

Regularity results for equations of ultra-parabolic type

by velocity averaging

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Larger Sobolev gains, with distinct hyperbolic and diffusive orders

Comparison for deterministic bounded kinetic solutions at the same integrability $1 < r < 9/5$

Configuration	Earlier EKM [4] Isotropic order	Present: (t, x') Hyperbolic order	Present: x'' Diffusive order
Exponents $L = N = 2$ $d = 2, \alpha = 1/2$	$s < \frac{1}{75}$	$s_h < \frac{1}{9}$	$s_p < \frac{2}{9}$
Rotating rank-one diffusion $d = 3, \alpha = 1/2$	$s < \frac{1}{87}$	$s_h < \frac{1}{9}$	$s_p < \frac{2}{9}$

Exact benchmark. In [4], Theorem 3.6 and Section 4, $s_{\text{EKM}}^* = \alpha/[3\alpha + 6(d+4)]$, where d is spatial dimension. EKM allows $r < 25/13$ and $r < 29/15$ in the two rows.

The table compares the displayed theorems under their common hypotheses: bounded kinetic solutions, spatially global integrability and finite defect measure.

What earlier results provide

Tadmor–Tao [1], Corollary 4.5

The complementary-power corollary assumes $N \geq 2L$. The case $L = N = 2$ lies outside that corollary.

Erceg–Mitrović [3], Corollary 21

Already removes that restriction for integer exponents and gives isotropic regularity. The present advance is the stronger, **distinct hyperbolic and diffusive orders**.

Nariyoshi [2], Theorems 2.1–2.4

Gives relative compactness under qualitative nondegeneracy with critical sources. The present quantitative condition yields **explicit positive Sobolev orders**.

Model and quantitative hypothesis

For $y = (t, x', x'') \in \mathbb{R}^{1+k+m}$, $k \geq 0$, $m \geq 1$, put

$$L_0 = \partial_t + f_h(\lambda) \cdot \nabla_{x'} - \sum_{i,j=1}^m a_{ij}(\lambda) \partial_{x_i'' x_j''},$$

where $a(\lambda) = a^T(\lambda) \geq 0$, $f_h, a \in C^{\max\{1, \kappa\}}(I)$ and $\kappa \in \mathbb{N}_0$. Let

$$\mathcal{P} = \{|\zeta_h|^2 + |\zeta_p|^4 = 1\},$$
$$\ell = i(\zeta_0 + f_h(\lambda) \cdot \zeta') + \zeta''^T a(\lambda) \zeta''.$$

Assume, for $0 < \delta \leq 1$ and $0 < \alpha \leq 1$,

$$\sup_{\zeta \in \mathcal{P}} |\{\lambda \in I : |\ell(\zeta, \lambda)| \leq \delta\}| \leq C_I \delta^\alpha.$$

Multiplier input

Uniform truncation [1,2] removes the cutoff-multiplier loss. Anisotropic frequency decomposition and interpolation give the new exponents.

The new directional estimates

For $1 < p < \infty$ and $D = 1 + k + m$, define

$$A_p = \int_I \rho h \, d\lambda, \quad \beta_p = \alpha \min\{1/p, 1/p'\}, \quad p' = \frac{p}{p-1},$$

where $\rho \in C_c^\infty(I)$.

Integrable source

If $L_0 h = \partial_\lambda^\kappa g$ and $h, g \in L^p(\mathbb{R}^D \times I)$, then

$$\|A_\rho\|_{\mathcal{H}_A^{s,p}} \lesssim_s \|h\|_p + \|g\|_p, \quad 0 < s < \frac{\beta_p}{\kappa + 1}.$$

Finite measure source

If $L_0 h = \partial_\lambda^\kappa \mu$, $h \in L^p(\mathbb{R}^D \times I)$ and $\mu \in \mathcal{M}(\mathbb{R}^D \times I)$, put

$$\theta_* = \frac{\beta_p}{\kappa + 1 + \beta_p}, \quad \frac{1}{r_*} = \frac{1 - \theta_*}{p} + \theta_*.$$

Then $A_\rho \in \mathcal{H}_{A, \text{loc}}^{s,r}$ for $0 < s < \theta_*$ and $1 < r < r_*$, giving simultaneously

$$s_h < \theta_* \quad \text{in } (t, x'), \quad s_p < 2\theta_* \quad \text{in } x''.$$

For fixed α, p, κ , the exponents have **no dimension loss**. Constants may depend on dimension.

