Computable type of an unglued space

Matea Čelar and Zvonko Iljazović

University of Zagreb
Faculty of Science, Department of Mathematics

27th July 2021

This work has been fully supported by Croatian Science Foundation under the project 7459 CompStruct

Computable topological spaces

Computable topological space is a triple $(X, \mathcal{T}, (I_i))$ where

- $(X, \mathcal{T})$ is a topological space
- $\{I_i \mid i \in \mathbb{N}\}$ is a base for $\mathcal{T}$
- there exist computably enumerable sets $\mathcal{C}$ and $\mathcal{D}$ such that:
  1. $(i,j) \in \mathcal{C} \;\Rightarrow\; I_i \subseteq I_j$;
  2. $(i,j) \in \mathcal{D} \;\Rightarrow\; I_i \cap I_j = \emptyset$;
  3. $x \in I_i \cap I_j \;\Rightarrow\; \exists k(x \in I_k$ and $(k,i),(k,j) \in \mathcal{C})$;
  4. $x \neq y \;\Rightarrow\; \exists i,j(x \in I_i,\; y \in I_j$ and $(i,j) \in \mathcal{D})$.

Computable topological spaces

By J_j we denote the finite unions of basis elements, e.g.

$$J_j = \bigcup_{i \in [j]} I_i.$$

There exist c.e. sets $C, D \subseteq \mathbb{N}^2$ such that

- $(i, j) \in C$ implies $J_i \subseteq J_j$;
- $(i, j) \in D$ implies $J_i \cap J_j \neq \emptyset$.

Computable and semicomputable sets

Let S be a set in a computable topological space $(X, \mathcal{T}, (I_i))$.

- S is **computably enumerable** if it is closed and

$$\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$$

  is c.e.

- S is **semicomputable** if it is compact and

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

  is c.e.

- S is **computable** if S is semicomputable and computably enumerable.

Computable type

General question:

Under which (topological) conditions does the implication

$$S \text{ semicomputable} \quad \Rightarrow \quad S \text{ computable}$$

hold for a set S in a computable topological space?

Topological space A has **computable type** if the implication above holds whenever S is homeomorphic to A.

More generally, a topological pair (A, B) has computable type if the implication above holds whenever $T \subseteq S$ is semicomputable and (S, T) is homeomorphic to (A, B).

Spaces and pairs with computable type

Some known examples: ([5, 9, 4, 2])

- $(M, \partial M)$, where M is a (topological) manifold with boundary ∂M
- $(C, \{a, b\})$, where C is a continuum chainable from a to b
- Circularly chainable continua
- Finite graphs
- Pseudo-cubes

...

We want to investigate the relationship between computable type and certain topological constructions; namely, quotient spaces.

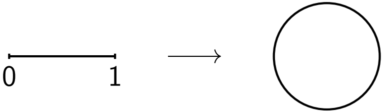
Quotient spaces

Let B be a subset of a topological space A. By A/B we denote the **quotient space** obtained by identifying points in B, i.e. the set

$$\{B\} \cup \{\{x\} \mid x \in A \setminus B\}$$

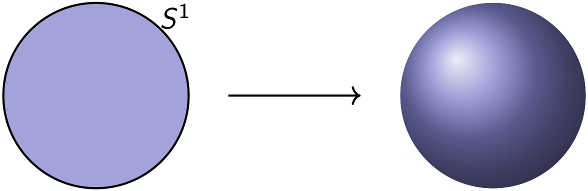
equipped with the topology defined by

$$V \subseteq A/B \text{ is open} \quad \Longleftrightarrow \quad \bigcup V \subseteq A \text{ is open.}$$



$$[0,1]/\{0,1\} \cong S^1$$

Quotients and computable type



B^2 $\qquad$ $B^2/S^1 \cong S^2$

(B^2, S^1) has computable type $\qquad$ S^2 has computable type

Problem statement

Does any of the implications

$$(A, B) \text{ has computable type} \quad \Rightarrow \quad A/B \text{ has computable type}$$

and

$$A/B \text{ has computable type} \quad \Rightarrow \quad (A, B) \text{ has computable type}$$

hold in general?

A/B has computable type $\Rightarrow (A, B)$ has computable type

Generally, **NO** – A/A is a one-point set, therefore it has computable type for any A.

