

Computable subcontinua and effective decomposability of arcs and arc-like continua

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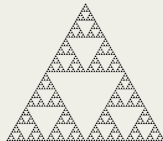
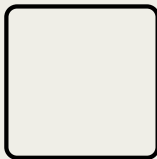


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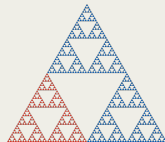
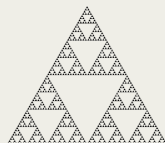
Continua

- **Continuum:** compact + connected metric space (with more than one point)



Decomposability

- **Decomposable** continuum: can be expressed as a union of two proper subcontinua (not disjoint!)

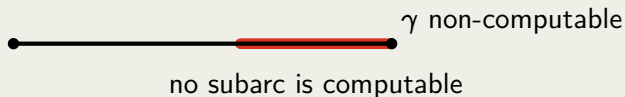


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More in: S.B. Nadler. *Continuum theory*. Marcel Dekker, Inc., New York, 1992; A. Illanes. *Continuum theory*. Springer Cham, 2025

Effective decomposability

- Computable set: can be effectively approximated by a finite set
- **Effectively decomposable** continuum: can be expressed as a union of two *computable* proper subcontinua
- Not all decomposable continua are effectively decomposable:



Question

When is a given (classically) decomposable computable effectively decomposable?

Computable metric space

- **Computable metric space:** (X, d, α) , where
 - (X, d) is a separable metric space
 - $\alpha : \mathbb{N} \rightarrow X$ is an enumeration of a dense subset
 - $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is a computable function
- $\{B_i\}_{i \in \mathbb{N}}$ effective enumeration of $\{B(\alpha_i, q) : i \in \mathbb{N}, q \in \mathbb{Q}^+\}$
- c.e. relations $(i, j \in \mathbb{N}, q, r \in \mathbb{Q}^+)$:
 - $d(\alpha_i, \alpha_j) > q + r \Rightarrow B(\alpha_i, r) \cap B(\alpha_j, q) = \emptyset$
 $B(\alpha_i, r)$ and $B(\alpha_j, q)$ are **formally disjoint**
 - $d(\alpha_i, \alpha_j) + q < r \Rightarrow B(\alpha_i, r) \subseteq B(\alpha_j, q)$
 $B(\alpha_j, q)$ **formally contains** $B(\alpha_i, r)$

Computability of sets

- A compact set S in (X, d, α) is **computable** if:
 - there is an effective procedure which, on input k , outputs a finite set of special points α_i which approximates S with precision 2^{-k} ;
 - equivalently: $\{\langle i_1, \dots, i_n \rangle : S \subseteq B_{i_1} \cup \dots \cup B_{i_n}\}$ and $\{i \in \mathbb{N} : S \cap B_i \neq \emptyset\}$ are c.e. sets
- If X is (compact and) computable in (X, d, α) , we say that (X, d, α) is **effectively compact**

More in: Downey R. G. and Melnikov A. G. "Computably compact metric spaces". In: *The Bulletin of Symbolic Logic* 29.2 (2023), pp. 170–263

Back to effective decomposability

- From now on, (X, d, α) is a computable metric space such that (X, d) is a continuum
- If $X = A \cup B$ where A and B are computable proper subcontinua of X , then X is a computable set

Definition

An effectively compact computable metric space (X, d, α) is **effectively decomposable** if X can be expressed as a union of two of its computable proper subcontinua.

Classical decomposability: a characterisation

Theorem

A continuum is decomposable if and only if it contains a proper subcontinuum with non-empty interior.

$\Rightarrow$ Easy.

$\Leftarrow$ If D is a proper subcontinuum of X with non-empty interior.

