Computability of non-compact manifolds

Matea Čelar
University of Zagreb

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COMPUTABILITY OF COMPACT SETS

A compact set S in $\mathbb{R}^2$ is **computable** if it is empty or there is an algorithm which, on input $k \in \mathbb{N}$, outputs a finite set of rational points which approximate S with precision 2^{-k}.

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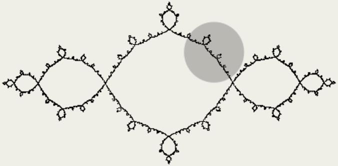

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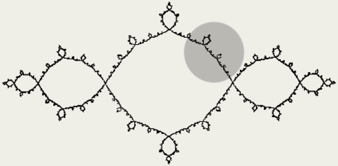
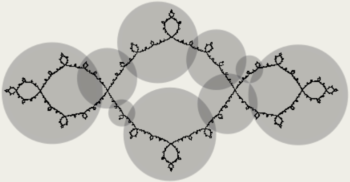


S is **semicomputable**: we can effectively list all finite unions of rational open balls which cover S



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(ii) we can effectively list all finite unions of rational open balls which cover $S \cap \hat{B}_i$, uniformly over all rational closed balls $\hat{B}_i$

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WHEN IS SEMICOMPUTABILITY ENOUGH?

Not all semicomputable sets are computably enumerable. In fact, there is a semicoputable set with no computable points.

However, semicomputability of a set can automatically imply computable enumerability, under some additional topological conditions.

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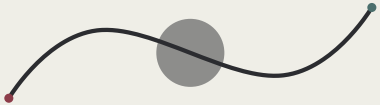

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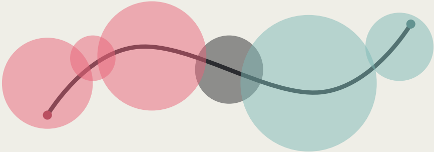

$$B_i \cap S \neq \emptyset \iff \begin{array}{c} \bigcup_u B_u \cup B_i \cup \bigcup_v B_v \text{ covers } S \\ a \in \bigcup_u B_u, \quad b \in \bigcup_v B_v \\ \bigcup_u B_u \text{ and } \bigcup_v B_v \text{ are formally disjoint} \end{array}$$

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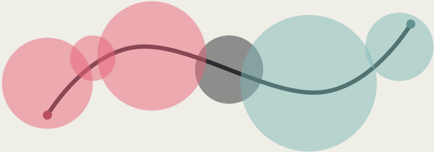

$$B_i \cap S \neq \emptyset \quad \Longleftrightarrow \quad \begin{array}{c} \bigcup_u B_u \cup B_i \cup \bigcup_v B_v \text{ covers } S \\ a \in \bigcup_u B_u, \quad b \in \bigcup_v B_v \\ \bigcup_u B_u \text{ and } \bigcup_v B_v \text{ are formally disjoint} \end{array}$$

In this case, computable enumerability is reducible to semicomputability!

Theorem 1 (Burnik and Iljazović 2014)

Let (X, d, α) be a computable metric space. Let M be a 1-manifold with boundary in this space such that M has finitely many components. Suppose M and ∂M are semicomputable. Then M is computable.

K. Burnik and Z. Iljazović (2014). "Computability of 1-manifolds". In: Logical Methods in Computer Science 10.2:8, pp. 1–28.

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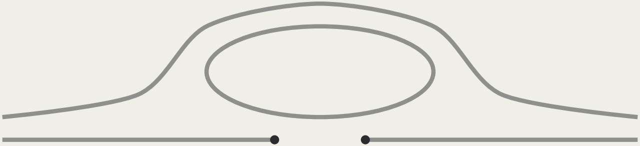

Theorem 2 (Iljazović 2020)

Let (X, d, α) be a computable metric space and let S be a semicomputable set in this space. Suppose S, as a subspace of (X, d), is a generalized graph such that the set E of all endpoints of S is semicomputable in (X, d, α). Then S is computable in (X, d, α).

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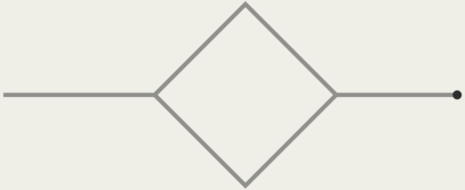

Theorem 3 (Iljazović and Sušić 2018)

Let (X, d, α) be a computable metric space and let K be a manifold with boundary in this space. Suppose K and ∂K are semicomputable. Then K is computable if there exists an open set U in K such that $\overline{U}$ is compact in K and $K \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0,1)$ or $\mathbb{H}^n$.

