

On homogenisation of linear nonlocal problems in fractional divergence form

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Motivation: Stationary diffusion equation

$\Omega \subseteq \mathbb{R}^d$ open and bounded, $f \in H^{-1}(\Omega)$.

$$\begin{cases} -\operatorname{div}(\mathbf{A}\nabla u) = f, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Physical interpretation:

- u – temperature / electric potential
- \mathbf{A} – conductivity (material coefficient)
- f – source term
- $\mathbf{A}\nabla u$ – heat/electric flux

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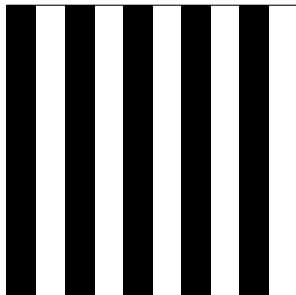
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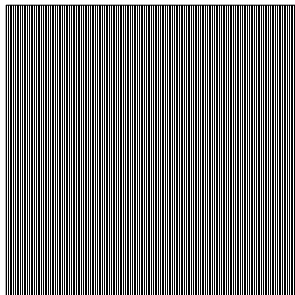
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$\varepsilon(n)$



\mathbf{A}_n

$n \rightarrow \infty$
→



\mathbf{A}_∞

Question (Homogenisation): If \mathbf{A}_n oscillates highly, what is the effective coefficient \mathbf{A}_∞ ?

H-convergence (classical; Murat–Tartar, 1970s) 1/2

For $0 < \alpha < \beta$, $\Omega \subseteq \mathbb{R}^d$ open and bounded, we define the set of **admissible coefficients**:

$$\mathcal{M}(\alpha, \beta; \Omega) := \left\{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}^{d \times d}) : \mathbf{A}(x)\xi \cdot \xi \geq \alpha|\xi|^2, \mathbf{A}^{-1}(x)\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \right\}.$$

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H-convergence (on $\mathcal{M}(\alpha, \beta; \Omega)$)

(\mathbf{A}_n) **H-converge** to \mathbf{A} ($\mathbf{A}_n \xrightarrow{H} \mathbf{A}$) if for all $f \in H^{-1}(\Omega)$

$$u_n \xrightarrow{H_0^1(\Omega)} u \quad \text{and} \quad \mathbf{A}_n \nabla u_n \xrightarrow{L^2(\Omega; \mathbb{R}^d)} \mathbf{A} \nabla u,$$

where

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$$-\operatorname{div}(\mathbf{A}_n \nabla u_n) = f \quad \text{and} \quad -\operatorname{div}(\mathbf{A} \nabla u) = f.$$

Key properties:

- **Metrisability and compactness:** $(\mathcal{M}(\alpha, \beta; \Omega), \tau_H)$ is compact and metrisable.
- Works for **non-symmetric** coefficients and **extends** G -convergence.

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The H-limit is non-trivial: For $d = 1$ we have

$$A_n \xrightarrow{H} A \iff \frac{1}{A_n} \xrightarrow{*} \frac{1}{A} \text{ in } L^\infty(\Omega).$$

The H-limit is the *harmonic mean*.

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$-\frac{d}{dx}(A_n u'_n) = f$ implies $A_n u'_n$ converges *strongly* in L^2 (the 1D advantage)

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Limitation: Each **new** class of PDEs **requires** developing a separate theory.

Fractional divergence form problem

New equation: fractional divergence form

$$\begin{cases} -\operatorname{div}^s(\mathbf{A} \nabla^s u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

Nonlocal operators ($s \in (0, 1)$):

- $\nabla^s, \operatorname{div}^s$ – Riesz fractional gradient and divergent
 \rightsquigarrow a little bit more on this in a moment, but much more was in S14.02
- Natural function spaces: $H_0^s(\Omega)$ and its dual $H^{-s}(\Omega)$
- Well-posedness in terms of weak solutions

$$\int_{\mathbb{R}^d} \mathbf{A}(x) \nabla^s u(x) \cdot \nabla^s v(x) dx =_{H^{-s}(\Omega)} \langle f, v \rangle_{H_0^s(\Omega)}, \quad v \in H_0^s(\Omega).$$

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Remark:

Although the support of u is in Ω , $\nabla^s u$ is not supported in Ω since ∇^s is a nonlocal operator. Thus, \mathbf{A} should be defined on the whole \mathbb{R}^d .

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M. Caponi, A. Carbotti, and A. Maione. *H-compactness for nonlocal linear operators in fractional divergence form*, Calc. Var. Partial Differential Equations, **64**.290 (2025).

