Computable approximations of semicomputable graphs

Joint work with Vedran Čačić, Marko Horvat and Zvonko Iljazović

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Introduction

A compact set S in $\mathbb{R}$ is

- **computable** if it is empty or there is an algorithm which, on input $k \in \mathbb{N}$, outputs a finite set of rational points which approximate S with precision 2^{-k};
- **semicomputable** if its complement can be effectively exhausted by rational open intervals.

These notions can be defined in more general spaces: $\mathbb{R}^n$, computable metric spaces, computable Hausdorff spaces, ...
Regardless of the ambient space, it holds that

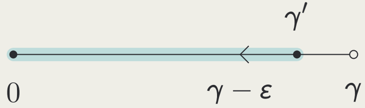
$$S \text{ computable} \quad \Rightarrow \quad S \text{ semicomputable.}$$

Introduction

Does the converse hold?

No! Even in $\mathbb{R}$: if γ is a left-computable number which is not computable, the segment $[0, \gamma]$ is a semicomputable set which is not computable.

However, for any $\varepsilon > 0$, one can find $\gamma' \in \langle \gamma - \varepsilon, \gamma \rangle$ such that $[0, \gamma']$ is computable.



Therefore, $[0, \gamma]$ can be inner approximated by a computable set $[0, \gamma']$, which is also a line segment (so the nice topological properties have been preserved).

Main question

Is there always a computable inner approximation?

No! In fact, there are semicomputable sets which do not contain any computable points, so they have no computable subsets.

Question

Which (topological) conditions enable semicomputable sets to have "nice" computable inner approximations?

Related topic: (Strong) computable type

A topological space X has **(strong) computable type** if

$$\begin{array}{c} S \text{ semicomputable} \\ (\text{relative to } O) \end{array} \implies \begin{array}{c} S \text{ computable} \\ (\text{relative to } O) \end{array}$$

holds for any $S \cong X$ (and any oracle O).

Examples: closed manifolds, polyhedra, ...

Obviously, if X has strong computable type, then any semicomputable $S \cong X$ has a computable inner approximation: S itself!

Strong computable type of pairs

A pair (X, A) of a topological space X and its subspace A has (strong) computable type if

$$\begin{array}{c} S \text{ and } T \text{ semicomputable} \\ \text{(relative to } O\text{)} \end{array} \implies \begin{array}{l} S \text{ computable} \\ \text{(relative to } O\text{)} \end{array}$$

holds for any $(S, T) \cong (X, A)$ (and any oracle O).

Examples: manifolds with boundary, simplicial complexes, graphs with endpoints, ...

Intuitively: if both S and its "boundary" are semicomputable, then S is computable.

What if the boundary of a semicomputable set is not semicomputable? Is it possible "cut out" the "bad" boundary and obtain a new, semicomputable boundary?

Preliminaries

The setting

A **computable metric space** is a triple (X, d, α), where (X, d) is a metric space and $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ is a dense sequence in (X, d) such that the function $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

Let $i \mapsto (c_i, r_i)$ be a fixed computable enumeration of $\{\alpha_n\}_{n\in\mathbb{N}} \times \mathbb{Q}^+$. Then

$$(I_i = B(c_i, r_i))_{i\in\mathbb{N}} \quad \text{and} \quad (\widehat{I}_i = \overline{B}(c_i, r_i))_{i\in\mathbb{N}}$$

are effective enumerations of, respectively, open and closed balls centered at α_i with rational radii.

(Families of) rational open sets

Let $j \mapsto ((j)_0, (j)_1, \ldots, (j)_{\bar{j}})$ be a fixed effective enumeration of all nonempty finite sequences in $\mathbb{N}$ and let $[j] := \{(j)_0, \ldots, (j)_{\bar{j}}\}$.

For $j, l \in \mathbb{N}$ let

$$J_j := \bigcup_{i \in [j]} I_i \qquad \widehat{J}_j := \bigcup_{i \in [j]} \widehat{I}_i \qquad J_{[l]} := \bigcup_{j \in [l]} J_j$$

The function $\mathrm{fdiam} : j \mapsto \max_{u,v \in [j]} d(c_u, c_v) + 2 \max_{u \in [j]} r_u$ is computable.

It provides an upper bound for the diameter of $\widehat{J}_j$, i.e. $\mathrm{diam}(\widehat{J}_j) \leq \mathrm{fdiam}(j)$.

Distance between sets

Let (X, d) be a metric space, $A \subseteq X$ and $\varepsilon > 0$.

