

Computable inner approximation of topological graphs

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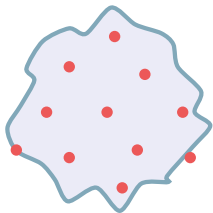


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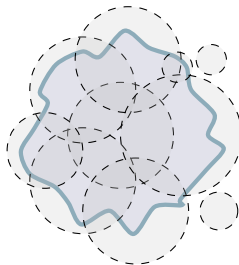
Computable set

$k \rightarrow \{p_1, \dots, p_{n_k}\}$ computable
 $d_H(S, \{p_1, \dots, p_{n_k}\}) < 2^{-n}$



Semicomputable set

$k \rightarrow \{(p_1, r_1), \dots, (p_{n_k}, r_{n_k})\}$ computable
 $S \subseteq B(p_1, r_1) \cup \dots \cup B(p_{n_k}, r_{n_k})$



Computability in metric spaces

- Computable metric space (X, d, α) : metric space $(X, d) + \{\alpha_i\}_{i \in \mathbb{N}}$ dense subset such that $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is a computable function
- Basic open balls $B(\alpha_i, q)$, $q \in \mathbb{Q}^+$

A compact set S in (X, d, α) is

- **semicomputable** if $\{(i_1, q_1, \dots, i_n, q_n) : S \subseteq B(\alpha_{i_1}, q_1) \cup \dots \cup B(\alpha_{i_n}, q_n)\}$ is computably enumerable (as a subset of $(\mathbb{N} \times \mathbb{Q}^+)^*$);
- **computable** if there is a computable function $k \mapsto (i_1, i_2, \dots, i_{n_k})$ such that $d_H(S, \{\alpha_{i_1}, \dots, \alpha_{i_{n_k}}\}) < 2^{-k}$.

Computable points in (semi)computable sets

- $x \in X$ is a computable point if there is a computable function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $d(x, \alpha_{\varphi(k)}) < 2^{-k}$, $\forall k \in \mathbb{N}$
- Computable points are **dense in computable sets**
- There are semicomputable sets with **no computable points**
- Nonempty convex semicomputable sets in $\mathbb{R}^n$ (Le Roux and Ziegler, 2008) and in finite-dimensional computable Banach spaces (Neumann, 2015) contain computable points
- Computable points are dense in semicomputable manifolds and polyhedra (Iljazović and Validžić, 2017)

An “almost computable” line segment

Let $H \subseteq \mathbb{N}$ be the Halting set and let $\gamma = \sum_{n \in H} 2^{-n}$. γ is a non-computable point in $\mathbb{R}$. The line segment $S = [\gamma, 1]$ is semicomputable, but not computable.



Computable points are dense in S .

Moreover, for each $\varepsilon > 0$, we can find a computable subarc $C = [c, 1]$ of S such that $d_H(S, C) < \varepsilon$.

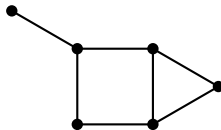
Inner approximation

Let $\mathcal{A}$ be a collection of compact sets. A compact set B is **inner approximated** by $\mathcal{A}$ if, for any $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $A \subseteq B$ and $d_H(A, B) < \varepsilon$.

- Semicomputable decomposable chainable continua are inner approximated by computable continua chainable between computable points (Iljazović, 2009)
- There is a semicomputable dendrite which is not inner approximated by computable dendrites; there is a semicomputable copy of the Cantor fan which contains no computable points (Kihara, 2012)

Topological graphs

- We consider **topological graphs**: spaces homeomorphic to finite unions of line segments intersecting only in common endpoints



- All but finitely many points on a topological graph have neighbourhoods homeomorphic to $\mathbb{R}$
- Points with neighbourhoods homeomorphic to $\mathbb{R}$ also have computable neighbourhoods (Iljazović and Validžić, 2017)

Computable arcs in semicomputable sets

Theorem 1 (Čačić, Č, Horvat, and Iljazović, 2026)

Suppose an open neighbourhood N of a point x in a semicomputable set S is homeomorphic to $\mathbb{R}$. Then there exist computable points $a, b \in N$ and a computable neighbourhood $N' \subseteq N$ of x in S which is an arc from a to b .

Proof. Technical; the computable arc is constructed as a nested intersection of chains consisting of finite unions of basic open balls. □

This homeomorphism is generally not computable!

More in: V. Čačić, M. Čelar, M. Horvat, and Z. Iljazović (2026). "Computable Approximations of Semicomputable Graphs". In: *Logical Methods in Computer Science* 22, 28 (1)

Inner approximation by arcs

- Theorem 1 guarantees there are arbitrarily small computable subarcs around any non-vertex point of a topological graph
- Finite unions of computable sets are computable

⇒ **Corollary.** Semicomputable compact graphs are inner approximated by finite unions of computable arcs with computable endpoints.



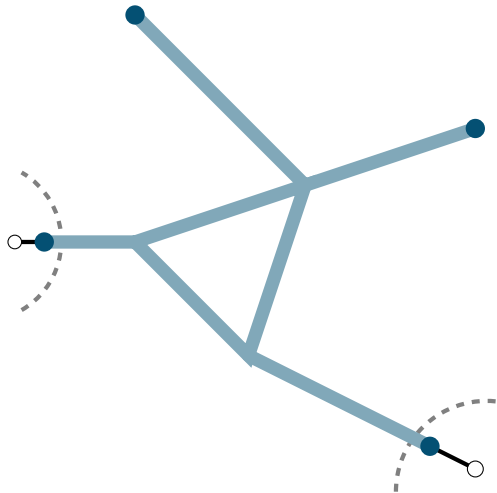
Inner approximation by homeomorphic copies

Theorem 2 (Čačić, Č, Horvat, and Iljazović, 2026)

Let S be a semicomputable topological graph in a computable metric space. Then S is inner approximated by computable graphs homeomorphic to S with computable endpoints.

Sketch of proof: “Cut out” basic open balls around non-computable endpoints, then add computable arcs to “patch” sections of edges which are no longer connected.

The resulting set is a semicomputable topological graph with computable endpoints, which is known to be computable (Iljazović, 2020). □

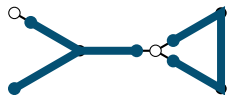
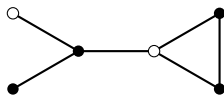


Non-compact graphs and 1-manifolds

- Definitions of semicomputable and computable sets can be generalized to also apply to certain non-compact sets
 - Using these generalized definitions, Theorem 2 also holds for closed “non-compact topological graphs”, i.e. finite unions of arcs and rays intersecting only in common endpoints
- ⇒ **Corollary.** Semicomputable 1-manifolds with finitely many components are inner approximated by their computable homeomorphic copies with computable endpoints.

Computable vertices?

- The approximating computable graphs can share all computable endpoints with the original graph and differ only around non-computable endpoints
- It is not always possible to construct an approximating graph homeomorphic to the original one with all computable vertices
- It is possible to construct approximating graphs with all computable vertices, but topology will be preserved only around computable ones



Closing remarks

- Semicomputable sets generally do not need to contain any computable points, but certain topological properties can render them “almost computable” = inner approximable by computable sets
- Semicomputable graphs are inner approximated by computable graphs
- When constructing the approximating family of sets, there is a tradeoff between nice topological properties and nice effective properties

Thank you for your attention!

Questions? Comments?

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