

Regularity of velocity averages in kinetic equations with heterogeneity

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Introduction

Kinetic equation with **heterogeneous** drift and a **singular** source:

$$\partial_t h + \operatorname{div}_x (f(x, \lambda) h) = \partial_\lambda^\kappa \gamma, \quad (t, x) \in \mathbb{R}^{1+d}, \quad \lambda \in \mathbb{R}^m.$$

Appears throughout kinetic theory (Boltzmann, Vlasov) and in the kinetic formulation of conservation laws.

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- **Heterogeneous:** the drift $\mathfrak{f} = \mathfrak{f}(x, \lambda)$ depends on space x , not only on velocity λ
- **Singular right-hand side:** γ is only a Radon measure, and we allow $\partial_\lambda^\kappa \gamma$ for a multi-index κ

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Goal: regularity, in **fractional Sobolev spaces**, of the velocity average

$$\langle h, \rho \rangle(t, x) = \int_{\mathbb{R}^m} h(t, x, \lambda) \rho(\lambda) d\lambda.$$

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Conservation laws

Scalar conservation law, entropy solution $u \in L^\infty(\mathbb{R}_+^{d+1})$:

$$\partial_t u + \operatorname{div}_x (f(u)) = 0.$$

Kinetic formulation. Set $h(t, x, \lambda) := \operatorname{sgn}(u(t, x) - \lambda)$. Then

$$\partial_t h + \operatorname{div}_x (f'(\lambda) h) = \partial_\lambda \zeta, \quad \zeta \geq 0 \text{ a Radon measure}$$

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For $M \geq \|u\|_{L^\infty}$,

$$\int_{\mathbb{R}} h(t, x, \lambda) \chi_{[-M, M]}(\lambda) d\lambda = 2 u(t, x).$$

A non-degeneracy condition

$$\partial_t h + \operatorname{div}_x (f(x, \lambda) h) = \partial_\lambda^\kappa \gamma, \quad (t, x) \in \mathbb{R}^{1+d}, \quad \lambda \in \mathbb{R}^m.$$

Example. $f(x, \lambda) = \lambda_0$, for a fixed λ_0 , and $\gamma = 0$. Then

$$\partial_t h + \lambda_0 \cdot \nabla_x h = 0.$$

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$\rightsquigarrow$ no gain in regularity

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A condition is needed to eliminate this example

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Non-degeneracy condition.

There exists $\alpha > 0$ such that for any $\delta > 0$, $K \subset\subset \mathbb{R}^d$, and $L \subset\subset \mathbb{R}^m$:

$$\operatorname{ess\,sup}_{x \in K} \sup_{(\tau, \xi_1, \dots, \xi_d) \in \mathbb{S}^d} \operatorname{meas} \left\{ \lambda \in L : |\tau + \xi \cdot \mathfrak{f}(x, \lambda)| < \delta \right\} \leq \delta^\alpha.$$

Homogeneous theory

$$\partial_t h + \mathfrak{f}(\lambda) \cdot \nabla_x h = \partial_\lambda^\kappa \gamma, \quad (t, x) \in \mathbb{R}^{1+d}, \quad \lambda \in \mathbb{R}^m.$$

Contributions:

Agoshkov (1984), Golse–Lions–Perthame–Sentis (1988), DiPerna–Lions–Meyer (1991), Bézard (1994), Lions–Perthame–Tadmor (1994), De Vore–Petrova (2001), Jabin–Perthame (2002), Westdickenberg (2002), Jabin–Vega (2004), Tadmor–Tao (2007), Arsénio–Masmoudi (2019), Jabin–Lin–Tadmor (2022), and many more.

Homogeneous theory

$$\partial_t h + f(\lambda) \cdot \nabla_x h = \partial_\lambda^\kappa \gamma, \quad (t, x) \in \mathbb{R}^{1+d}, \quad \lambda \in \mathbb{R}^m.$$

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- Results for $h \in L^p$, $p \in (1, \infty)$.
- For $d \geq 2$ the **maximal gain** of regularity is $\min\{\frac{1}{p}, 1 - \frac{1}{p}\}$.

