Computable type of certain quotient spaces

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Computable topological spaces

Computable topological space is a second countable Hausdorff space $(X, \mathcal{T})$ equipped with an enumeration of its basis $(I_i)_{i\in\mathbb{N}}$ such that there exist computably enumerable sets $\mathcal{C}$ and $\mathcal{D}$ with the following properties:

i. $\mathcal{C} \subseteq \{(i,j) \mid I_i \subseteq I_j\}$;

ii. $\mathcal{D} \subseteq \{(i,j) \mid I_i \cap I_j = \emptyset\}$;

iii. $x \in I_i \cap I_j \;\Rightarrow\; \exists k(x \in I_k \wedge (k,i) \in \mathcal{C} \wedge (k,j) \in \mathcal{C})$;

iv. $x \neq y \;\Rightarrow\; \exists i \exists j(x \in I_i \wedge y \in I_j \wedge (i,j) \in \mathcal{D})$.

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Question

Under which (topological) conditions is a semicomputable set computable?

Computable type

Definition

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The following spaces have been shown to have computable type:

- topological manifolds ([Iljazović and Sušić, 2018])
- chainable and circularly chainable continua [Čičković et al., 2019]
- some finite simplicial complexes [Amir and Hoyrup, 2022]
- pseudo-cubes [Horvat et al., 2020]

Pairs and quotients

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Let B be a subset of a topological space A. The **quotient space** A/B is the set $\{B\} \cup \{\{x\} \mid x \in A \setminus B\}$ equipped with the finest topology which makes the cannonical projection map continuous.

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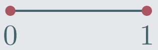

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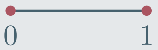

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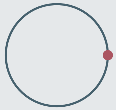

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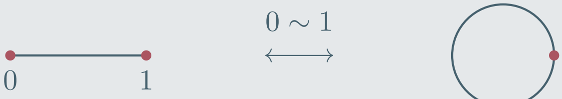

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The main goal

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Let A be a topological space and let B be a compact subset of A such that $\operatorname{Int}_A B = \emptyset$. If A/B has computable type, then (A, B) also has computable type.

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Sketch of proof.

- Given a semicomputable set B in a computable topological space, we can define a natural structure of computability on the corresponding quotient space
- The natural quotient map $q : X \to X/B$ preserves semicomputability
- The inverse image of q preserves computable enumerability ($\operatorname{Int}_A B = \emptyset$ is crucial in this step)

$$(A, B) \text{ has computable type} \quad \underset{\operatorname{Int}_A B = \emptyset}{\overset{?}{\Longleftrightarrow}} \quad A/B \text{ has computable type}$$

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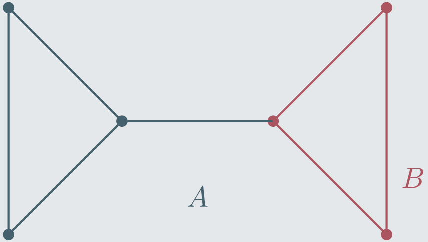
Counterexample I



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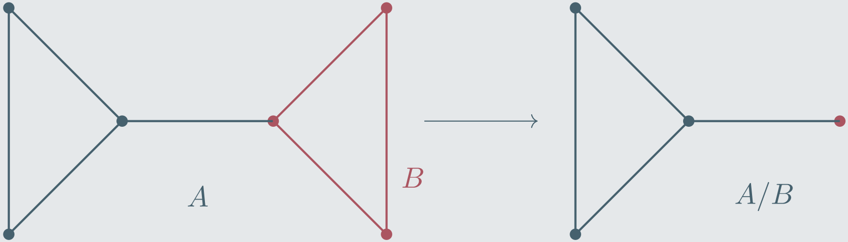

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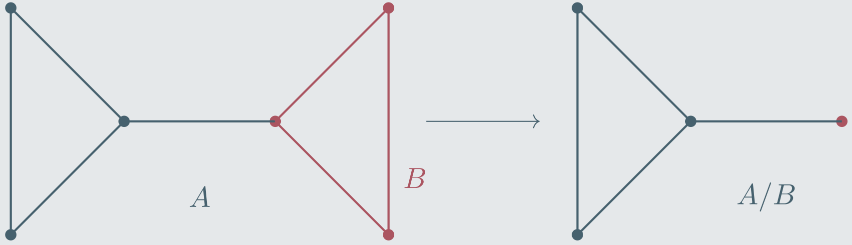

(A, B) has computable type

A/B does not have computable type

Counterexample II

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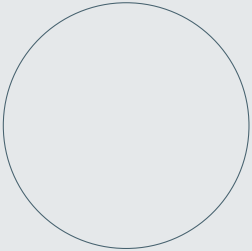

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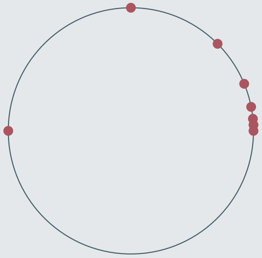