Let $A_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon)$ be the **ε-neighbourhood** of A.

For a pair (A, B) of (nonempty) closed, bounded subsets of X we define

$$d_H(A, B) = \inf\{\varepsilon \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}.$$

The function d_H is a metric on the space of all closed, bounded subsets of X. It is called the **Hausdorff distance**.

(Semi)computable sets

A set S in (X, d, α) is

- **semicomputable**, if $S \cap \hat{B}$ is compact for each closed ball $\hat{B}$ and the set

$$\{(i, j) \in \mathbb{N}^2 \mid S \cap \hat{I}_i \subseteq J_j\}$$

  is c.e.

- **computable**, if it is semicomputable and the set

$$\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$$

  is c.e.

(Semi)computable compact sets

For a compact set S

- semicomputability is equivalent to

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

  being c.e.
- computability is equivalent to the existence of a c.e. function $k \mapsto j_k$ such that

$$d_H(S, \{\alpha_{(j_k)_0}, \ldots, \alpha_{(j_k)_{\overline{j_k}}}\}) < 2^{-k}$$

Computing with rational open sets

The following relations are c.e.:

- $I_i \diamond I_j :\Leftrightarrow d(c_i, c_j) > r_i + r_j$
- $I_i \subseteq_\forall I_j :\Leftrightarrow d(c_i, c_j) + r_i < r_j$

- $J_i \diamond J_j :\Leftrightarrow (\forall u \in [i])(\forall v \in [j])(I_u \diamond I_v)$
- $J_i \subseteq_\forall J_j :\Leftrightarrow (\forall u \in [i])(\exists v \in [j])(I_u \subseteq_\forall I_v)$

- $J_{[i]} \subseteq_\forall J_{[j]} :\Leftrightarrow (\forall u \in [i])(\exists v \in [j])(J_u \subseteq_\forall J_v)$.

It holds that

- $I_i \diamond I_j$ implies $\widehat{I_i} \cap \widehat{I_j} = \emptyset$
- $I_i \subseteq_\forall I_j$ implies $\widehat{I_i} \subseteq I_j$
- $J_i \diamond J_j$ implies $\widehat{J_i} \cap \widehat{I_j} = \emptyset$
- $J_i \subseteq_\forall J_j$ implies $\widehat{J_i} \subseteq J_j$.

Separation by J_js

Lemma 1 (Iljazović 2009)

Let K and U be subsets of (X, d) such that K is nonempty and compact, U is open and $K \subseteq U$. Let $\varepsilon > 0$. Then there exists $j \in \mathbb{N}$ such that $K \subseteq J_j$, $\widehat{J_j} \subseteq U$ and $\mathrm{fdiam}(j) < \mathrm{diam}\, K + \varepsilon$.

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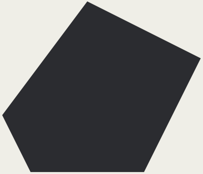

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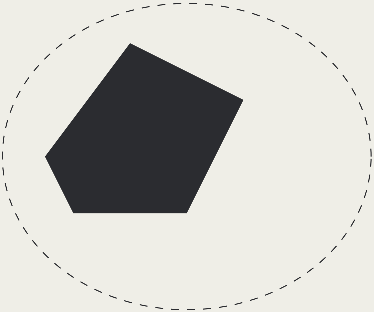

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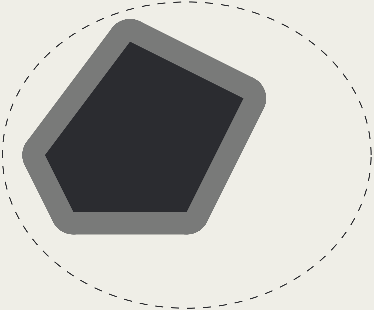

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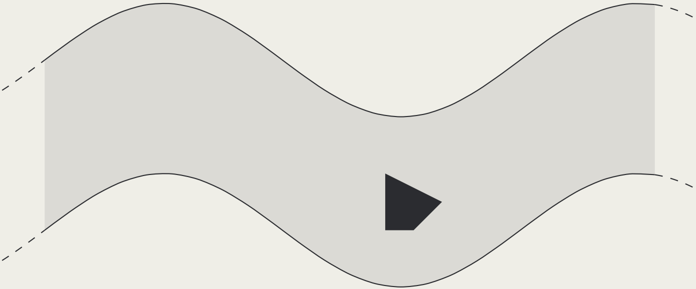

