



Microlocal defect measures and wave-front sets

Ivana Vojnović

University of Novi Sad

Based on joint work with Ivan Ivec

ivana.vojnovic@dmi.uns.ac.rs



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H-measures (microlocal defect measures)

In the analysis of nonlinear PDEs it is important to understand oscillation and concentration effects in sequences of weakly convergent L^p functions. By Vitali's convergence theorem if (f_n) is a sequence in $L^p(\Omega)$, for $|\Omega| < \infty$ and $1 \leq p < \infty$ with following assumptions:

- $\forall \varepsilon > 0 \exists \delta > 0$ such that $\int_A |f_n|^p < \varepsilon \quad \forall n$ and $\forall A \subset \Omega$ measurable and $|A| < \delta$ (**non-concentrating sequence**)
- $f_n \rightarrow f$ a.e. (**non-oscillating sequence**),

then $f \in L^p(\Omega)$ and $f_n \rightarrow f$ in $L^p(\Omega)$. Many tools have been developed to study oscillation and concentration problems, some of them are defect measures, Young measures, generalized Young measures, H-measures.

Example - H-measures

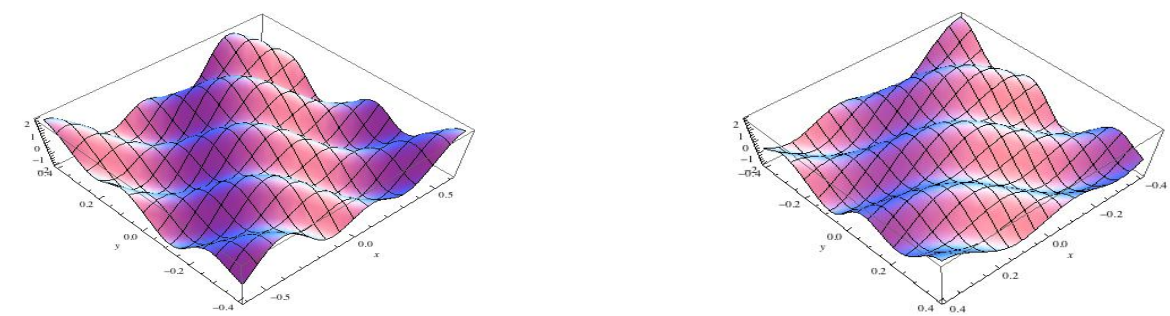
H-measures were introduced independently by Tartar and Gérard (under name microlocal defect measures) around 1990. Tartar called them H-measures because they appeared in connection with certain problems in homogenization. They are restricted to sequences converging weakly in L^2 (or L^2_{loc}).

H-measures depend on the variables x and ξ , and allow us to describe both the location and the directions of oscillations and concentrations of L^2 weakly convergent sequences.

The sequence $u_n = \cos(2n\pi(x - y)) + \cos(2n\pi(2x + 3y))$ has the H-measure

$$\mu = dx dy \otimes \left(\delta_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} + \delta_{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)} + \delta_{\left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)} + \delta_{\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)} \right).$$

It is clear from this example that the H-measure of given sequence contains information about direction of oscillations.



Existence theorem for H-measures

Scalar H-measure - definition (the existence theorem, [4])

Let $u_n, v_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$. Then there exist subsequences $u_{n'}, v_{n'}$ and Radon measure $\mu \in \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})$, such that $\forall \varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$, $\psi \in C(\mathbb{S}^{d-1})$,

$$\langle \mu, \varphi_1 \varphi_2 \psi \rangle := \lim_{n' \rightarrow \infty} L^2 \langle \mathcal{A}_\psi(\varphi_1 u_{n'}), \varphi_2 v_{n'} \rangle_{L^2} = \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'}) \overline{\mathcal{F}(\varphi_2 v_{n'})} \psi \left(\frac{\xi}{|\xi|} \right) d\xi.$$

- S^{d-1} is the unit sphere in $\mathbb{R}^d$
- $\mathcal{A}_\psi$ is the Fourier multiplier operator (bounded operator from L^p to L^p , $1 < p < \infty$) with symbol ψ : $(\mathcal{A}_\psi u)(x) := \mathcal{F}^{-1}(\psi(\xi)(\mathcal{F}u)(\xi))(x)$, $\mathcal{F}$ denotes the Fourier transform

If $u_n \rightarrow 0$ strongly, then $\mu = 0$. Conversely, if $\mu = 0$, then $u_n \rightarrow 0$ strongly in L^2_{loc} . The localization principle provides additional information about the support of the measure μ .

Localization principle for H-measures

Let $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, $A_i \in C_0(\mathbb{R}^d)$ and $\sum_{i=1}^d \partial_{x_i}(A_i(x)u_n(x)) = f_n(x) \xrightarrow{W_{\text{loc}}^{-1,2}} 0$.

Then the following relation holds:

$$P(x, \xi) \mu(x, \xi) = \sum_{j=1}^d A_j(x) \xi_j \mu(x, \xi) = 0.$$

This implies

$$\text{supp } \mu \subset \text{ch} P,$$

where $\text{ch} P = \{(x, \xi) : P_1(x, \xi) = 0\}$ is the characteristic variety of the first-order differential operator P .

Compactness results, spaces of symbols

Let us recall compactness results regarding subsets of $L^p(\mathbb{R}^d)$ spaces, $1 \leq p < \infty$. Since $L^p(\mathbb{R}^d)$ is a complete metric spaces, a subset K is compact if and only if it is closed and totally bounded. The following theorem holds.