Case 1: $X \setminus \text{Int } D$ connected $\Rightarrow X = D \cup (X \setminus \text{Int } D)$

Case 2: $X \setminus \text{Int } D = A \cup B$ disconnected $\Rightarrow X = (A \cup D) \cup (B \cup D)$

More details in A. Illanes. *Continuum theory*. Springer Cham, 2025

Effective decomposability via subcontinua

- If (X, d, α) is effectively decomposable, e.g. into A and B , then both A and B are computable subcontinua of X
- Suppose (X, d, α) has a computable proper subcontinuum D with non-empty interior

Case 1: $X \setminus \text{Int } D$ disconnected

- $X \setminus \text{Int } D = A \cup B$, $A \cap B = \emptyset$
- $A \subseteq U_A$, $B \subseteq U_B$, U_A and U_B formally disjoint finite unions of B_i
- $D \cup A = D \cup U_A = D \cup (X \setminus U_B) \rightsquigarrow$ computable!

Case 2: $X \setminus \text{Int } D$ connected

- $X \setminus \text{Int } D$ is a subcontinuum
- it is not necessarily computable
- it is computable if $\text{Int } D$ is c.e. open

Effective decomposability via subcontinua

Proposition

An effectively compact continuum is decomposable if it contains a computable proper subcontinuum with non-empty, c.e. open interior.

Effective decomposability via subcontinua

Proposition

An effectively compact continuum is decomposable if it contains a computable proper subcontinuum with non-empty, c.e. open interior.

Can these conditions be replaced by purely topological ones?

Continua containing a copy of $\mathbb{R}$

- We consider a space (X, d) which contains a copy of $[0, 1]$ as a subcontinuum with non-empty interior
- Equivalently, it contains a copy of $\mathbb{R}$ as an open subset
- Examples: arcs, circles, (topological) graphs, ...
- Always classically decomposable

Constructing a computable subarc

Theorem 1 (Čačić, Č, Horvat, Iljazović 2025)

Let (X, d, α) be an effectively compact computable metric space. Suppose a point $x \in X$ has an open neighbourhood N which is homeomorphic to $\mathbb{R}$. Then there exist computable points $a, b \in N$ and a computable neighbourhood $N' \subseteq N$ of x which is an arc from a to b .

More in: V. Čačić et al. "Computable Approximations of Semicomputable Graphs". In: *Logical Methods in Computer Science* 22, 28 (1 2026). ISSN: 1860-5974. DOI: 10.46298/lmcs-22(1:28)2026

Corollary

We can find arbitrarily small computable arcs around any point in N , and we can find computable arcs arbitrarily close to any arc in X .

Theorem 2 (Č, Iljazović, Jelić, Tarandek)

Let (X, d, α) be an effectively compact computable metric space. Suppose U is an open set in (X, d) homeomorphic to $\mathbb{R}$. Then (X, d, α) is effectively decomposable.

Proof idea:

- Fix a homeomorphism $f : \mathbb{R} \rightarrow U$
- If $X \setminus U$ is disconnected, we take any computable arc A in U ; its interior is not empty
- As before, if $X \setminus \text{Int } A = F \cup G$, then $A \cup F$ and $A \cup G$ are computable.

Theorem 2 (Č, Iljazović, Jelić, Tarandek)

Let (X, d, α) be an effectively compact computable metric space. Suppose U is an open set in (X, d) homeomorphic to $\mathbb{R}$. Then (X, d, α) is effectively decomposable.

- If $X \setminus U$ is connected, we take computable arcs

$$A_{-1} \approx f([-1, 0]), \quad A_0 \approx f([0, 1]) \quad \text{and} \quad A_1 \approx f([1, 2])$$

- Let $f([0, 1]) \subseteq U_A \subseteq f(\langle -1, 2 \rangle)$
- A_0 and $(X \setminus U_A) \cup A_{-1} \cup A_1$ are computable proper subcontinua

Chainable continua and effective decomposability

- **Open chain:** a finite sequence $U_1, \dots, U_n$ of open sets such that $U_i \cap U_j \neq \emptyset$ iff $|i - j| < 1$
- **Chainable** continuum: for each $\varepsilon > 0$ can be covered by an open chain such that $\max \text{diam } U_i < \varepsilon$.
- There are indecomposable chainable continua: Knaster continuum, pseudo-arc

Theorem 3 (Č, Iljazović, Jelić, Tarandek)

Let (X, d, α) be an effectively compact computable metric space such that (X, d) is a decomposable chainable continuum. Then (X, d, α) is effectively decomposable.

Thank you!

Questions?

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