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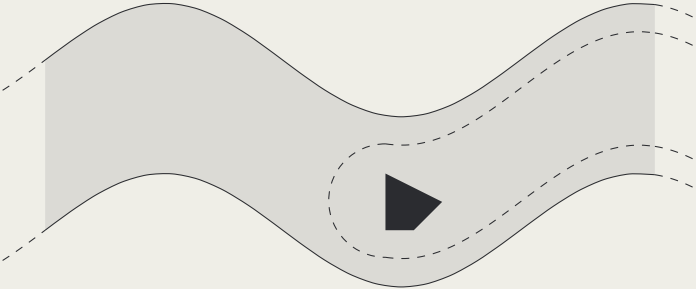

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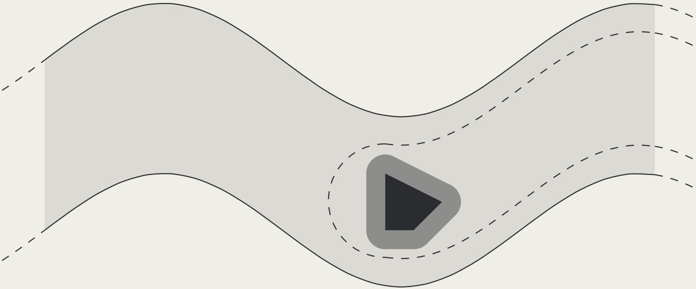

Known results: starting point

Theorem 3 (Iljazović 2009)

Let (X, d, α) be a computable metric space and let S be a semicomputable chainable and decomposable continuum in this space. Then for each $\varepsilon > 0$ there exists a computable subcontinuum K of S such that $d_H(S, K) < \varepsilon$. Moreover, K can be chosen so that it is chainable from a to b, where a and b are computable points.

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Theorem 4 (Iljazović and Jelić 2024)

Let (X, d, α) be a computable metric space and let S be a semicomputable continuum chainable from a to b, where a is a computable point. Then for each $\varepsilon > 0$ there exists a computable point $\hat{b} \in S$ and computable subcontinuum K of S chainable from a to $\hat{b}$ such that $d(b, \hat{b}) < \varepsilon$ and $d_H(S, K) < \varepsilon$.

Known results: limits

Theorem 5 (Kihara 2012)

There exists a contractible, locally contractible, co-c.e. planar curve which is not inner approximated by computable continua.

Fact (Kihara 2012)

There exists a co-c.e. homeomorphic copy of the Cantor fan which contains no computable point.

Main results

Computable subarc theorem

Theorem 6

Let S be a semicomputable set in a computable metric space (X, d, α). Suppose a point $x \in S$ has an open neighborhood N in S which is homeomorphic to $\mathbb{R}$. Then there exist computable points $a, b \in N$ and a computable neighbourhood $N' \subseteq N$ of x in S which is an arc from a to b.

Proof. Let x be a point in a semicomputable set S and let $f : \mathbb{R} \to N$ be a homeomorphism, where N is an open neighbourhood of x in S. WLOG $f(0) = x$.

Goal: Find a computable arc $N' \subseteq N$ with computable endpoints a and b which contains x in its interior.

Proof of CST: first try

Theorem 7 (Iljazović and Validžić 2017)

Suppose (X, d, α) is a computable metric space, S a semicomputable set in this space and $x \in S$ a point which has a neighbourhood in S homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N} \setminus \{0\}$. Then x has a computable compact neighbourhood in S.

This theorem ensures x has a computable neighbourhood

… however, it is not necessarily an arc with computable endpoints!

Computable points are dense in computable sets, so x has a neighbourhood which is an arc with computable endpoints

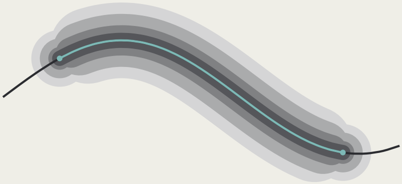
… however, is is not necessarily computable!

Proof of CST: general idea

Let's start over.

Idea: Construct N' as an intersection of a nested sequence of suitably chosen $J_{[I]}$s

- working with rational open sets ensures the resulting neighbourhood and its endpoints will be computable;
- (effective) topological properties ensure the resulting set will be an arc.

