

Homogenisation of a heterogeneous Boltzmann equation in the L^p setting

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Boltzmann equation

$$\partial_t u + \lambda \cdot \nabla_x u = Q(u, u), \quad (t, x, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_\lambda^3.$$

Describes how the distribution of particles of dilute gas in position–velocity space changes with time.

Gain–loss structure

$$Q(u, u) = Q^+(u, u) - Q^-(u, u) = Q^+(u, u) - uL[u].$$

Equivalently,

$$\partial_t u + \lambda \cdot \nabla_x u + L[u]u = Q^+(u, u).$$

- $\lambda \cdot \nabla_x u$: free transport / drift.
- $L[u]u$: loss term, interpreted as opacity.
- $Q^+(u, u)$: gain term from collisions producing particles with vel. λ .

Main difficulty

Collision operator Q is nonlinear, which greatly complicates the theory of existence and uniqueness of the equation under various initial conditions.

DiPerna–Lions existence theory

Reference: DiPerna–Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. of Math. 130 (1989).

Assumptions on initial data

$$u_0 \geq 0, \quad \iint u_0 (1 + |x|^2 + |\lambda|^2 + |\log u_0|) \, dx \, d\lambda < \infty.$$

Conclusion

There exists a global nonnegative *renormalized solution*

$$u \in C([0, \infty); L^1(\mathbb{R}_x^3 \times \mathbb{R}_\lambda^3)),$$

with

$$u(1 + |x|^2 + |\lambda|^2 + |\log u|) \in L_{\text{loc}}^\infty([0, \infty); L_{x,\lambda}^1).$$

Why homogenisation?

We consider particles moving through a material whose microscopic structure varies rapidly in space.

$$\text{small scale of the material} = \varepsilon, \quad 0 < \varepsilon \ll 1.$$

Physical picture

Imagine a particle beam passing through a composite material:

$$A|B|A|B|A|B|\dots$$

Different parts of the material interact with the particles differently, changing their direction or velocity at different rates.

Question

If the layers become extremely thin, can we replace this complicated medium by one effective homogeneous medium?

Homogenisation problem

For a sequence $\varepsilon_n \rightarrow 0$, consider the problems

$$L_n u_n = f_n,$$

where $f_n \rightarrow f$ in $W^{-1,p}$.

Goal

Find an effective operator $\tilde{L}$ such that

$$u_n \rightharpoonup u \quad \text{in } L^p$$

and the limit u solves

$$\tilde{L}u = f.$$

Interpretation

Instead of tracking every microscopic layer of the material, we want one macroscopic equation describing the averaged effect of the medium.

Linear Boltzmann equation

A typical linear transport model is

$$\partial_t u_\varepsilon + \lambda \cdot \nabla_x u_\varepsilon + \sigma_\varepsilon u_\varepsilon = K_\varepsilon u_\varepsilon.$$

where u_ε converges to u in some sense.

Main simplification

The collision operator is linear:

$$K_\varepsilon u_\varepsilon - \sigma_\varepsilon u_\varepsilon,$$

Under suitable assumptions on the coefficients and initial data, the linear Boltzmann equation is well posed in L^p , $1 \leq p < +\infty$.

Homogenisation of the linear Boltzmann equation is not a new topic. There are several papers by Bernard, Golse, Salvarani, Hutridurga etc. However, they usually use two-scale convergence in L^2 or L^∞ weak-* convergence.

Why $1 < p < 2$ is interesting

Many homogenisation papers for various reasons assume that Ω and $\mathcal{V}$ are bounded. Furthermore, they work with finite time interval $(0, T)$. With these assumptions for $1 < p < 2$ we have that

$$L^2((0, T) \times \Omega \times \mathcal{V}) \subseteq L^p((0, T) \times \Omega \times \mathcal{V}) \subseteq L^1((0, T) \times \Omega \times \mathcal{V}).$$

Interpretation

- L^1 : physically natural space, but nonreflexive.
- L^2 : Hilbert space structure, powerful microlocal and homogenisation tools.
- $L^p, 1 < p < 2$: reflexive, but closer to L^1 .

Motivation

A homogenisation theory in $L^p, 1 < p < 2$, would move the linear theory closer to the natural DiPerna–Lions L^1 framework.

