Non-compact manifolds and computable type

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A set S in a computable metric space is *semicomputable compact* if it is compact and it is possible to effectively enumerate all finite unions of basic open balls which cover S. If it is also possible to effectively enumerate all basic open balls which intersect S, we say that S is *computable compact*. Topology plays an important role in determining the relationship between semicomputability and computability. This is expressed in the notion of *computable type*: topological space A is said to have computable type if any semicomputable compact set in a computable metric space which is homeomorphic to A is necessarily computable compact. More generally, topological pair (A, B) of a space A and its subspace B has computable type if, whenever $f : A \to X$ is an embedding of A in a computable metric space such that $f(A)$ and $f(B)$ are semicomputable compact, then $f(A)$ is computable compact. In recent years, results regarding computable type have been obtained for certain classes of topological spaces, most notably compact manifolds and simplicial complexes [Ilj13, AH22].

In view of the above definitions, the study of computable type is restricted to compact spaces. However, using a more general notion of semicomputability can yield similar results for non-compact spaces. A set S in a computable metric space is *semicomputable* if its intersection with any closed ball is compact and it is possible to effectively enumerate all finite unions of basic open balls which cover $S \cap \hat{B}_i$, uniformly over rational closed balls $\hat{B}_i$. In [BI14], it was shown that a semicomputable 1-manifold in a computable metric space must be computable. This approach was later extended to generalized graphs [Ilj20]. In [IS18], it was shown that a semicomputable manifold M (of arbitrary dimension) is computable if there exists a relatively compact open set U in M such that $M \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B^n$.

In this talk, we survey the techniques used to obtain the aforementioned results and develop them further to study more general non-compact manifolds. In particular, we consider *neighborhoods of infinity* (that is, complements of compact sets) of non-compact manifolds, and show how topological properties of neighborhoods of infinity influence computable type. For example, we show that the infinite cylinder $S^1 \times \mathbb{R}$ (and $S^n \times \mathbb{R}$ in general) has computable type.

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