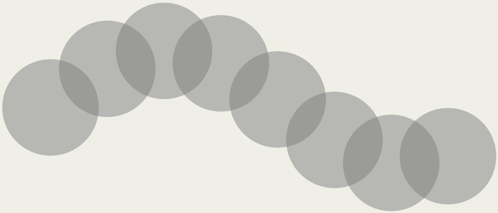


We will work with formal chains.

Interlude: chains

Let $\mathcal{C} = (C_0, \dots, C_n)$ be a finite sequence of nonempty subsets of X.

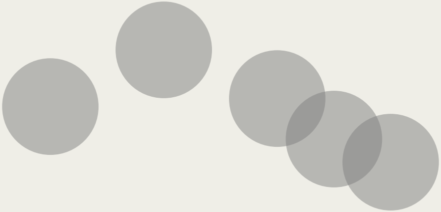
- $\mathcal{C}$ is a **chain** if, for all i, j, $C_i \cap C_j \neq \emptyset \Leftrightarrow |i - j| \leq 1$. Each set C_i is said to be the (ith) **link** of $\mathcal{C}$.
- The **mesh** of $\mathcal{C}$ is the number $\operatorname{mesh}(\mathcal{C}) := \max_{i=0}^{n} \operatorname{diam} C_i$.



Formal chains

Let $l \in \mathbb{N}$.

- $(J_{(l)_0}, \ldots, J_{(l)_{\bar{l}}})$ is a **formal chain** if $J_{(l)_i} \diamond J_{(l)_j}$ for all $i, j \in \{0, \ldots, \bar{l}\}$ such that $|i - j| > 1$.
- The function $\mathrm{fmesh} : l \mapsto \max_{j \in [l]} \mathrm{fdiam}(j)$ is computable.



Proof of CST: Setting the stage

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By Lemma 2, there exists a semicomputable compact set S' such that

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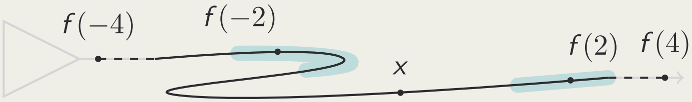
By Theorem 8, $f(-2)$ and $f(2)$ have computable compact neighbourhoods in S'.



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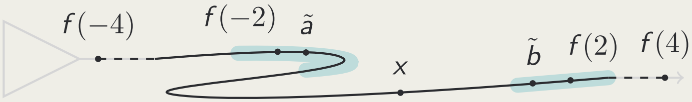
Since computable points are dense in computable sets, one can find computable points $\tilde{a}$ and $\tilde{b}$ arbitrarily close to $f(-2)$ and $f(2)$, respectively.



Proof of CST: Induction base

Claim 1

There exist $p, l, q \in \mathbb{N}$ such that

(i) $(J_p, J_{(l)_0}, \ldots, J_{(l)_{\bar{l}}}, J_q)$ is a formal chain;

(ii) $S' \subseteq J_p \cup J_{[l]} \cup J_q$

(iii) $\mathrm{fmesh}(l) < \frac{\varepsilon}{2}$

(iv) $\tilde{a} \in J_p$

(v) $\tilde{b} \in J_q$

(vi) $d\big(\tilde{a}, J_{(l)_0}\big) < \frac{\varepsilon}{2}$

(vii) $d\big(\tilde{b}, J_{(l)_{\bar{l}}}\big) < \frac{\varepsilon}{2}$.

The set $\Omega := \{l \in \mathbb{N} \mid (\exists p, q)((\text{i})\text{-}(\text{v}) \text{ holds})\}$ is c.e.

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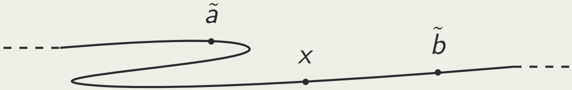

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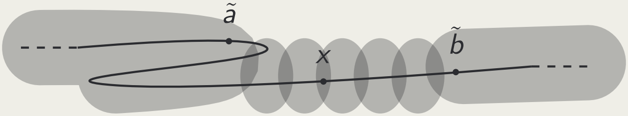

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Proof of CST: Induction step

Claim 2

For each $l \in \Omega$, there exists $l' \in \Omega$ such that

(i) $J_{[l']} \subseteq_\forall J_{[l]}$

(ii) $\mathrm{fmesh}(l') < \frac{1}{2}\,\mathrm{fmesh}(l)$

(iii) $J_{(l')_0} \subseteq_\forall J_{(l)_0}$

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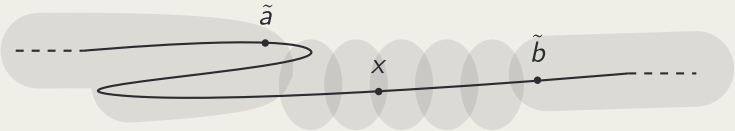

