# Computable type of certain quotient spaces


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# Computable topological spaces

**Computable topological space** is a second countable Hausdorff space $(X, \mathcal{T})$ equipped with an enumeration of its base $(I_i)_{i\in\mathbb{N}}$ such that there exist r.e. sets $\mathcal{C}$ and $\mathcal{D}$ with the following properties:

i. $\mathcal{C} \subseteq \{(i,j) \mid I_i \subseteq I_j\}$;

ii. $\mathcal{D} \subseteq \{(i,j) \mid I_i \cap I_j = \emptyset\}$;

iii. $x \in I_i \cap I_j \;\Rightarrow\; \exists k(x \in I_k \wedge (k,i) \in \mathcal{C} \wedge (k,j) \in \mathcal{C})$;

iv. $x \neq y \;\Rightarrow\; \exists i \exists j(x \in I_i \wedge y \in I_j \wedge (i,j) \in \mathcal{D})$.

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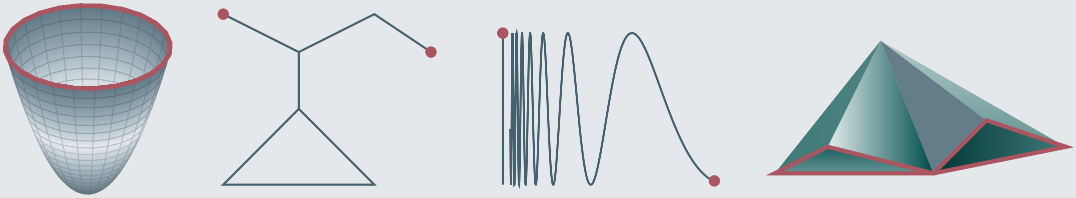
**Definition**

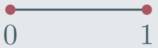
Topological pair $(A, B)$ has **computable type** if, for every semicomputable copy $(S, T)$ of $(A, B)$ in a computable topological space, $S$ is computable.

# Computable type: examples



[Iljazović and Sušić, 2018, Čičković et al., 2019, Amir and Hoyrup, 2022, Horvat et al., 2020]

# Pairs and quotients




$([0,1],\{0,1\})$ has computable type

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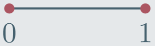


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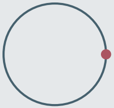

$\mathbb{S}^1$ has computable type

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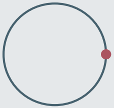
$$\xleftrightarrow{0 \sim 1}$$



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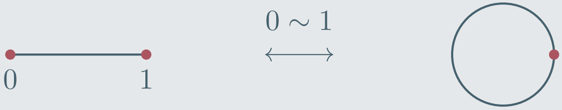


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**Definition**

Let $B$ be a subset of a topological space $A$. The **quotient space** $A/B$ is the set $\{B\} \cup \{\{x\} \mid x \in A \setminus B\}$ equipped with the finest topology which makes the cannonical projection map continuous.

# The main goal

$$\begin{tikzcd} \begin{array}{c}(A, B) \text{ has} \\ \text{computable type}\end{array} \arrow[r, Rightarrow, shift left, "?"] & \begin{array}{c}A/B \text{ has} \\ \text{computable type}\end{array} \arrow[l, Rightarrow, shift left, "?"] \end{tikzcd}$$

# The main goal

$$\text{$(A, B)$ has computable type} \quad \underset{?}{\overset{?}{\rightleftarrows}} \quad \text{$A/B$ has computable type}$$

# "Unbinding" a quotient space

**Theorem**

*Let $A$ be a topological space and let $B$ be a compact subset of $A$ such that $\operatorname{Int}_A B = \emptyset$. If $A/B$ has computable type, then $(A, B)$ also has computable type.*

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***Sketch of proof.***

- Given a semicomputable set $B$ in a computable topological space, we can define a natural structure of computability on the corresponding quotient space
- The natural quotient map $q : X \to X/B$ preserves semicomputability
- The inverse image of $q$ preserves computable enumerability ( $\operatorname{Int}_A B = \emptyset$ is crucial in this step)

$$\begin{tikzcd}[column sep=huge] \text{$(A, B)$ has computable type} \arrow[r, Rightarrow, shift left, "?"] & \text{$A/B$ has computable type} \arrow[l, Rightarrow, shift left, "\operatorname{Int}_A B = \emptyset"] \end{tikzcd}$$

