Computability of manifolds with cylindrical ends

Joint work with Zvonko Iljazović

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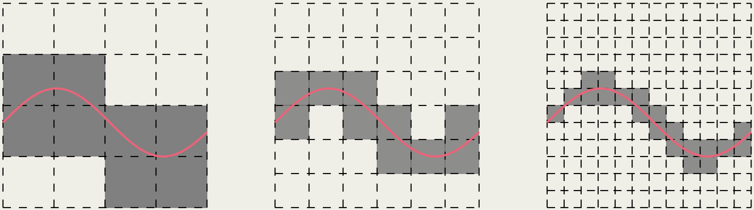
COMPUTABILITY OF COMPACT SETS

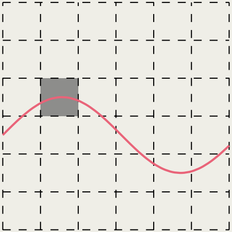
A compact set S in $\mathbb{R}^2$ is **computable** if it is empty or there is an algorithm which, on input $k \in \mathbb{N}$, outputs a finite set of rational points which approximate S with precision 2^{-k}.



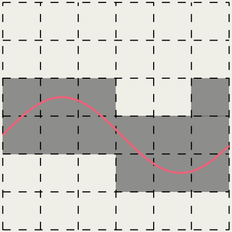
COMPUTABILITY OF COMPACT SETS

Equivalently, S is computable if it has the following two properties:

S is **computably enumerable**: we can effectively list all rational open balls which intersect S



S is **semicomputable**: we can effectively list all finite unions of rational open balls which cover S



COMPUTABILITY OF NON-COMPACT SETS

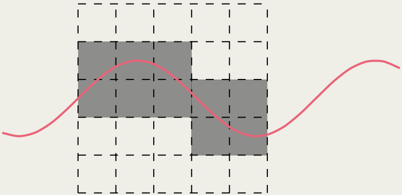
What about non-compact sets?

Definition of computable enumerability can remain the same. However, we need to modify the notion of semicomputability.

S is **semicomputable** if

(i) $S \cap \hat{B}$ is compact for each closed ball B

(ii) we can effectively list all finite unions of rational open balls which cover $S \cap \hat{B}_i$, uniformly over all rational closed balls $\hat{B}_i$



WHEN IS SEMICOMPUTABILITY ENOUGH?

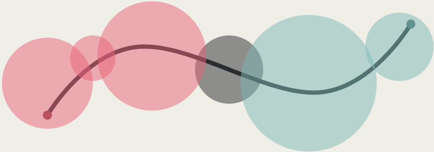
Not all semicomputable sets are computably enumerable. In fact, there is a semicoputable set with no computable points.

However, semicomputability of a set can automatically imply computable enumerability, under some additional topological conditions.

Our goal is to find such conditions.

EXAMPLE: ARC

Suppose A is a **semicomputable arc** with **computable endpoints** a and b



$$B_i \cap A \neq \emptyset \iff \begin{array}{c} \bigcup_u B_u \cup B_i \cup \bigcup_v B_v \text{ covers } A \\ a \in \bigcup_u B_u, \quad b \in \bigcup_v B_v \\ \bigcup_u B_u \text{ and } \bigcup_v B_v \text{ are formally disjoint} \end{array}$$

In this case, computable enumerability is reducible to semicomputability!

EXAMPLE: LINE

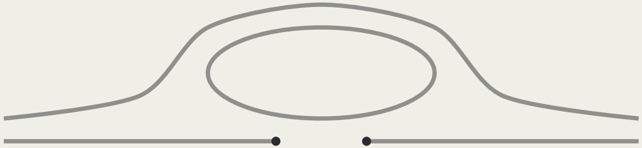
Non-compact case (L is a line):

$$I_i \cap L \neq \emptyset \quad \iff \quad \begin{array}{c} \mathcal{H}_\ell \text{ is a formal chain} \\ L \cap \hat{B}(a,n) \subseteq \bigcup \mathcal{H}_\ell^{p \leq q} \\ L \cap \hat{B}(a,m) \subseteq \bigcup \mathcal{H}_\ell \\ \mathcal{H}_\ell^{p \leq q} \subseteq_F B(a,m) \\ p < e < q < \overline{\ell} \wedge m > 1 \wedge p \leq w \leq q \\ \mathrm{fmesh}(\ell) < 2^{-(k_0+k+3)} \\ I_A \text{ and } \mathcal{H}_\ell^{e \leq \overline{\ell}} \text{ are formally disjoint} \\ I_B \text{ and } \mathcal{H}_\ell^{0 \leq e} \text{ are formally disjoint} \\ J_{(\ell)_w} \subseteq_F I_i \end{array}$$

Theorem 1 (Burnik and Iljazović 2014)

Let (X, d, α) be a computable metric space. Let M be a 1-manifold with boundary in this space such that M has finitely many components. Suppose M and ∂M are semicomputable. Then M is computable.