Boltzmann-like equation - assumptions

Let $\Omega, \mathcal{V} \subseteq \mathbb{R}^d$ be nonempty domains such that $\mathcal{V}$ is bounded. Consider the following sequence of first order differential operators on $\Omega \times \mathcal{V}$

$$L_n = a(x, \lambda) \cdot \nabla_x + \sigma \left(x, \frac{x}{\varepsilon_n}, \lambda \right) \mathcal{K},$$

where

- $\varepsilon_n \longrightarrow 0$
- $\sigma \in C(\Omega \times \mathbb{T}^d \times \mathcal{V})$
- $a : \Omega \times \mathcal{V} \longrightarrow \mathbb{C}^d$ is a vector function with analytic components
- $\mathcal{K} : L^2(\mathcal{V}) \longrightarrow L^2(\mathcal{V})$ is a self-adjoint continuous linear operator

Theorem ($p = 2$)

Let $u_n \rightharpoonup u$ in $L^2(\Omega \times \mathcal{V})$ and $f_n \rightarrow f$ in $H_{\text{loc}}^{-1}(\Omega \times \mathcal{V})$ such that

$$L_n u_n = f_n.$$

If, for any $q \in \mathbb{Z}^d \setminus \{0\}$ such that $\hat{\sigma}(\cdot, q, \cdot)\mathcal{K} \not\equiv 0$, we have that $a \cdot q \not\equiv 0$ (i.e. the principal symbol of L_n for fixed q is not identically zero), then $\tilde{L}u = f$, where

$$\tilde{L} = a(x, \lambda) \cdot \nabla_x + \hat{\sigma}(x, 0, \lambda)\mathcal{K}.$$

Here $\hat{\sigma}(\cdot, q, \cdot)$ denotes the q -th Fourier coefficient of σ with respect to the second variable.

What needs to be proven?

Since

$$L_n u_n = f_n, \quad u_n \rightharpoonup u,$$

the transport term passes to the limit. Thus, it remains to prove

$$\int_{\Omega \times \mathcal{V}} \sigma \left(x, \frac{x}{\varepsilon_n}, \lambda \right) \mathcal{K} u_n \varphi \, dx \, d\lambda \longrightarrow \int_{\Omega \times \mathcal{V}} \hat{\sigma}(x, 0, \lambda) \mathcal{K} u \varphi \, dx \, d\lambda ,$$

for all $\varphi \in \mathcal{D}(\Omega \times \mathcal{V})$.

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Main difficulty

Both factors contain weakly converging sequences, so weak convergence alone is not sufficient to pass to the limit in their product.

Sketch of the proof

The first step is to approximate the λ -dependence of the opacity by

$$\sigma^N\left(x, \frac{x}{\varepsilon_n}, \lambda\right) := \sum_j \sigma_j\left(x, \frac{x}{\varepsilon_n}\right) \chi_{K_j}(\lambda).$$

Here $\{K_j : j \in A(N)\}$ is a finite partition of $\mathcal{V}$ consisting of measurable sets of diameter $\leq \frac{1}{N}$, and

$$\sigma_j(x, y) = \sigma(x, y, \lambda_j), \quad \lambda_j \in K_j^N.$$

By density, we may assume that

$$\varphi = \varphi_1 \boxtimes \varphi_2 \in \mathcal{D}(\Omega) \boxtimes \mathcal{D}(\mathcal{V}).$$

Sketch of the proof (cont.)

For the approximating opacity σ^N , we have the exact identity

$$\begin{aligned} \int_{\Omega \times V} \sigma^N \left(x, \frac{x}{\varepsilon_n}, \lambda \right) \mathcal{K} u_n \varphi \, dx \, d\lambda \\ = \sum_{j \in A(N)} \int_{\Omega \times V} \sigma_j \left(x, \frac{x}{\varepsilon_n} \right) u_n \varphi_1 \mathcal{K}(\chi_{K_j^N} \varphi_2) \, dx \, d\lambda. \end{aligned}$$

By density, it remains to prove that the sequences (u_n) and $(\sigma_j(x, \frac{x}{\varepsilon_n}) \mathcal{K}(\chi_j^N \varphi_2))$ are orthogonal for each j .

Therefore let μ be the H-measure defined by the sequence $(u_n - u)$ and ν be the H-measure defined by the sequence $((\sigma_j(x, \frac{x}{\varepsilon_n}) - \hat{\sigma}_j(x, 0)) \mathcal{K}(\chi_j^N \varphi_2))$.

Condition in the statement of the theorem implies that these two H-measures are mutually singular, which proves the statement (Gérard's result).

Theorem (Gérard)

Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $L^2_{\text{loc}}(\Omega)$. Furthermore, let μ and ν be H-measures defined by some subsequences of $(u_n - u)$ and $(v_n - v)$. If they are mutually singular ($\mu \perp \nu$), then on a subsequence we have that $u_n v_n \xrightarrow{*} uv$ in $\mathcal{D}'(\Omega)$.

