# PRINCIPAL SUBSPACES FOR THE AFFINE LIE ALGEBRAS IN TYPES $D$, $E$ AND $F$

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ABSTRACT. We consider the principal subspaces of certain level $k \geqslant 1$ integrable highest weight modules and generalized Verma modules for the untwisted affine Lie algebras in types $D$, $E$ and $F$. Generalizing the approach of G. Georgiev we construct their quasi-particle bases. We use the bases to derive presentations of the principal subspaces, calculate their character formulae and find some new combinatorial identities.

## 1. INTRODUCTION

Starting with J. Lepowsky and S. Milne [30], the fascinating connection between Rogers–Ramanujan-type identities and affine Kac–Moody Lie algebras was extensively studied; see, e.g., [31–33, 35] and references therein. The principal subspaces of standard modules, i.e. of integrable highest weight modules for the affine Lie algebras, introduced by B. L. Feigin and A. V. Stoyanovsky [16], present a remarkable example of this interplay between combinatorics and algebra. In particular, their so-called quasi-particle bases provide an interpretation of the sum sides of various Rogers–Ramanujan-type identities; see [4–7, 16, 20, 34]. Aside from quasi-particle bases, numerous research directions are focused on other aspects of principal subspaces and related structures such as certain generalized principal subspaces [2], Feigin–Stoyanovsky's type subspaces [3, 22, 38], realizations of Jack symmetric functions [8], presentations of principal subspaces [9–12, 36, 37, 39, 40], Rogers–Ramanujan-type recursions [13, 14], Koszul complexes [24], principal subspaces for quantum affine algebras and double Yangians [26–28] etc. The key ingredient that all the aforementioned studies have in common is the application of vertex-operator theoretic methods.

Let $\Lambda_0, \ldots, \Lambda_l$ be the fundamental weights of the untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$ associated with the simple Lie algebra $\mathfrak{g}$ of rank $l$. In this paper, we consider the principal subspaces $W_{N(k\Lambda_0)}$ of the generalized Verma modules $N(k\Lambda_0)$ and the principal subspaces $W_{L(k\Lambda_0)}$ of the standard modules $L(k\Lambda_0)$ of highest weights $k\Lambda_0$ for $\widetilde{\mathfrak{g}}$ in types $D$, $E$ and $F$. The main result is a construction of the quasi-particle bases $\mathfrak{B}_{N(k\Lambda_0)}$ and $\mathfrak{B}_{L(k\Lambda_0)}$ of the corresponding principal subspaces. It is presented in Theorem 3.1, which we formulate so

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that it includes the corresponding bases of the principal subspaces $W_{N(k\Lambda_0)}$ and $W_{L(k\Lambda_0)}$ for all untwisted affine Lie algebras $\widetilde{\mathfrak{g}}$. The bases in the remaining types $A$, $B$, $C$ and $G$ were given by several authors, as we explain below.

**Theorem 3.1.** *For any positive integer $k$ the set $\mathfrak{B}_V$ forms a basis of the principal subspace $W_V$ of the $\widetilde{\mathfrak{g}}$-module $V = N(k\Lambda_0), L(k\Lambda_0)$.*

The bases $\mathfrak{B}_V$ are expressed in terms of monomials of certain operators, called quasi-particles, applied on the highest weight vector, whose charges and energies satisfy certain difference conditions. Theorem 3.1 for $\mathfrak{g}$ of type $A_1$ goes back to Feigin and Stoyanovsky [16]. Next, G. Georgiev [20] constructed the quasi-particle bases of the principal subspaces $W_{L(\Lambda)}$, where $\mathfrak{g} = A_l$, for all rectangular weights $\Lambda$, i.e. for all integral dominant highest weights $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$. Finally, the bases $\mathfrak{B}_V$ from Theorem 3.1, where $V = N(k\Lambda_0), L(k\Lambda_0)$ and $\mathfrak{g} = B_l, C_l, G_2$, were obtained by the first author in [4–6]. The quasi-particle bases of the principal subspaces $W_{L(\Lambda_i)}$ for $\mathfrak{g} = A_l$ and $i = 0, 1, \ldots, l$ can be also recovered from the recent result of K. Kawasetsu [25]. Our proof of Theorem 3.1 in types $D$, $E$ and $F$ follows the approach in [20] and relies on [4, 5, 22]. In addition to Theorem 3.1, in Theorem 3.2 we construct quasi-particle bases of the principal subspaces $W_{L(\Lambda)}$ for all rectangular highest weights $\Lambda$ in types $D$ and $E$, thus generalizing [20].

