,p}(\Omega) \hookrightarrow X \text{ Banachov}, \quad \|u\|_X \leq C \cdot \|u\|_{m,p,\Omega}$$

$$(W_0^{m,p}(\Omega))$$

$X \ldots W^{j,q}(\Omega), \ j \leq m$

$W^{j,q}(\Omega_k), \ 1 \leq k < d$

Ω_k = presjek Ω s k-dimenzionalnom ravninom u $\mathbb{R}^d$

$u|_{\Omega_k} = ?$

$C^\infty(\Omega) \ni \varphi_j \longrightarrow u \quad (u \ W^{j,q}(\Omega))$

$\varphi_j|_{\Omega_k} \longrightarrow u^* \quad (u \ W^{j,q}(\Omega_k))$

$$\left.\begin{array}{l} C_b^j(\Omega) \ldots \text{ neprekinute i ograničene, zajedno s derivacijama do reda } j \\ C^j(\overline{\Omega}) \ldots \text{ uniformno neprekinute i ograničene} \end{array}\right\} \max_{|\alpha| \leq j} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)|$$

$C^{j,\lambda}(\overline{\Omega}) \ldots$ Hölder neprekinute reda λ

$$\|\varphi\|_{C^{j,\lambda}} = \|\varphi\|_{C^j} + \max_{|\alpha| \leq j} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)|}{|x-y|^\lambda}$$

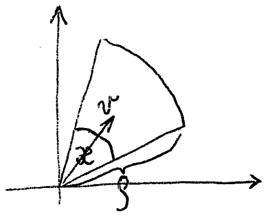
Za ulaganja $W^{m,p}(\Omega)$ prostora bitna je kvaliteta domene Ω.

$$W_0^{m,p}(\Omega) \underset{\substack{\uparrow \\ \text{izometrično } (u \mapsto \tilde{u})}}{\hookrightarrow} W^{m,p}(\mathbb{R}^d)$$

Kvaliteta domene Ω

Oznake i pojmovi:

- konačni konus C



$v \neq 0,\ \rho > 0,\ 0 < \varkappa \leq \pi,$

$$C = \left\{ x \in \mathbb{R}^d : x = 0 \text{ ili } 0 < |x| \leq \rho,\ \measuredangle(x,v) \leq \frac{\varkappa}{2} \right\}$$

$$x + C = C_x$$

- $\Omega_\delta = \left\{ x \in \Omega : d(x, \partial\Omega) < \delta \right\}$

- m-glatka transformacija

$$\Phi : \Omega \longrightarrow G, \quad \Psi = \Phi^{-1}$$

$$y = \Phi(x) = (\varphi_1(x), \dots, \varphi_d(x))$$

$$x = \Psi(y) = (\psi_1(y), \dots, \psi_d(y))$$

$$\varphi_j \in C^m(\overline{\Omega}), \quad \psi_j \in C^m(\overline{G})$$

u na Ω $\longrightarrow$ Au na G

$$Au(y) = u(\Psi(y))$$

Teorem.

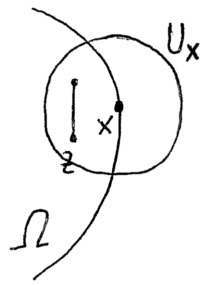
Φ m-glatka $(m \geq 1)$ $\Rightarrow$ $A \in L(W^{m,p}(\Omega), W^{m,p}(G)),$

$$A^{-1} \in L(W^{m,p}(G), W^{m,p}(\Omega))$$

Skica dokaza.

$$\partial^\alpha (A u_j)(y) = \sum_{\beta \leq \alpha} \underbrace{M_{\alpha\beta}(y)}_{\text{polinomi}} A(\partial^\beta u_j)(y)$$

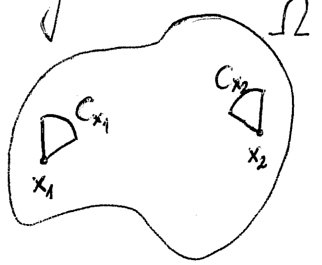
1) uvjet segmenta



pokrivač $\{U_x\}$ od $\partial\Omega$

bez smanjenja općenitosti:
lokalno konačan pokrivač
$\{U_1, U_2, U_3, \dots\}$

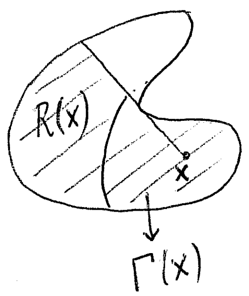
2) uvjet konusa



– svi konusi međusobno kongruentni

3) slabi uvjet konusa

$x \in \Omega$, $R(x)$ = unija zraka i segmenata s početkom u x, sadržanih u Ω



$$\Gamma(x) = \{y \in R(x) : |y - x| < 1\}$$

$$(\exists \delta > 0)(\forall x \in \Omega)\ \lambda_d(\Gamma(x)) \geq \delta$$

λ_d – Lebesgueova mjera

4) uniformni uvjet konusa

– postoji lokalno konačan otvoren pokrivač $\{U_j\}$ ruba $\partial\Omega$ i odgovarajući niz $\{C_j\}$ konačnih konusa t.d.