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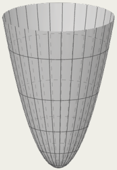
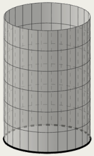
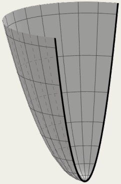

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$$I_i \cap S \neq \emptyset \quad \iff \quad \begin{array}{c} \mathcal{H}_\ell \text{ is a formal chain} \\ S \cap \hat{B}(a,n) \subseteq \bigcup \mathcal{H}_\ell^{p \leq q} \\ S \cap \hat{B}(a,m) \subseteq \bigcup \mathcal{H}_\ell \\ \mathcal{H}_\ell^{p \leq q} \subseteq_F B(a,m) \\ p < e < q < \overline{\ell} \wedge m > 1 \wedge p \leq w \leq q \\ \text{fmesh}(\ell) < 2^{-(k_0+k+3)} \\ I_A \text{ and } \mathcal{H}_\ell^{e \leq \overline{\ell}} \text{ are formally disjoint} \\ I_B \text{ and } \mathcal{H}_\ell^{0 \leq e} \text{ are formally disjoint} \\ J_{(\ell)_w} \subseteq_F I_i \end{array}$$

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Is there a better way to deal with non-compact sets?

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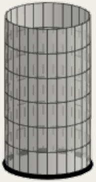

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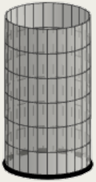

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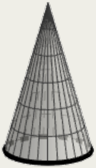

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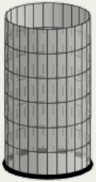

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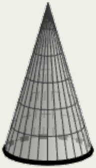

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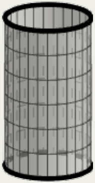

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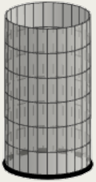

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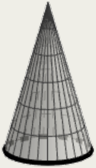

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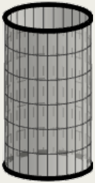

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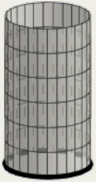

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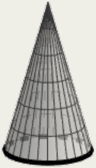

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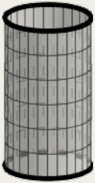

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What about computability?

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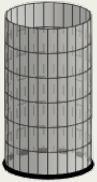

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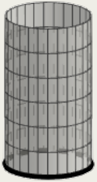

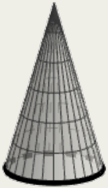

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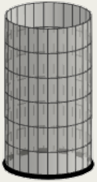

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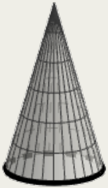

We can define a structure of a computable topological space on the compactification.

We call the newly defined computable topological space a **pseudocompactification** of the original computable metric space.

Theorem 4 (Iljazović and Sušić 2018)

Let (X, d, α) be a computable metric space and let $(Y, \mathcal{S}, (I_i))$ be its pseudocompactification. Let K be a semicomputable set in (X, d, α). Suppose the metric space (X, d) is unbounded.

(i) If K is compact in (X, d), then K is semicomputable in $(Y, \mathcal{S}, (I_i))$.

(ii) If K is not compact in (X, d), then $K \cup \{\infty\}$ is semicomputable in $(Y, \mathcal{S}, (I_i))$.

Theorem 5 (Iljazović and Sušić 2018)

Let (X, d, α) be a computable metric space and let $(Y, \mathcal{S}, (I_i))$ be its pseudocompactification. Suppose $K \subseteq X$ is such that $K \cup \{\infty\}$ is a c.e. set in $(Y, \mathcal{S}, (I_i))$. Then K is c.e. in (X, d, α).

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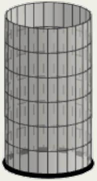

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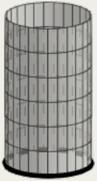

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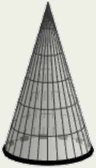

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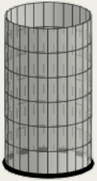

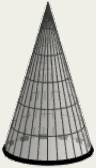

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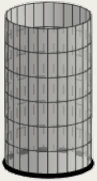

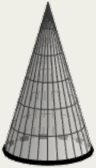

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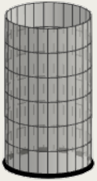

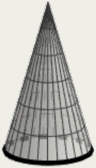

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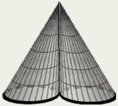

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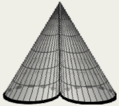
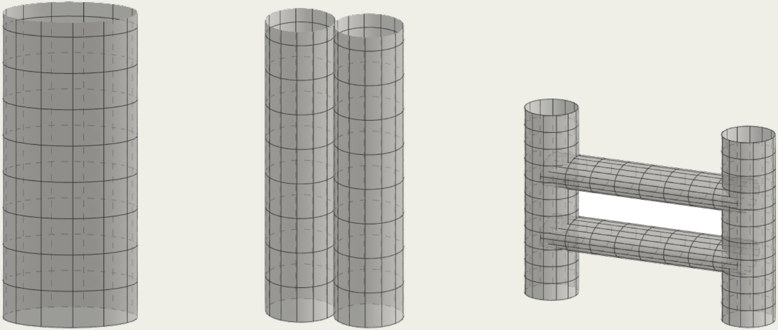


This allows us to conclude the same about a more general class of non-compact manifolds.

EXAMPLES



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Thank you!