H^s -convergence: characterisation

Caponi et al (2025)

For some $\mathbf{A}_0 \in \mathcal{M}(\alpha, \beta; \mathbb{R}^d)$ we define

$$\mathcal{M}(\alpha, \beta; \Omega, \mathbf{A}_0) := \{\mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbb{R}^d) : \mathbf{A}|_{\mathbb{R}^d \setminus \Omega} = \mathbf{A}_0|_{\mathbb{R}^d \setminus \Omega}\}.$$

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Our contribution:

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Consequences:

- **Compactness** and **metrisability**

$$d(A, B) = d_H(A|_{\Omega}, B|_{\Omega}) + d_*(A|_{\mathbb{R}^d \setminus \Omega}, B|_{\mathbb{R}^d \setminus \Omega})$$

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Consequences:

- Compactness and metrisability
- H^s -convergence is local

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Consequences:

- **Compactness** and **metrisability**
- H^s -convergence is *local*
- $d = 1$: The H^s -limit is the *harmonic mean* on Ω and the *arithmetic mean* on the complement.

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- H^s -convergence is *local*
- $d = 1$: The H^s -limit is the *harmonic mean* on Ω and the *arithmetic mean* on the complement. ($D^s(AD^s u) = 0 \not\Rightarrow AD^s u = \text{const.}$)

Fractional (nonlocal) differential operators in short

Fractional Differential Operators

Let $s \in (0, 1)$, $\mu_s = 2^s \pi^{-d/2} \Gamma\left(\frac{d+s+1}{2}\right) \Gamma\left(\frac{1-s}{2}\right)^{-1}$.

Riesz fractional gradient & divergence

$$\nabla^s \varphi(x) = \mu_s \int_{\mathbb{R}^d} \frac{(\varphi(x) - \varphi(y))(x - y)}{|x - y|^{d+s+1}} dy, \quad \varphi \in C_c^\infty(\mathbb{R}^d),$$

and div^s is defined analogously via the integration-by-parts formula.

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Key properties:

- **Canonical:** unique (up to scalar) translation/rotation-invariant, s -homogeneous operator $\mathcal{T} \rightarrow \mathcal{T}^d$.
- **Fourier symbols:** $\widehat{\nabla^s f}(\xi) = i\xi|\xi|^{s-1}\hat{f}(\xi)$.
- **Relation to fractional Laplacian:** $-\operatorname{div}^s(\nabla^s F) = (-\Delta)^s F$.



M. Šilhavý. *Fractional vector analysis based on invariance requirements (critique of coordinate approaches)*, Continuum Mech. Thermodyn. **32** (2019) 207–228.



T. Shieh and D. Spector. *On a new class of fractional partial differential equations*, Adv. Calc. Var **8** (2015) 321–336.

Fractional Sobolev Spaces

For $s \in (0, 1)$:

$$\|u\|_s^2 := \|u\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla^s u\|_{L^2(\mathbb{R}^d)^d}^2, \quad H^s(\mathbb{R}^d) = \text{Cl}_{\|\cdot\|_s} C_c^\infty(\mathbb{R}^d).$$

For open $\Omega \subseteq \mathbb{R}^d$:

$$H_0^s(\Omega) := \text{Cl}_{\|\cdot\|_s} C_c^\infty(\Omega) \subseteq H^s(\mathbb{R}^d), \quad H^{-s}(\Omega) := (H_0^s(\Omega))'.$$

Essential tools (valid for $d \geq 1$):

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① $I_{1-s} : H_0^s(\Omega) \rightarrow H_{\text{loc}}^1(\mathbb{R}^d)$ is linear, cont. and $\nabla(I_{1-s}u) = \nabla^s u$ a.e. in Ω .

I_α is the Riesz potential: $I_\alpha f(x) := \frac{\mu_{1-\alpha}}{d-\alpha} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy$.

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- 2 **Poincaré inequality:** $\|u\|_{L^2(\Omega)} \lesssim \|\nabla^s u\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}$, $u \in H_0^s(\Omega)$.

$$\|u\|_{L^2(\Omega)}^2 \lesssim [u]_{s, \mathbb{R}^d}^2 \approx \langle u, (-\Delta)^s u \rangle = \langle u, \text{div}^s \nabla^s u \rangle = \|\nabla^s u\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}^2.$$

Brasco et al. *The fractional Cheeger problem*, *Interfaces Free Bound.* **16.3** (2014).

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- 3 **Rellich–Kondrašov:** $H_0^s(\Omega) \xhookrightarrow{c} L^2(\Omega)$.