Theorem 1 (Kolmogorov-Riesz) A subset $F \subseteq L^p(\mathbb{R}^d)$ is totally bounded if and only if the following properties hold:

1. F is bounded, that is there is some $C > 0$ such that $\int_{\mathbb{R}^d} |f(x)|^p dx < C$, $f \in F$,
2. for every $\varepsilon > 0$ there is some $r > 0$ such that $\int_{|x| > r} |f(x)|^p dx < \varepsilon^p$, $f \in F$,
3. for every $\varepsilon > 0$ there is some $\rho > 0$ such that $\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p dx < \varepsilon^p$, $f \in F$, $|y| < \rho$.

In the case of $L^2(\mathbb{R}^d)$ spaces the third condition can be replaced by

$$\lim_{r \rightarrow \infty} \sup_{f \in F} \int_{|\xi| > r} |\mathcal{F}f(\xi)|^2 d\xi = 0. \quad (1)$$

Condition (1) motivates the introduction of the microlocal compactness wave front set of a sequence (u_n) . Regarding pseudodifferential calculus, $S^m = S^m_{1,0}$ denotes the Hörmander class of symbols, consisting of all smooth $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ it holds

$$(\forall x \in \mathbb{R}^d)(\forall \xi \in \mathbb{R}^d) \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|},$$

where $C_{\alpha, \beta}$ is a constant depending only on α and β .

For every symbol a introduced above we denote the corresponding pseudodifferential operator T_a by

$$T_a \varphi(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi) \hat{\varphi}(\xi) d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Definition 2 A symbol $a_l \in S^l$ is said to be homogeneous of order l if for all $x \in \mathbb{R}^d$ it holds that

$$a_l(x, \lambda \xi) = \lambda^l a_l(x, \xi) \quad (2)$$

for all $\lambda \geq 1$ and $|\xi| \geq 1$.

Definition 3 We say that a symbol $a \in S^m_{cl}$ is a classical symbol of order m if $a \in S^m$ and admits an asymptotic expansion

$$a \sim \sum_{k=0}^{\infty} a_{m-k} \quad (3)$$

such that each a_{m-k} is homogeneous of order $m-k$ and $a - \sum_{k=0}^N a_{m-k} \in S^{m-N-1}$ for all $N \geq 0$.

Sometimes it is enough to consider a decomposition $a = a_m + r_{m-1}$, where a_m is homogeneous of order m and $r_{m-1} \in S^{m-1}$. We denote by Ψ^m the class of pseudodifferential operators whose symbols admit such a decomposition, and refer to them as **polyhomogeneous operators** of order m . The function a_m is called the principal symbol of the operator. Clearly, $CL^m \subseteq \Psi^m$, where CL^m denotes the class of pseudodifferential operators whose symbols belong to the class S^m_{cl} .

If $A \in \Psi^m$, we define its characteristic set by

$$\text{char} A = \{(x, \xi) \in \mathbb{R}^d \times S^{d-1} : a_m(x, \xi) = 0\}.$$

Microlocal compactness wave front sets

Definition 4 Let (u_n) be a bounded sequence in $W^{k,2}(\mathbb{R}^d)$. We say that $(x_0, \xi_0) \in \mathbb{R}^d \times S^{d-1}$ is not in the microlocal compactness wave front set of (u_n) of order k ($k \geq 0$, $k \in \mathbb{Z}$), denoted by $MCWF_k(u_n)$, if there is a conical neighbourhood Γ of ξ_0 and a $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x_0) \neq 0$ such that

$$\sup_n \int_{\xi \in \Gamma, |\xi| > r} |\mathcal{F}(\varphi \partial^\alpha u_n)(\xi)|^2 d\xi \rightarrow 0 \text{ as } r \rightarrow \infty \text{ for every } |\alpha| \leq k. \quad (4)$$

Remark 5 It is immediate that $MCWF_{k_1}(u_n) \subseteq MCWF_{k_2}(u_n)$ for $k_1 \leq k_2$. Furthermore, the following lemma implies that $MCWF_k(u_n)$ is a closed subset of $\mathbb{R}^d \times S^{d-1}$. For $k = 0$, we denote $MCWF_0(u_n) = MCWF(u_n)$.

Lemma 6 Let (u_n) be a bounded sequence in $W^{k,2}(\mathbb{R}^d)$. Then

$$MCWF_k(u_n) = \bigcap \text{char} A,$$

where A runs through all pseudodifferential operators $A \in CL^0$ such that $(A \partial^\alpha u_n)$ is relatively compact in $L^2(\mathbb{R}^d)$ for every $|\alpha| \leq k$. Also, φ from Definition 4 can be chosen independently of ξ_0 .

In the following theorem, we reveal the true meaning of $MCWF_k(u_n)$ as a measure of how far the sequence (u_n) is from being relatively compact.

Theorem 7 Let (u_n) be a bounded sequence in $W^{k,2}(\mathbb{R}^d)$. If $(\partial^\alpha u_n)$ is relatively compact in $L^2(\mathbb{R}^d)$ for every $|\alpha| \leq k$, then $MCWF_k(u_n) = \emptyset$. Conversely, if $MCWF_k(u_n) = \emptyset$, then $(\varphi \partial^\alpha u_n)$ is relatively compact in $L^2(\mathbb{R}^d)$ for every $|\alpha| \leq k$ and each $\varphi \in C_c^\infty(\mathbb{R}^d)$.

In the next result we give connection between $MCWF(u_n)$ and $\text{supp } \mu$, where μ is an H-measure associated with the sequence (u_n) which converges weakly to 0 in $L^2(\mathbb{R}^d)$.

Theorem 8 For any sequence (u_n) converging weakly to 0 in $L^2(\mathbb{R}^d)$ it holds that

$$MCWF(u_n) = \text{supp } \mu.$$

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