Computable approximations of semicomputable graphs

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This talk is based on joint work with Vedran Čačić, Marko Horvat and Zvonko Iljazović.

In recent years, many works have examined the conditions which render semicomputable subsets of computable metric spaces computable [1, 2, 3, 4, 8].

Here, we focus on a more general problem: under which circumstances is it possible to approximate a semicomputable set by a computable one? This question was recently studied for chainable Hausdorff continua [9, 6]. In this talk, we study (generalized) topological graphs, which are obtained by gluing arcs and rays together at their endpoints. Our goal is to prove that a semicomputable generalized graph can be approximated with arbitrary precision by a computable subset which is itself a generalized graph with computable endpoints. The method we use ensures this is done by only "cutting out" small sections around non-computable endpoints and preserving all computable ones.

An important auxiliary result is the following: any point in a semicomputable set S which has a neighbourhood in S homeomorphic to $\mathbb{R}$ also has a neighbourhood in S which is a computable arc with computable endpoints. A similar result, proven in [7], states that any point x in a semicomputable set S which has a Euclidean neighbourhood in S also has a computable compact neighbourhood in S. Although it is very useful, this result does not guarantee that the computable neighbourhood is an arc with computable endpoints. On the other hand, since computable points are dense in computable sets, it is possible to find a neighbourhood of x which is an arc with computable endpoints, but this neighbourhood will not necessarily be computable, or even semicomputable.

In order to avoid these issues, our approach is to construct a neighbourhood as an intersection of a sequence of chains whose links have strictly decreasing diameters. This guarantees that our neighbourhood will simultaneously be computable and an arc with computable endpoints.

Another result from the literature states that a topological pair of a graph and the set of all its endpoints has computable type [5]. Together with the result mentioned above, this allows us to prove our main result:

Theorem 1. *Let (X, d, α) be a computable metric space and let S be a semicomputable generalized graph in this space. Then for each $\varepsilon > 0$ there exists a computable generalized graph T in (X, d, α) such that all endpoints of T are computable and such that* $T \subseteq S \subseteq \bigcup_{x \in T} B(x, \varepsilon)$.

Since any 1-manifold is (a homeomorphic copy of) a generalized graph, the following is an immediate consequence:

Corollary 2. *Let (X, d, α) be a computable metric space and let M be a semicomputable 1-manifold in this space such that M has finitely many connected components. Then for each $\varepsilon > 0$ there exists a computable 1-manifold N in (X, d, α) such that each point of ∂N is computable and such that $N \subseteq M \subseteq \bigcup_{x \in N} B(x, \varepsilon)$.*

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