Computability and adjunctions of Euclidean spaces

Matea Čelar and Zvonko Iljazović

A compact set S in a computable topological space $(X, \mathcal{T}, (I_i))$ is computably enumerable if we can effectively enumerate all base elements I_i which intersect S. A compact set S is semicomputable if we can effectively enumerate all finite unions of base elements which contain S. We say that a set is computable in a computable topological space if it is computably enumerable and semicomputable. This definition corresponds with the classical definition of a computable set in the Euclidean space and, more generally, in computable metric spaces.

We examine topological conditions under which semicomputability of a set implies its computable enumerability and, therefore, computability. For certain sets S it is sufficient to study such conditions locally around each point $x \in S$. This follows from the fact that computable enumerability is, in a way, a local property: every compact locally computably enumerable set is computably enumerable. A set is locally computably enumerable if it is computably enumerable at every point x. A set S is computably enumerable at x if there exists a neighborhood U of x in S and a c.e. set $\Omega \subseteq \mathbb{N}$ such that

$$I_i \cap U \neq \emptyset \;\Rightarrow\; i \in \Omega \quad \text{and} \quad i \in \Omega \;\Rightarrow\; I_i \cap S.$$

In this case we say that U is computably enumerable up to S. For example, if a point $x \in S$ has a neighborhood in S homeomorphic to $\mathbb{R}^n$, then S is computable at x. This implies that every semicomputable manifold in a computable topological space is computable. Similarly, it holds that a semicomputable manifold with semicomputable boundary is computable [1].

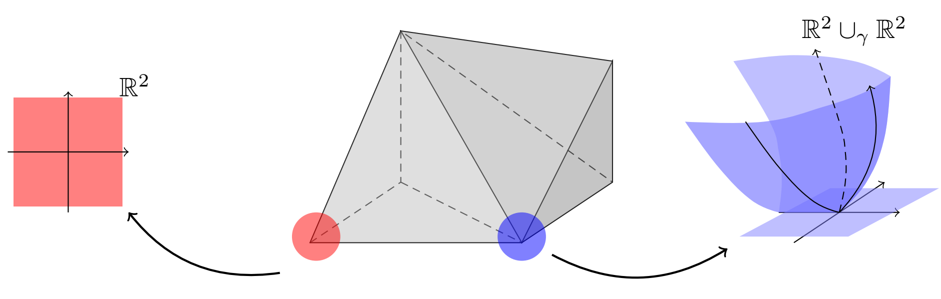

Figure 1: Neighborhoods of points in a polyhedron

We want to expand this result to a more general family of sets – for example, sets with topological type of polyhedra. Let P be a semicomputable topological polyhedron in a computable topological space. Some points in P have neighborhoods homeomorphic to $\mathbb{R}^n$ or to upper half-space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \mid x_n \geq 0\}$. The results mentioned above guarantee that P is computably enumerable at every such point. However, other points in P have neighborhoods homeomorphic to multiple copies of $\mathbb{R}^n$ or $\mathbb{H}^n$ attached (or "glued together") along some subspace (Figure 1). In certain such cases, a similar result holds.

The concept of attaching two spaces together is formalized by the notion of adjunction space. Let X and Y be topological spaces, let A be a subspace of X and let $f : A \to Y$ be a continuous function. By $X \cup_f Y$ we denote the space obtained from the disjoint union of X and Y by identifying a with $f(a)$ for all $a \in A$, and we call this space adjunction space. We have the following theorem.

Theorem 1. *Let $n \geq 2$ and let A be a closed subset of $\mathbb{R}^{n-1} \times \{0\}$ such that $(0, \ldots, 0) \in A$. Let Y be a locally compact topological space and let $\gamma : A \to Y$ be an embedding such that $\gamma(A)$ is closed in Y. Suppose $(X, \mathcal{T}, (I_i))$ is a computable topological space and $f : \mathbb{R}^n \cup_\gamma Y \to X$ is an embedding such that $f(\mathbb{R}^n \cup_\gamma Y)$ is an open subset of a semicomputable set S. Then the set $f([-1, 1]^n)$ is c.e. up to S.*

References

[1] Z. Iljazović and I. Sušić. Semicomputable manifolds in computable topological spaces. *Journal of Complexity*, 45:38-114, 2018.