Computability of non-compact manifolds

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A set S in a computable metric space is *semicomputable* if

(i) its intersection with any closed ball is compact; and

(ii) it is possible to effectively enumerate all finite unions of basic open balls which cover $S \cap \overline{B}_i$, uniformly over rational closed balls $\overline{B}_i$.

If it is also possible to effectively enumerate all basic open balls that intersect S, we say that S is *computable*. Topological properties can play an important role in determining computability of a semicomputable set. Namely, if any semicomputable set homeomorphic to some space A is computable, we say that A has *computable type*. More generally, topological pair (A, B) of a space A and its subspace B is said to have computable type if, whenever $f : A \to X$ is an embedding of A in a computable metric space such that $f(A)$ and $f(B)$ are semicomputable, then $f(A)$ is computable.

The study of computable type is usually restricted to compact spaces, most notably manifolds and simplicial complexes [3, 1]. However, a more general approach can yield similar results for non-compact spaces. So far, in [2] it was shown that a semicomputable 1-manifold in a computable metric space must be computable, and this was later extended to generalized graphs [5]. In [4], it was shown that a semicomputable manifold M (of arbitrary dimension) is computable if there exists a relatively compact open set U in M such that $M \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B^n$.

In this talk, we begin with a survey of techniques used to obtain aforementioned results for non-compact spaces. Next, we focus on the notion of pseudocompactification from [4], which allows us to utilize familiar results from the compact setting. Then, we consider *neighborhoods of infinity* (that is, complements of compact sets) of non-compact manifolds, and show how topological properties of neighborhoods of infinity influence computable type. We prove the following:

Theorem 1. *Suppose S is a semicomputable set in a computable metric space which contains a closed neighborhood of infinity homeomorphic to $Q \times [0, 1\rangle$ for* *some compact space Q. If the cone $\mathsf{C}(Q)$ has local computable type, then S is computable at infinity.*

As a direct consequence, we get that the infinite cylinder $S^1 \times \mathbb{R}$ (and $S^n \times \mathbb{R}$ in general) has computable type.

This talk is based on joint work with Zvonko Iljazović.

References

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