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Theorem

$\mu \perp \nu$ if and only if for any $\varepsilon > 0$ there are open sets $U, V \subseteq \Omega$ such that $U \cup V = \Omega$ and

$$(\forall \varphi_1 \in C_c(\Omega))(\forall \varphi_2 \in C_c(\Omega)) \quad \text{supp } \varphi_1 \subseteq U \text{ \& \; } \text{supp } \varphi_2 \subseteq V \Rightarrow$$

$$|\langle \mu, \varphi_1 \rangle| \leq \varepsilon \|\varphi_1\|_{C_b(\Omega)},$$

$$|\langle \nu, \varphi_2 \rangle| \leq \varepsilon \|\varphi_2\|_{C_b(\Omega)}.$$

Sketch of proof (cont.)

More precisely, by the localisation principle,

$$\operatorname{supp} \mu \subset \{(x, \lambda, \xi, \eta) : a(x, \lambda) \cdot \xi = 0\}.$$

For arbitrary $\varepsilon > 0$ by defining

$$U_\varepsilon = \{(x, \lambda, \xi, \eta) \in S^*(\Omega \times \mathcal{V}) : |a(x, \lambda) \cdot \xi| > \varepsilon\} \text{ and}$$

$$V_\varepsilon = \{(x, \lambda, \xi, \eta) \in S^*(\Omega \times \mathcal{V}) : |a(x, \lambda) \cdot \xi| < 2\varepsilon\} \text{ we get the desired result.}$$

Theorem ($p = 2$)

Let $u_n \rightharpoonup u$ in $L^2(\Omega \times \mathcal{V})$ and $f_n \rightarrow f$ in $H_{\text{loc}}^{-1}(\Omega \times \mathcal{V})$ such that

$$L_n u_n = f_n.$$

$\tilde{L}u = f$ if and only if for any $q \in \mathbb{Z}^d \setminus \{0\}$ such that $\hat{\sigma}(\cdot, q, \cdot)\mathcal{K} \not\equiv 0$, we have that $a \cdot q \neq 0$.

Homogenisation of the linear Boltzmann equation ($p = 2$)

Theorem

Let $\sigma \in C(\Omega \times T^d \times \mathcal{V})$, $\kappa \in L^\infty(\mathcal{V} \times \mathcal{V})$. Suppose $u_n \rightharpoonup u$ in $L^2((0, T) \times \Omega \times \mathcal{V})$ such that

$$\partial_t u_n + \lambda \cdot \nabla_x u_n + \sigma\left(x, \frac{x}{\varepsilon_n}, \lambda\right) \left(u_n - \int_{\mathcal{V}} \kappa(v, \lambda) u_n(t, x, v) dv\right) = 0$$

for each $n \in \mathbb{N}$. Then

$$\partial_t u + \lambda \cdot \nabla_x u + \hat{\sigma}(x, 0, \lambda) \left(u - \int_{\mathcal{V}} \kappa(v, \lambda) u(t, x, v) dv\right) = 0.$$

$$1 < p < 2?$$

Ideally, one would take an H-distribution defined by some subsequences of $(u_n - u)$ and $\Phi_p(u_n - u)$, where

$$\Phi_p(z) = |z|^{p-2}z,$$

and an H-measure defined by some subsequence of $(v_n - v)$ and apply the following:

Conjecture

Let $u_n \rightharpoonup u$ in $L^p_{\text{loc}}(\Omega)$ and $v_n \rightharpoonup v$ in $L^q_{\text{loc}}(\Omega)$ for some $1 < p < 2$ and $q > p'$. Additionally, assume μ is an H-distribution defined by a pair of pure sequences $(u_n - u)$ and $(\Phi_p(u_n - u))$, while ν is the H-measure defined by pure sequence $(v_n - v)$, and that $\mu \perp \nu$.

Then $u_n \bar{v}_n \xrightarrow{*} u \bar{v}$ (vaguely) in the space of Radon measures $C_c(\Omega)'$ (and thus also weakly in the sense of distributions).

However, this does not work, since the conjecture is false.

Counterexample to the conjecture

Issue: the H-distribution generated by sequences

$$(u_n - u), (\Phi_p(u_n - u))$$

should be viewed as a cross-defect object rather than as a defect object associated with the sequence $(u_n - u)$ alone and therefore it need not encode all microlocal oscillations of $(u_n - u)$.

This loss of information can be seen explicitly in the periodic setting (trigonometric polynomials). Suppose

$$u_n(x) = U(nx), \quad \Phi_p(u_n)(x) = H(nx).$$

In this case the corresponding H-distribution μ is of the form

$$\mu = dx \otimes \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \widehat{U}(m) \overline{\widehat{H}(m)} \delta_{m/|m|}.$$

The Nemytskii map Φ_p may annihilate nonzero Fourier modes of $(u_n - u)$.

Thank you for your attention!