However, if the interior of B in A is empty, this implication is true.

Theorem

Let A be a topological space and let B, C be subsets of A such that $\operatorname{Int}_A B = \emptyset$ and $B \cap C = \emptyset$. If $(A/B, C)$ has computable type, then $(A, B \cup C)$ also has computable type.

Sketch of proof

The main idea of the proof is: given a computable topological space $(X, \mathcal{T}, (I_i))$ and a semicomputable set B in it, we can define a structure of computable topological space on the quotient space X/B such that the natural quotient map preserves (semi)computability.

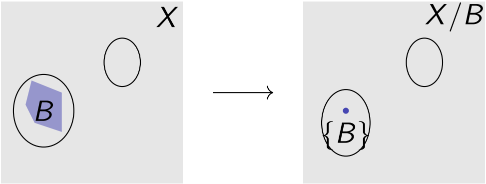
More precisely, we want to define a new computable topological space $(Y, \mathcal{S}, (I'_i))$ such that

1. $(Y, \mathcal{S})$ is the space X/B (with the quotient topology)
2. if S is semicomputable in $(X, \mathcal{T}, (I_i))$, then S/B is semicomputable in $(Y, \mathcal{S}, (I'_i))$
3. if $B \subseteq S$, $\text{Int}_S B = \emptyset$ and S/B is c.e. in $(Y, \mathcal{S}, (I'_i))$, then S is c.e. in $(X, \mathcal{T}, (I_i))$.

Sketch of proof

We consider the c.e. set

$$\Omega = \{j \in \mathbb{N} \mid B \subseteq J_j\} \cup \{j \in \mathbb{N} \mid (\exists k \in \mathbb{N})(B \subseteq J_k \wedge (j,k) \in D)\}.$$

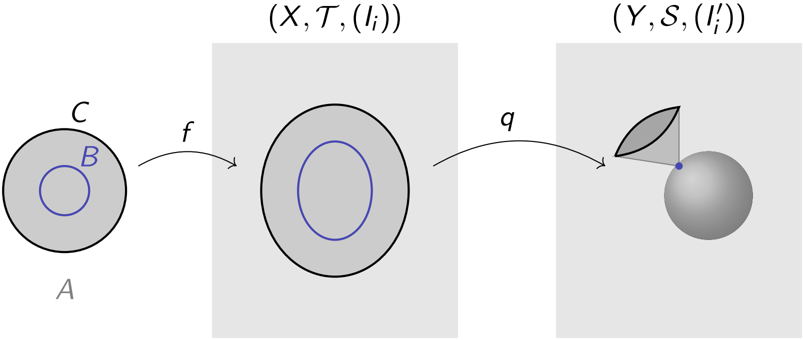


Let $\phi : \mathbb{N} \to \Omega$ be a recursive surjection. The sequence $(I_i)'$ defined by

$$I_i' = J_{\phi(i)}/B$$

forms a basis for the quotient topology such that $(Y, \mathcal{S}, (I_i'))$ is a computable topological space with the desired properties.

Sketch of proof



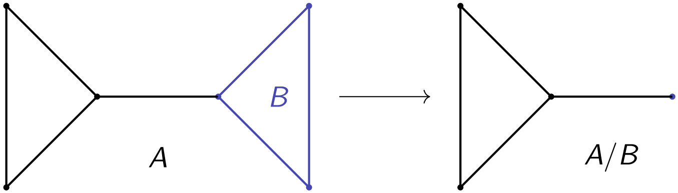
$f(A), f(B), f(C)$ semicomputable $\Rightarrow$ $f(A)/f(B), f(C)$ semicomputable

$\Downarrow$

$f(A)$ c.e. $\Leftarrow$ $f(A)/f(B)$ c.e.

(A, B) has computable type $\Rightarrow A/B$ has computable type

Again, this generally does not hold.