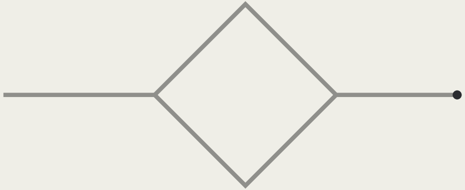
K. Burnik and Z. Iljazović (2014). "Computability of 1-manifolds". In: Logical Methods in Computer Science 10.2:8, pp. 1–28.



Theorem 2 (Iljazović 2020)

Let (X, d, α) be a computable metric space and let S be a semicomputable set in this space. Suppose S, as a subspace of (X, d), is a generalized graph such that the set E of all endpoints of S is semicomputable in (X, d, α). Then S is computable in (X, d, α).

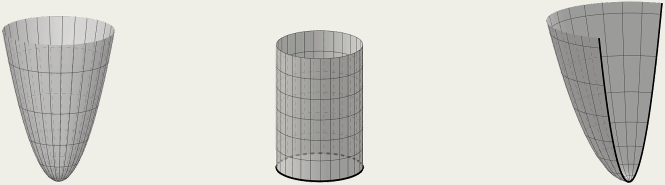
Z. Iljazović (2020). "Computability of graphs". In: Mathematical Logic Quarterly 66, pp. 51–64.



Theorem 3 (Iljazović and Sušić 2018)

Let (X, d, α) be a computable metric space and let K be a manifold with boundary in this space. Suppose K and ∂K are semicomputable. Then K is computable if there exists an open set U in K such that $\overline{U}$ is compact in K and $K \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0,1)$ or $\mathbb{H}^n$.

Z. Iljazović and I. Sušić (2018). "Semicomputable manifolds in computable topological spaces". In: Journal of Complexity 45, pp. 83–114.



FOCUSING ON "INFINITE" PARTS

Techniques developed for studying compact sets can be easily applied to compact subsets of non-compact sets.

Therefore, we can focus only on the complements of compact sets, i.e. **neighbourhoods of infinity**.

Corollary 4

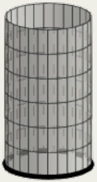
Let M be a manifold with boundary in (X, d, α). Suppose M and ∂M are semicomputable. If there exists an open set U in M such that $\overline{U}$ is compact in M and $M \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0, 1)$, then M is computable at infinity.

What is special about $\mathbb{R}^n \setminus B(0, 1)$?

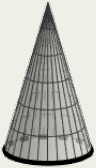
Its compactification is $\mathbb{R}^n$!

COMPACTIFICATION

From the standpoint of general topology, a **compactification** of a space is an embedding as a dense subspace of a compact space.



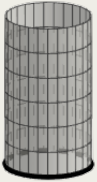
(a) A one-sided infinite cylinder



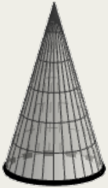
(b) One-point compactification

What about computability?

PSEUDOCOMPACTIFICATION



We start with a non-compact space equipped with a structure of computable metric space

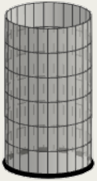


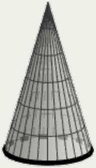
We can define a structure of a computable topological space on the compactification.

We call the newly defined computable topological space a **pseudocompactification** of the original computable metric space.

Z. Iljazović and I. Sušić (2018). "Semicomputable manifolds in computable topological spaces". In: Journal of Complexity 45, pp. 83–114.

GENERAL ARGUMENT





S semicomputable $\quad\Rightarrow\quad$ S^∞ semicomputable

$\Downarrow$

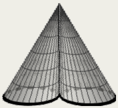
S c.e. $\quad\Longleftarrow\quad$ S^∞ c.e.

MORE CYLINDERS

Generally, spaces obtained by compactifying cylinders are (locally) cones or wedges of cones.

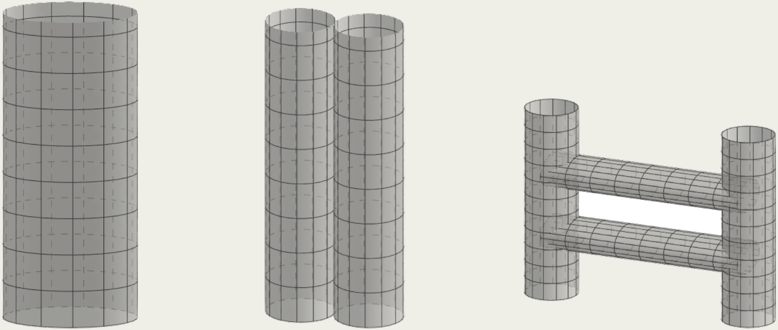
We know that semicomputable (compact) spaces obtained by glueing manifolds together are computably enumerable.

More in Matea Čelar and Zvonko Iljazović (Oct. 2021). "Computability of glued manifolds". In: Journal of Logic and Computation 32.1, pp. 65–97. ISSN: 0955-792X. DOI: 10.1093/logcom/exab063.



This allows us to conclude the same about a more general class of non-compact manifolds – manifolds with **cylindrical ends**.

EXAMPLES



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Thank you!