Why the directions have different orders

The norm uses the principal scaling:

$$\|v\|_{\mathcal{H}_A^{s,r}} = \left\| \mathcal{F}^{-1} \left[(1 + |\eta_h|^2 + |\eta_p|^4)^{s/2} \tilde{v} \right] \right\|_r.$$

This controls order s in hyperbolic variables and order $2s$ in the diffusive variables x'' .

The examples in the comparison

Complementary convection and diffusion

$$\partial_t h + \lambda^L \partial_{x'} h - |\lambda|^N \partial_{x''}^2 h = \partial_\lambda \mu.$$

For integer $L \geq 1$ and real $N \geq 2$, set $M_0 = \max\{L, N\}$. Then $\alpha = 1/M_0$. For $h \in L^2$ and finite μ ,

$$s_h < \frac{1}{4M_0 + 1}, \quad s_p < \frac{2}{4M_0 + 1}, \quad 1 < r < \frac{4M_0 + 1}{2M_0 + 1}.$$

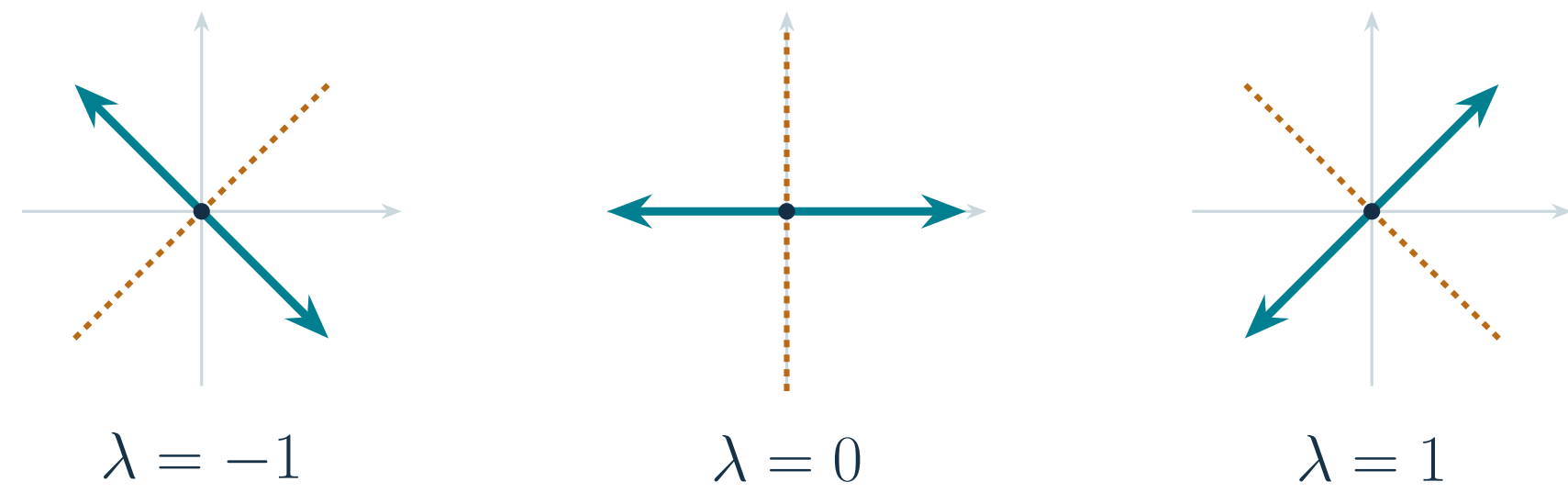
For $L = N = 2$, this gives $s_h < 1/9$, $s_p < 2/9$ and $1 < r < 9/5$, as in the table.

Rank one with a rotating kernel

Take $k = 1$, $m = 2$ and

$$f_h(\lambda) = \lambda, \quad a(\lambda) = \begin{pmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{pmatrix}.$$

The kernel span $\{(-\lambda, 1)\}$ rotates with λ .



Solid teal: diffusion range. Dashed amber: kernel.

Here $\alpha = 1/2$. **Both parabolic coordinates gain the order $< 2/9$ for a finite measure source and $h \in L^2$.**