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- Various generalisations: e.g. mixed integrability and regularity in space (x) and velocity (λ) .
- In general, obtained regularity (and integrability) **depends** on α .

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$$m = 1, \kappa = 1, \quad \sup_{(\tau, \xi_1, \dots, \xi_d) \in S^d} \text{meas} \left\{ \lambda \in L : |\tau + \xi \cdot \mathfrak{f}(\lambda)| < \delta \right\} \leq \delta^\alpha$$

$$\langle h, \rho \rangle \in W_{\text{loc}}^{s,p}, \quad p = \frac{4+\alpha}{2+\alpha}, \quad s < \frac{\alpha}{\alpha+2}$$

For $\alpha = 1$, the result is not optimal ($s < 1/3$).

Heterogeneous case: known so far

Compactness

- Proofs based on H-measures (microlocal defect measures): Tartar (1990), Gérard (1991).
- Robust: rough coefficients (a qualitative non-degeneracy condition).
- Compactness for any $p > 1$.
- No regularity rate.
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Regularity

- Gérard (1990), Gérard–Golse (1992).
- Genuine regularity estimates (a quantitative non-degeneracy cond.)
- L^2 only.
- Smooth coefficients in x .
- Source $\gamma \in H^{-\sigma}$, $\sigma < 1$ (Radon measures not covered)
 $\Rightarrow$ Out of reach: conservation laws, gas dynamics, and Boltzmann-type sources.

Main result

$$\partial_t h + \operatorname{div}_x (f(x, \lambda) h) = \partial_\lambda^\kappa \gamma, \quad (t, x) \in \mathbb{R}^{1+d}, \quad \lambda \in \mathbb{R}^m.$$

Assumptions

- $p > 1$ and $h \in L_{\text{loc}}^p(\mathbb{R}^{1+d+m})$.
- For a.e. $x \in \mathbb{R}^d$, $\lambda \mapsto f(x, \lambda)$ in $C^\kappa(\mathbb{R}^m; \mathbb{R}^d)$.
- $x \mapsto f(x, \lambda)$ in $L_{\text{loc}}^\infty(\mathbb{R}^m; W_{\text{loc}}^{\bar{s}, \bar{p}}(\mathbb{R}^d))$, for suitable $\bar{s}$ and $\bar{p}$.
 $\rightsquigarrow$ some discontinuities in the spatial variable are allowed
- $\gamma = \gamma(t, x\lambda)$ is a Radon measure.
- $\operatorname{ess\,sup}_{x \in K} \sup_{(\tau, \xi_1, \dots, \xi_d) \in S^d} \operatorname{meas} \left\{ \lambda \in L : |\tau + \xi \cdot f(x, \lambda)| < \delta \right\} \leq \delta^\alpha$

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There exists $r_0 \in (1, p)$ and a strictly positive, decreasing function $\beta_0 = \beta_0(r)$ such that

$$(\forall r \in (1, r_0)) (\forall \beta < \beta_0(r)) (\forall \rho \in C_c^\infty(\mathbb{R}^m)) \quad \langle h, \rho \rangle \in W_{\text{loc}}^{\beta, r}(\mathbb{R}^{1+d}).$$

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No claim of optimality: β_0 is well below the homogeneous benchmark $1/2$.

Application: Heterogeneous scalar conservation laws with nonlinear discontinuous flux and isentropic gas dynamics.

Idea of proof: “non-asymptotic blow-up”

Why the direct route fails. For $f = f(x, \lambda)$, the Fourier transform in x no longer separates h from the coefficient

$\rightsquigarrow$ would need **genuine** pseudodifferential calculus (this causes the limitation of the previous works)

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$\leadsto$ would need **genuine** pseudodifferential calculus (this causes the limitation of the previous works)

Our strategy

- ❶ **Localise** the kinetic equation around a fixed point y
 - ❷ Near y , the drift looks **homogeneous**: $\mathbf{f}_y(\lambda) := \mathbf{f}(y, \lambda)$
 - ❸ Apply homogeneous-type estimates there, then **integrate over** y
- error of “freezing” controlled by a new **commutator estimate** for Fourier multipliers
 - five error terms, several free parameters, balanced via the Marcinkiewicz multiplier theorem

Thank you for your attention!

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