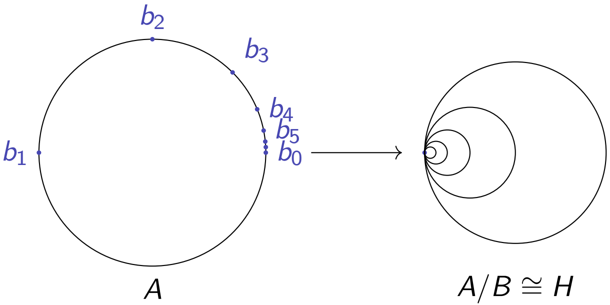


(A, B) has computable type $\qquad A/B$ does not have computable type

Could we consider some additional conditions: B is 'sufficiently small', $\{B\} \in A/B$ is computable?

Counterexample

Let $A = S^1$ and let $B = \{b_i \mid i \in \mathbb{N}\}$ where $b_i = (\cos \frac{2\pi}{2^i}, \sin \frac{2\pi}{2^i})$.



The space A/B is homeomorphic to the **Hawaiian earring** H – a union of circles in the Euclidean plane with center $(\frac{1}{n+1}, 0)$ and radius $\frac{1}{n+1}$.

$(H, \{(0,0)\})$ does not have computable type

Let (λ_i) be a computable sequence of real numbers such that $\lambda_i = 0$ is not decidable and such that $0 \leq \lambda_i \leq \frac{1}{4}$ for each $i \in \mathbb{N}$ ([7]).
Let $O = (0,0)$ and for each $i \in \mathbb{N}$ let

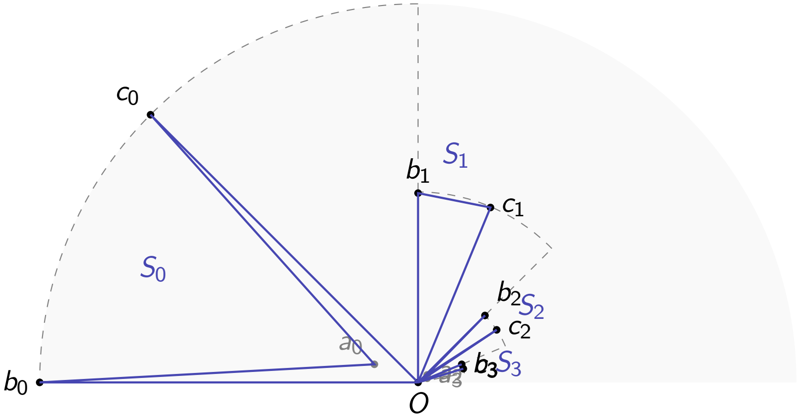
$$b_i = \frac{1}{2^i} E\left(\frac{1}{2^i}\right), \quad c_i = \frac{1}{2^i} E\left(\frac{3}{2^{i+2}}\right) \quad \text{and} \quad a_i = \frac{\lambda_i}{2^i} E\left(\frac{7}{2^{i+3}}\right)$$

where $E(t) = (\cos \pi t, \sin \pi t)$.
For $i \in \mathbb{N}$ let

$$T_i = \overline{Ob_i} \cup \overline{b_i c_i} \cup \overline{Oc_i}, \quad T_i' = \overline{Ob_i} \cup \overline{a_i b_i} \cup \overline{a_i c_i} \cup \overline{Oc_i}$$

and

$$S_i = \begin{cases} T_i', & \text{if } \lambda_i > 0, \\ T_i, & \text{if } \lambda_i = 0 \end{cases} .$$



c_0
S_1
b_1
c_1
S_0
b_2
S_2
c_2
a_0
b_3
S_3
b_0
O

Let

$$S = \bigcup_{i \in \mathbb{N}} S_i.$$

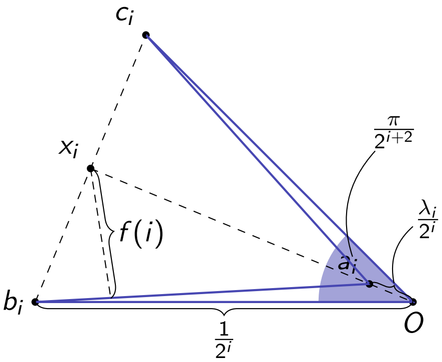
This is a co-c.e. set in $\mathbb{R}^2$ (and therefore semicomputable, [3]), and it is homeomorphic to H.

