Computable type of an unglued space

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A compact subset S of Euclidean space $\mathbb{R}^n$ is computable if S can be effectively approximated by a finite set of rational points with any given precision. On the other hand, a compact subset S of $\mathbb{R}^n$ is semicomputable if $\mathbb{R}^n \setminus S$ can be effectively exhausted by rational balls.

A semicomputable set need not be computable, in fact there exists a nonempty semicomputable subset of $\mathbb{R}$ which does not contain any computable point [4]. However, it turns out that under certain conditions we can conclude that a semicomputable set is computable. Topology plays an important role in this regard and that was first shown in [3]: a semicomputable set in $\mathbb{R}^n$ is computable if it is homeomorphic to the unit sphere S^m for some $m \in \mathbb{N}$. A semicomputable set homeomorphic the the unit ball B^m need not be computable, but the following result was proved in [3]: if S is semicomputable in $\mathbb{R}^n$ and $f : B^m \to S$ is a homeomorphism such that $f(S^{m-1})$ is also semicomputable in $\mathbb{R}^n$, then S is computable.

The notions of a computable and a semicomputable set can be easily defined in a more general ambient space - a computable topological space. The question is under what conditions the implication

$$S \text{ semicomputable } \Rightarrow S \text{ computable} \tag{1}$$

holds in a computable topological space X.

We say that a topological space A has computable type if (1) holds whenever S is homeomorphic to A (in any computable topological space X). It is known that S^m has computable type (for any m) and, in fact, any compact manifold has computable type [2]. On the other hand, $[0, 1]$ does not have computable type.

Let B be a subspace of a topological space A. We say that the pair (A, B) has computable type if (1) holds whenever there exists a homeomorphism $f : A \to S$ such that $f(B)$ is a semicomputable set in X. It is known that (B^n, S^{n-1}) has computable type, moreover $(M, \partial M)$ has computable type for any compact manifold M with boundary [2]. In particular, $([0, 1], \{0, 1\})$ has computable type.

In this paper we consider a subspace B of a topological space A and the space A/B obtained from A by identifying x and y for any $x, y \in B$. More precisely, A/B is a quotient space of A determined by the partition $\{\{x\} \mid x \in A \setminus B\} \cup \{B\}$. It is a well-known topological fact that B^m/S^{m-1} is homeomorphic to S^m, in particular $[0, 1]/\{0, 1\}$ is homeomorphic to S^1 (see Figure 1).

We prove the following.

Theorem 1. *Let B be a subspace of a topological space A such that the interior of B in A is empty. Suppose A/B has computable type. Then (A, B) has computable type.*

For example, as a consequence of Theorem 1, we immediately get that the fact that S^m has computable type implies that (B^m, S^{m-1}) has computable type.

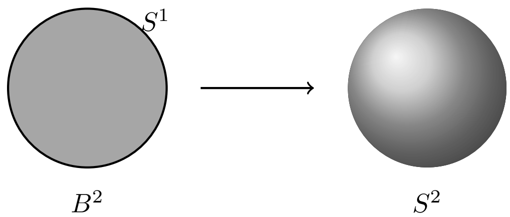

Figure 1: S^2 is obtained from B^2 by identifying points on S^1

Note that the assumption in Theorem 1 that the interior of B in A is nonempty cannot be omitted. Namely, for any topological space A we have that A/A has computable type (since A/A is a one-point set), but (A, A) does not have computable type in general (for example if $A = [0, 1]$).

We also examine some further connections between computability of a space and its quotient space. In particular, we examine computability of a space obtained by gluing two spaces together along certain subspaces.

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