(i) $(\forall j \in \mathbb{N})\ \operatorname{diam} U_j < M < \infty$

(ii) $(\exists \delta > 0)\ \Omega_\delta \subseteq \bigcup_j U_j$

(iii) $Q_j := \bigcup_{x \in \Omega \cap U_j} (x + C_j) \subseteq \Omega$

(iv) $(\exists R \in \mathbb{N})$ svaka kolekcija $R+1$ skupova Q_j ima prazan presjek

5) strogi lokalno Lipschitzov uvjet
- lokalno konačan otvoren pokrivač $\{U_j\}$

$(\exists \delta, M>0)$ ($\exists$ funkcije f_j od $d-1$ varijabli)

(i) $(\exists R \in \mathbb{N})$ svaka kolekcija $R+1$ skupova U_j ima prazan presjek

(ii) $(\forall x, y \in \Omega_\delta)$ $|x-y|<\delta \Rightarrow (\exists j)$ $x, y \in V_j := \{z \in U_j : d(z, \partial U_j) > \delta\}$

(iii) $|f_j(\xi) - f_j(\eta)| \leq M \cdot |\xi - \eta|$

(iv) postoji koordinatni sustav na U_j t.d.

$\Omega \cap U_j$ ... $y_d < f_j(y_1, \dots, y_{d-1})$

6) uvjet uniformne C^m-regularnosti
$\{U_j\}$, $\{\phi_j\}$ m-glatke transformacije

$$\phi_j(U_j) = K(0,1)$$
$$\psi_j = \phi_j^{-1}$$

(i) kolekcija $R+1$ skupova U_j ima prazan presjek

(ii) $(\exists \delta > 0)$ $\Omega_\delta \subset \bigcup_{j=1}^{\infty} \psi_j\left(K(0, \tfrac{1}{2})\right)$

(iii) $\phi_j(U_j \cap \Omega) = \{y \in K(0,1) : y_d > 0\}$

(iv) $(\forall x \in U_j)$ $|(\partial^\alpha \varphi_{j,k})(x)| \leq M$

$(\forall y \in B)$ $|(\partial^\alpha \psi_{j,k})(y)| \leq M$

$0 < |\alpha| \leq m$

uvjet uniformne C^m-regularnosti ($m \geq 2$)
$\Rightarrow$ strogi lokalno Lipschitzov uvjet
$\Rightarrow$ uniformni uvjet konusa $\Rightarrow$ uvjet konusa $\Rightarrow$ slabi uvjet konusa
$\Rightarrow$ uvjet segmenta

TEOREM (Soboljevljeva ulaganja) $\rightarrow$ Soboljev, $\rightarrow$ Morrey

Ω, Ω_k, $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, $p \in [1, \infty\rangle$

1. dio (pretpostavljamo uvjet konusa)

A) $\boxed{m \cdot p > d}$ ili ($m = d$, $p = 1$)

$$W^{j+m,p} \hookrightarrow C_b^j$$
$$\hookrightarrow W^{j,q}(\Omega_k), \quad p \leq q \leq \infty$$

Posebno: $W^{m,p} \hookrightarrow L^q$, $p \leq q \leq \infty$

B) $\boxed{m \cdot p = d}$

$$W^{j+m,p} \hookrightarrow W^{j,q}(\Omega_k), \quad p \leq q < \infty$$

Posebno: $W^{m,p} \hookrightarrow L^q$, $p \leq q < \infty$

C) $\boxed{m \cdot p < d}$ i $[(d - mp < k \leq d)$ ili $(p = 1,\ d - m \leq k \leq d)]$

$$W^{j+m,p} \hookrightarrow W^{j,q}(\Omega_k), \quad p \leq q \leq p^*, \quad \frac{d}{kp} - \frac{1}{p^*} = \frac{m}{p}$$

Posebno: $W^{m,p} \hookrightarrow L^q$, $p \leq q \leq p^*$, $\boxed{\frac{1}{p} - \frac{1}{p^*} = \frac{m}{d}}$

2. dio (pretpostavljamo strogi lokalno Lipschitzov uvjet)

A) $W^{j+m,p} \hookrightarrow C^j(\overline{\Omega})$

$$mp > d > (m-1)p$$

$$W^{j+m,p} \hookrightarrow C^{j,\lambda}(\overline{\Omega}), \quad 0 < \lambda \leq m - \frac{d}{p}$$

$$d = (m-1)p$$

$$W^{j+m,p} \hookrightarrow C^{j,\lambda}(\overline{\Omega}), \quad \begin{array}{l} 0<\lambda<1 \\ 0<\lambda\leq 1 \ (p=1) \end{array}$$

3. dio (proizvoljna domena)

tvrdnje vrijede za odgovarajuće W_0 prostore

Primjedbe

1. $\left.\begin{array}{l} \lambda_d(\Omega) < \infty \\ \lambda_d(\Omega_k) < \infty \end{array}\right\}$ uvjet $p \leq q$ možemo ispustiti

. Dovoljno je teorem dokazati za $j=0$.

Pr.) $W^{m,p} \hookrightarrow L^q$

$$u \in W^{j+m,p} \Rightarrow \partial^\alpha u \in W^{m,p}, \quad |\alpha| \leq j$$

$$\Rightarrow \partial^\alpha u \in L^q$$

$$\|u\|_{j,q} = \Big(\sum_{|\alpha|\leq j} \|\partial^\alpha u\|_q^q\Big)^{\frac{1}{q}}$$

$$\leq K_1 \cdot \Big(\sum_{|\alpha|\leq j} \|\partial^\alpha u\|_{m,p}^p\Big)^{\frac{1}{p}} \leq K_2 \cdot \|u\|_{j+m,p}$$