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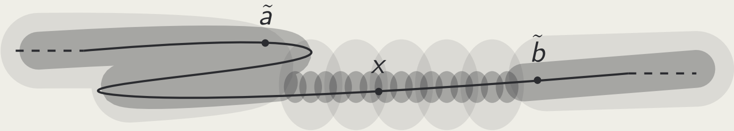

Proof of CST: final construction

Claim 1 and Claim 2 allow us to define the sequence $(l_n)_{n\in\mathbb{N}}$ of natural numbers such that, for each n, (i)-(iv) from Claim 2 hold for $l = l_n$ and $l' = l_{n+1}$.

The set

$$N' := \bigcap_{n\in\mathbb{N}} \left(\mathrm{Cl}(J_{(l_{n+1})_0} \cap S') \cup \cdots \cup \mathrm{Cl}(J_{(l_{n+1})_{\overline{l_{n+1}}}} \cap S')\right)$$

is an arc with endpoints a to b, where

$$a \in \bigcap_{n\in\mathbb{N}} J_{(l_n)_0} \quad \text{and} \quad b \in \bigcap_{n\in\mathbb{N}} J_{(l_n)_{\overline{l_n}}}.$$

a and b are uniquely determined due to Cantor's intersection theorem.

Proof of CST: computability of the subarc

The arc defined in this way is indeed computable: for each $k \in \mathbb{N}$, the centers of rational open balls forming $J_{[l]}$s approximate N' with precision 2^{-k}.

A similar argument shows that a and b are computable points. $\square$

Nipping lemma

Lemma 8

Let S be a semicomputable set in a computable metric space (X, d, α). Suppose a point $x \in S$ has an open neighborhood N in S such that there exists a homeomorphism $f : [0, 1\rangle \to N$ such that $f(0) = x$. Let $\varepsilon > 0$. Then there exists $a \in \langle 0, 1\rangle$ such that $f(a)$ is a computable point, $S \setminus f([0, a\rangle)$ is a semicomputable set and $f([0, a]) \subseteq B(x, \varepsilon)$.

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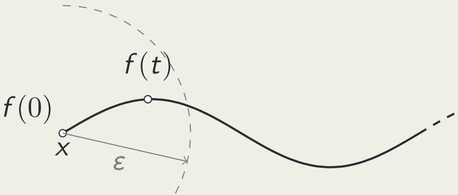

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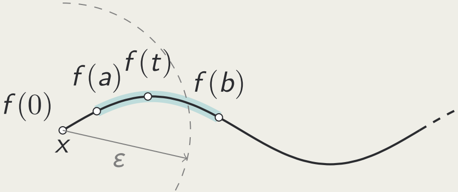
Let S be a semicomputable set in a computable metric space (X, d, α). Suppose a point $x \in S$ has an open neighborhood N in S such that there exists a homeomorphism $f : [0, 1\rangle \to N$ such that $f(0) = x$. Let $\varepsilon > 0$. Then there exists $a \in \langle 0, 1\rangle$ such that $f(a)$ is a computable point, $S \setminus f([0, a\rangle)$ is a semicomputable set and $f([0, a]) \subseteq B(x, \varepsilon)$.

Proof. Choose $t \in \langle 0, 1\rangle$ such that $f([0, t]) \subseteq B(x, \varepsilon)$.



Proof of the Nipping lemma

$f(t)$ has a neighbourhood homeomorphic to $\mathbb{R}$, so by CST it has a computable neighbourhood which is an arc with computable endpoints $f(a)$ and $f(b)$ (where WLOG $a < t < b$). Obviously $f([0, a]) \subseteq f([0, t]) \subseteq B(x, \varepsilon)$.

Choose J_m such that $f([0, a]) \subseteq J_m$ and $J_m \cap S \subseteq f([0, b\rangle)$.

$S \setminus J_m$ is semicomputable, so

$$(S \setminus J_m) \cup f([a, b]) = S \setminus f([0, a\rangle)$$

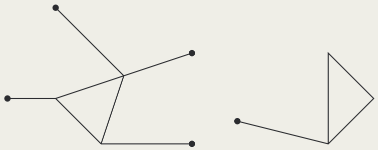
is semicomputable. □

Applications

Finite topological graphs

Let K be a nonempty finite family of (non-degenerate) line segments in $\mathbb{R}^n$, each two of which intersect at most at a common endpoint.