Consider the computable sequence (x_i) in $\mathbb{R}^2$ defined by

$$x_i = \frac{1}{2}(b_i + c_i)$$

and the function

$$f : \mathbb{N} \to \mathbb{R}, \quad f(i) = d(x_i, S).$$



Note that

$$f(i) = d(x_i, S_i) = \begin{cases} 0, & \text{if } \lambda_i = 0 \\ d(x_i, \overline{a_i b_i}), & \text{if } \lambda_i > 0. \end{cases}$$

Furthermore, note that

$$d(x_i, \overline{a_i b_i}) \geq \underbrace{\frac{\sqrt{2}}{2} \min\{d(x_i, a_i), d(x_i, b_i)\}}_{:=g(i)}.$$

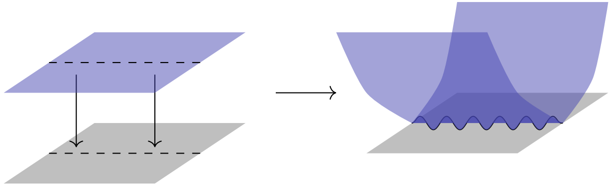
Obviously $g : \mathbb{N} \to \mathbb{R}$ is a strictly positive computable function, so we have

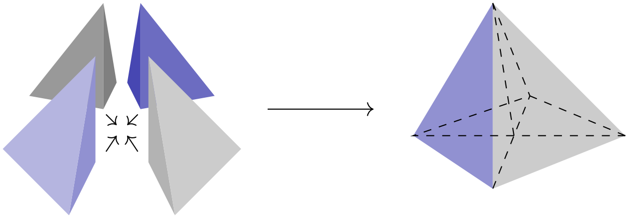
$$\mathbb{N} \setminus \left\{ i \in \mathbb{N} \mid f(i) > \frac{g(i)}{2} \right\} = \left\{ i \in \mathbb{N} \mid f(i) < \frac{g(i)}{2} \right\} = \{ i \in \mathbb{N} \mid \lambda_i = 0 \}.$$

We can see that the set S cannot be computable – otherwise f would be a computable function, which would imply that the above set is computable, contradictory to the choice of (λ_i).

Conclusion and further research motivation

- A/B has computable type $\Rightarrow$ (A, B) has computable type holds whenever the interior of B in A is empty
- The converse need not be true, even if the interior of B in A is empty
- It is possible to consider more specific families of topological spaces; for example, some interesting results can be proven for (more general) quotients of topological mainfolds





References I

- V. Brattka and G. Presser, *Computability on subsets of metric spaces*, Theoretical Computer Science (2003), no. 305, 43–76.
- Marko Horvat, Zvonko Iljazović, and Bojan Pažek, *Computability of pseudo-cubes*, Annals of Pure and Applied Logic **171** (2020), no. 8, 102823.
- Z. Iljazović, *Compact manifolds with computable boundaries*, Logical Methods in Computer Science **9** (2013), no. 4:19, 1–22.
- Z. Iljazović, *Computability of graphs*, Mathematical Logic Quarterly **66** (2020), 51–64.
- Z. Iljazović and I. Sušić, *Semicomputable manifolds in computable topological spaces*, Journal of Complexity **45** (2018), 83–114.
- J.S. Miller, *Effectiveness for embedded spheres and balls*, Electronic Notes in Theoretical Computer Science **66** (2002), 127–138.

References II

M.B. Pour-El and J.I. Richards, *Computability in analysis and physics*, Springer, Berlin, 1989.

E. Specker, *Der satz vom maximum in der rekursiven analysis*, Constructivity in Mathematics (A. Heyting, ed.), North Holland Publ. Comp., Amsterdam, 1959, pp. 254–265.

E. Čičković, Z. Iljazović, and L. Validžić, *Chainable and circularly chainable semicomputable sets in computable topological spaces*, Archive for Mathematical Logic **58** (2019), 885–897.

K. Weihrauch, *Computable analysis*, Springer, Berlin, 2000.

______, *Computable separation in topology, from T_0 to T_2*, Journal of Universal Computer Science **16** (2010), no. 18, 2733–2753.

K. Weihrauch and T. Grubba, *Elementary computable topology*, Journal of Universal Computer Science **15** (2009), no. 6, 1381–1422.

Thank you!