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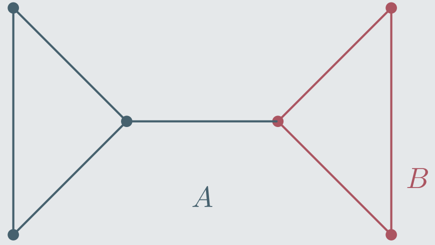


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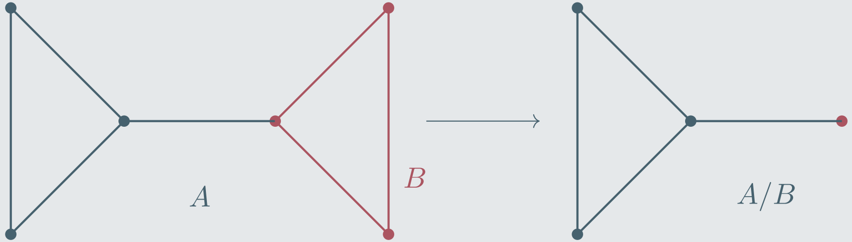


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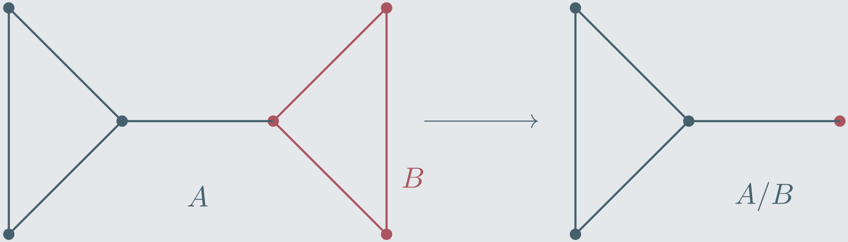


$(A, B)$ has computable type

$A/B$ does not have computable type

# Counterexample II

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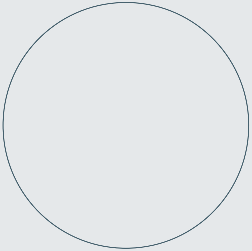

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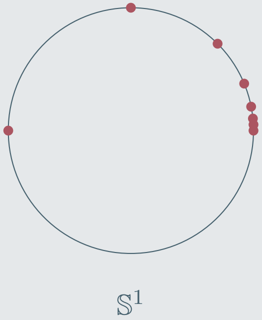

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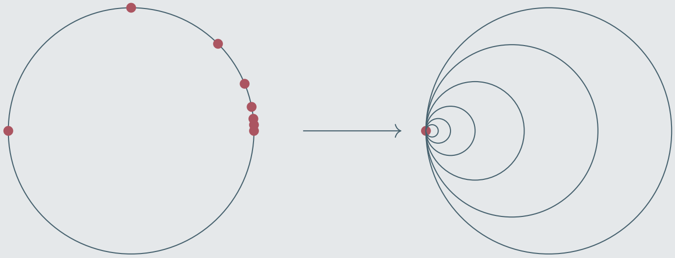

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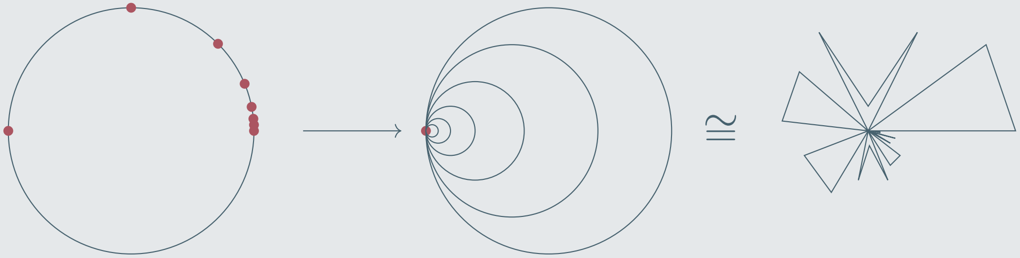


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# Local computability

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Suppose $T \subseteq S$. We say that $T$ is **computably enumerable up to** $S$ if there exists a r.e. set $\Omega \subseteq \mathbb{N}$ such that $\{i \in \mathbb{N} \mid I_i \cap T \neq \emptyset\} \subseteq \Omega \subseteq \{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$.