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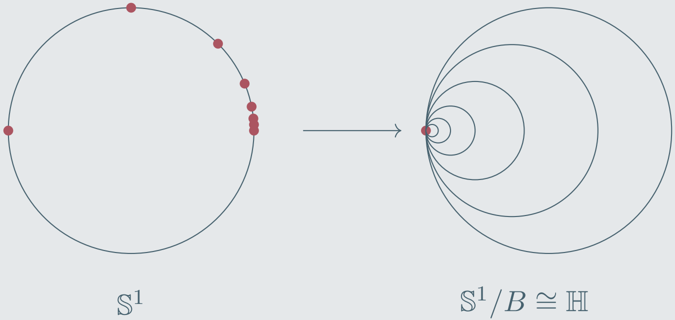

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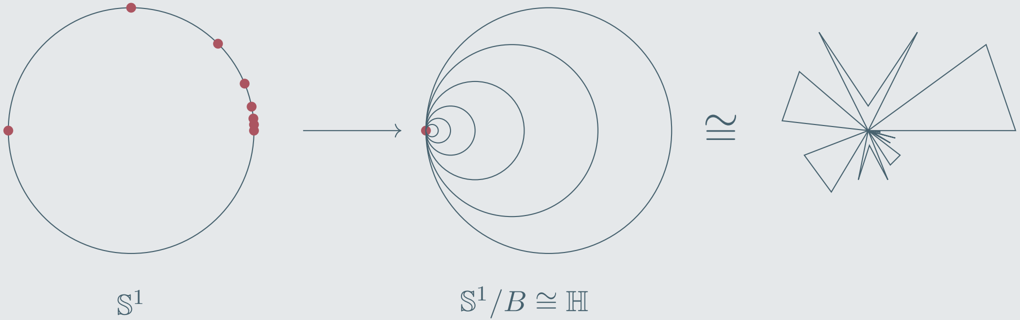

Local computability

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Suppose $T \subseteq S$. We say that T is **computably enumerable up to** S if there exists a c.e. set $\Omega \subseteq \mathbb{N}$ such that $\{i \in \mathbb{N} \mid I_i \cap T \neq \emptyset\} \subseteq \Omega \subseteq \{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$.

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We say that S is **computably enumerable at** $x \in S$ if x has a neighborhood in S which is c.e. up to S.

Definition

Topological space A has **local computable type** if a semicomputable set is c.e. at any point which has a neighborhood homeomorphic to A.

Local computable type of $\mathbb{R}^n/K$

Theorem

Let K be a compact subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ has finitely many connected components. Then $\mathbb{R}^n/K$ has local computable type.

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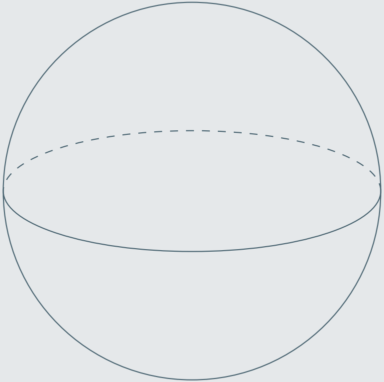
Sketch of proof.

- Suppose S is semicomputable and $x_0 \in S$ has a neighborhood homeomorphic to $\mathbb{R}^n/K$. WLOG we can assume x_0 is the image of the "collapsed" set K; otherwise, it has an Euclidean neighborhood.
- The image of any *unbounded* connected component of $\mathbb{R}^n \setminus K$ contains a neighborhood of x_0 which is c.e. up to S.
- The image of *any* connected component can be expressed as an image of an unbounded component under a similar map.

Example



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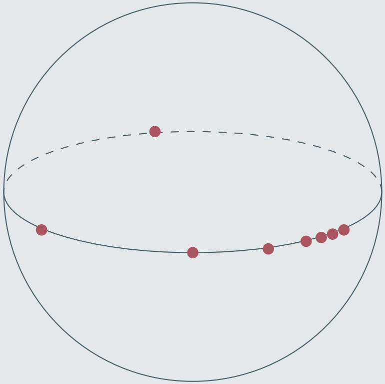

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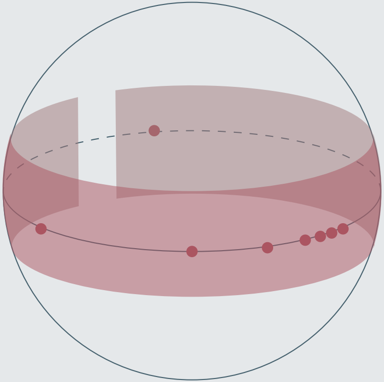

References

- Amir, D. E. and Hoyrup, M. (2022).
  Computability of finite simplicial complexes.
- Horvat, M., Iljazović, Z., and Pažek, B. (2020).
  Computability of pseudo-cubes.
  Annals of Pure and Applied Logic, 171(8):102823.
- Iljazović, Z. and Sušić, I. (2018).
  Semicomputable manifolds in computable topological spaces.
  Journal of Complexity, 45:83–114.
- Čičković, E., Iljazović, Z., and Validžić, L. (2019).
  Chainable and circularly chainable semicomputable sets in computable topological spaces.
  Archive for Mathematical Logic, 58:885–897.

Thank you!