Any topological space G homeomorphic to $\bigcup_{I \in K} I$ is called a **graph**.



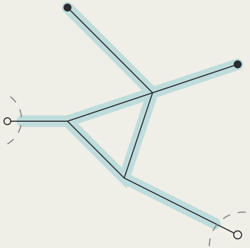
Equivalently, a topological space G is a graph if and only if there exists a nonempty finite family $\mathcal{A}$ of subspaces of G such that each $A \in \mathcal{A}$ is an arc, $G = \bigcup_{A \in \mathcal{A}} A$ and each two elements of $\mathcal{A}$ intersect at most at a common endpoint.

Application to graphs

Theorem 9

Let (X, d, α) be a computable metric space and let S be a semicomputable graph in this space. Then for each $\varepsilon > 0$ there exists a computable graph T in (X, d, α) such that all endpoints of T are computable and such that $d_H(S, T) < \varepsilon$.

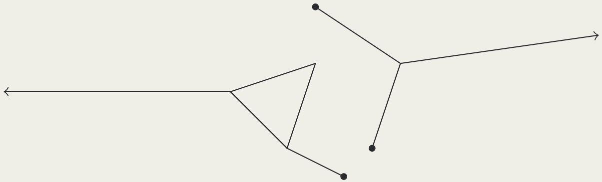
Proof. By induction on the number of non-computable endpoints: each application od the Nipping lemma reduces the number of non-computable endpoints by one. □



Generalized graphs

Let $a, v \in \mathbb{R}^n$. The set $\{a + tv \mid t \in [0, \infty\rangle\}$ is called a **ray** in $\mathbb{R}^n$. The point a is the (only) **endpoint** of this ray.

Let K be a nonempty finite family of line segments and rays in $\mathbb{R}^n$ each two of which intersect at most at a common endpoint. Any topological space G homeomorphic to $\bigcup K$ is called a **generalized graph**.



Application to generalized graphs

Theorem 10

Let (X, d, α) be a computable metric space and let S be a semicomputable generalized graph in this space. Then for each $\varepsilon > 0$ there exists a computable generalized graph T in (X, d, α) such that all endpoints of T are computable and such that $T \subseteq S \subseteq \bigcup_{x \in T} B(x, \varepsilon)$.

Special case: 1-manifolds

A 1**-manifold with boundary** is a second countable Hausdorff space in which every point has an open neighborhood homeomorphic to $[0, \infty\rangle$ or $\mathbb{R}$.
The **boundary** of a manifold is the set of all points which only admit open neighbourhoods homeomorphic to $[0, \infty\rangle$.

Classification of 1-manifolds

Each component of a 1-manifold with boundary is homeomorphic to one of the following:

$$\mathbb{R}, \quad [0, \infty\rangle, \quad \mathbb{S}^1, \quad \text{or} \quad [0, 1].$$

Therefore, a 1-manifold with finitely many components is a generalized graph!

Bonus result

The following is an immediate consequence of Theorem 7:

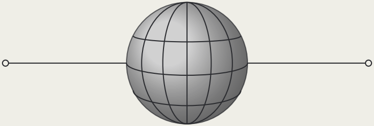
Corollary 11

Let (X, d, α) be a computable metric space and let M be a semicomputable 1-manifold in this space such that M has finitely many connected components. Then for each $\varepsilon > 0$ there exists a computable 1-manifold N in (X, d, α) such that each point of ∂N is computable and such that $N \subseteq M \subseteq \bigcup_{x \in N} B(x, \varepsilon)$.

More possible applications

This result is a nice supplement to the study of (strong) computable type.

In general, if (X, A) has computable type, A is finite and each point in A has a neighbourhood in X homeomorphic to $[0, 1\rangle$, then the nipping lemma can be used to show that any semicomputable set homeomorphic to X can be inner approximated by a computable subset (with computable endpoints).



Closing remarks

Conclusion

We proved that any point in a semicomputable set which has a neighbourhood homeomorphis to $\mathbb{R}$ also has a neighbourhood which is a computable arc with computable endpoints.

Using this, we proved that every semicomputable generalized graph in a computable metric space can be approximated, with arbitrary precision, by its computable subgraph with computable endpoints.

This approach can be applied to a larger class of semicomputable sets.

Future work

Higher dimensions: manifolds and simplicial complexes

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Thank you!