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We say that $S$ is **computably enumerable at** $x \in S$ if $x$ has a neighborhood in $S$ which is c.e. up to $S$.

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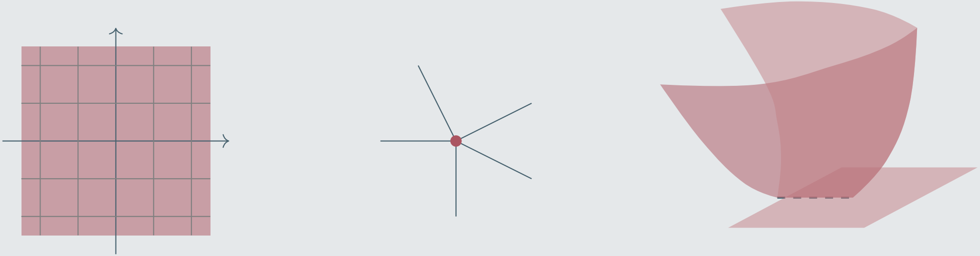
We say that $S$ is **computably enumerable at** $x \in S$ if $x$ has a neighborhood in $S$ which is c.e. up to $S$.

**Definition**

Topological space $A$ has **local computable type** if a semicomputable set is c.e. at any point which has a neighborhood homeomorphic to $A$.

# Local computable type: examples



# Local computable type of $\mathbb{R}^n/K$

**Theorem**

*Let $K$ be a compact subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ has finitely many connected components. Then $\mathbb{R}^n/K$ has local computable type.*

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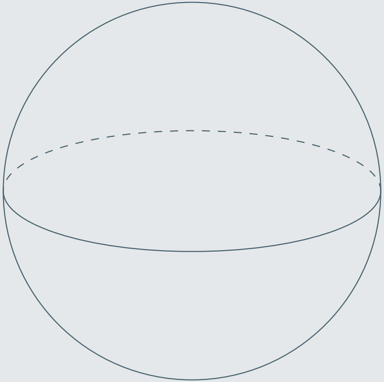
***Sketch of proof.***

- Suppose $S$ is semicomputable and $x_0 \in S$ has a neighborhood homeomorphic to $\mathbb{R}^n/K$. WLOG we can assume $x_0$ is the image of the "collapsed" set $K$; otherwise, it has an Euclidean neighborhood.
- The image of any *unbounded* connected component of $\mathbb{R}^n \setminus K$ contains a neighborhood of $x_0$ which is c.e. up to $S$.
- The image of *any* connected component can be expressed as an image of an unbounded component under a similar map.

## Example



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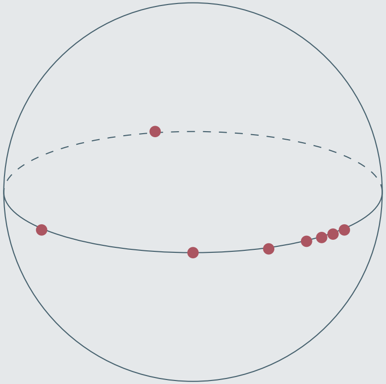

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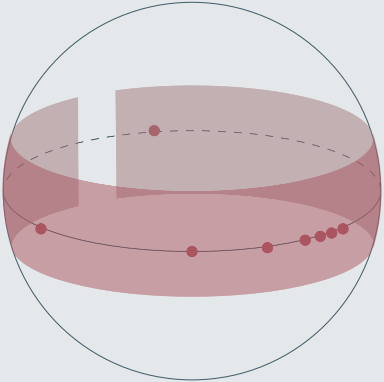

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# Thank you!