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- 2 **Poincaré inequality:** $\|u\|_{L^2(\Omega)} \lesssim \|\nabla^s u\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}$, $u \in H_0^s(\Omega)$.
- 3 **Rellich–Kondrašov:** $H_0^s(\Omega) \xrightarrow{c} L^2(\Omega)$.
- 4 **Compactness on the complement:** for $\Omega \Subset \Omega_1$, the map $H_0^s(\Omega) \ni u \mapsto \nabla^s u|_{\Omega_1^c} \in L^2(\Omega_1^c; \mathbb{R}^d)$ is compact.¹

¹C. Kreisbeck and H. Schönberger. *Quasiconvexity in the fractional calculus of variations: characterization of lower semicontinuity and relaxation*, *Nonlinear Analysis* **215** (2022).

Back to the main result

Key Ideas in the Proof

H^s -convergence (on $\mathcal{M}(\alpha, \beta; \mathbb{R}^d)$)

(\mathbf{A}_n) H^s -converge to \mathbf{A} ($\mathbf{A}_n \xrightarrow{H^s} \mathbf{A}$) if for all $f \in H^{-s}(\Omega)$

$$u_n \xrightarrow{H_0^s(\Omega)} u \quad \text{and} \quad \mathbf{A}_n \nabla^s u_n \xrightarrow{L^2(\mathbb{R}^d; \mathbb{R}^d)} \mathbf{A} \nabla u,$$

where

$$-\operatorname{div}^s(\mathbf{A}_n \nabla^s u_n) = f \quad \text{and} \quad -\operatorname{div}^s(\mathbf{A} \nabla^s u) = f.$$

$$\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \iff \mathbf{A}_n|_{\Omega} \xrightarrow{H} \mathbf{A}|_{\Omega} \quad \text{and} \quad \mathbf{A}_n|_{\mathbb{R}^d \setminus \Omega} \xrightarrow{*} \mathbf{A}|_{\mathbb{R}^d \setminus \Omega}$$

Key Ideas in the Proof

- 1 $I_{1-s} : H_0^s(\Omega) \rightarrow H_{\text{loc}}^1(\mathbb{R}^d)$ is linear, cont. and $\nabla(I_{1-s}u) = \nabla^s u$ a.e. in Ω .
- 2 **Poincaré inequality:** $\|u\|_{L^2(\Omega)} \lesssim \|\nabla^s u\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}$, $u \in H_0^s(\Omega)$.
- 3 **Rellich–Kondrašov:** $H_0^s(\Omega) \xhookrightarrow{c} L^2(\Omega)$.
- 4 **Compactness on the complement:** for $\Omega \Subset \Omega_1$, the map $H_0^s(\Omega) \ni u \mapsto \nabla^s u|_{\Omega_1^c} \in L^2(\Omega_1^c; \mathbb{R}^d)$ is *compact*.

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Idea of the proof.

- $\mathbf{A}_n \nabla^s u_n \longrightarrow \mathbf{A} \nabla^s u$ in $L^2(\Omega^c)$ due to (4).
- The analysis of $\mathbf{A}_n \nabla^s u_n$ on Ω uses that $\mathbf{A}_n|_{\Omega} \xrightarrow{H} \mathbf{A}|_{\Omega}$, and hinges on (1) (enabling the use of classical and fractional div-curl lemmas). The same idea is present in (Caponi et al, 2025).

Nonlocal H -Convergence (Waurick, 2018/2025)

Motivation: Extend from L^∞ coefficients to operators $a \in \mathcal{L}(L^2(\Omega)^d)$.

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① Let $\mathcal{H} := L^2(\Omega)^d$, $\mathcal{H}_0 \subseteq \mathcal{H}$ closed, $\mathcal{H}_1 = \mathcal{H}_0^\perp$.

Decompose $a \in \mathcal{L}(\mathcal{H})$ as $a = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$.

$\mathfrak{M}(\mathcal{H}_0, \mathcal{H}_1) \dots a \in \mathcal{L}(\mathcal{H})$ such that $a_{00}^{-1} \in \mathcal{L}(\mathcal{H}_0)$ and $a^{-1} \in \mathcal{L}(\mathcal{H})$.

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- ③ $\mathcal{M}(\alpha, \beta; \mathbb{R}^d) \subseteq \mathfrak{M}(\text{ran}(\nabla^s_\Omega), \text{ran}(\nabla^s_\Omega)^\perp)$, where ∇^s_Ω denotes ∇^s restricted on $H_0^s(\Omega)$.

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\implies H^s -convergence is a **special case** of nonlocal H -convergence when restricted to local (multiplicative) operators.

ApplMath²⁶

13th Conference on Applied
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