Next, in Theorem 4.1, we derive presentations of the principal subspaces $W_{L(k\Lambda_0)}$ for all types of $\mathfrak{g}$, i.e. we give the vector space isomorphisms

$$W_{L(k\Lambda_0)} \cong U(\mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}])/I_{L(k\Lambda_0)},$$

where $\mathfrak{n}_+$ is the subalgebra of $\mathfrak{g}$ spanned by all positive root vectors and $I_{L(k\Lambda_0)}$ is a certain left ideal in $U(\mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}])$. Moreover, we provide explicit formulae for the generators of $I_{L(k\Lambda_0)}$. The presentations of principal subspaces of standard $\widetilde{\mathfrak{g}}$-modules $L(\Lambda)$ for the level $k$ integral dominant highest weights $\Lambda$ were established by Feigin and Stoyanovsky [16] for $\mathfrak{g} = A_1$ and $k = 1$. Furthermore, the presentations were proved by C. Calinescu, Lepowsky and A. Milas [9–11] for $\mathfrak{g} = A_1$ and $k \geqslant 1$ and for $\mathfrak{g} = A, D, E$ and $k = 1$, and by C. Sadowski [39] for $\mathfrak{g} = A_2$ and $k \geqslant 1$. The proofs in [9–11, 39] are sometimes referred to as *a priori proofs* as they do not rely on the detailed underlying structure, such as bases of the standard modules or of the principal subspaces. Finally, Sadowski [40] proved the general case $\mathfrak{g} = A_l$ for all $k \geqslant 1$ using Georgiev's quasi-particle bases [20]. In contrast with [9–11, 39], our proof employs the sets $\mathfrak{B}_{L(k\Lambda_0)}$ from Theorem 3.1, thus solving a simpler problem. In addition, using the quasi-particle bases from Theorem 3.2 we obtain presentations of the principal subspaces $W_{L(\Lambda)}$ for all rectangular highest weights $\Lambda$ in types $D$ and $E$; see Theorem 4.2. It is worth noting that, aside from the aforementioned cases covered in [9–11, 39], the a priori proof of these presentations is still lacking.

In the end, we use the bases from Theorems 3.1 and 3.2 to explicitly write the character formulae for the principal subspaces. Moreover, let $R_+$ be the set of positive roots of $\mathfrak{g}$ and let $\mu_i = \nu_i/\nu_{i'}$, where the numbers $\nu_j$ and $i'$ are given by (3.3). By regarding two different bases for $W_{N(k\Lambda_0)}$ we find

**Theorem 7.3.** *For any untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$ we have*

$$\frac{1}{\prod_{\alpha\in R_+}(\alpha;q)_\infty} = \sum_{\substack{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(m)}\geqslant\cdots\geqslant 0\\ \vdots \\ r_l^{(1)}\geqslant\cdots\geqslant r_l^{(m)}\geqslant\cdots\geqslant 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t\geqslant 1} {r_i^{(t)}}^2 - \sum_{i=2}^{l}\sum_{t\geqslant 1}\sum_{p=0}^{\mu_i-1} r_{i'}^{(t)} r_i^{(\mu_i t-p)}}}{\prod_{i=1}^{l}\prod_{j\geqslant 1}(q;q)_{r_i^{(j)}-r_i^{(j+1)}}} \prod_{i=1}^{l} y_i^{n_i},$$

*where $n_i = \sum_{t\geqslant 1} r_i^{(t)}$ for $i=1,\ldots,l$ and the sum on the right hand side goes over all descending infinite sequences of nonnegative integers with finite support.*

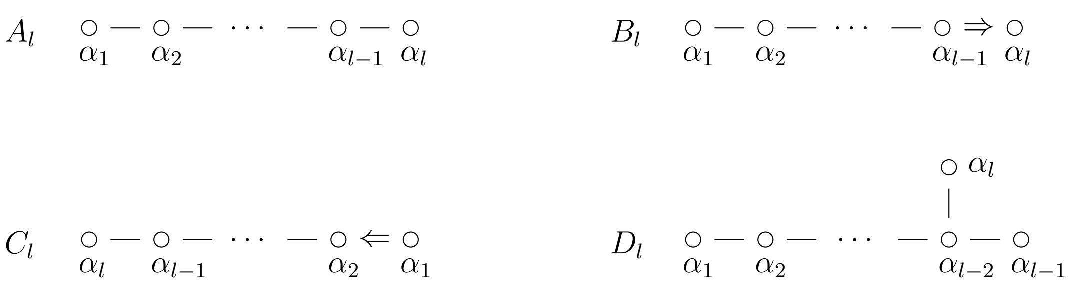
The theorem produces three new families of combinatorial identities which correspond to types $D$, $E$ and $F$, while the remaining identities, for types $A$, $B$, $C$ and $G$, are already well-known; see [4–6, 20].

## 2. Preliminaries

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $l$ equipped with a nondegenerate invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ and let $\mathfrak{h}$ be its Cartan subalgebra. As the restriction of the form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ is nondegenerate, it defines a symmetric bilinear form on the dual $\mathfrak{h}^*$. Let $\Pi = \{\alpha_1,\ldots,\alpha_l\}\subset\mathfrak{h}^*$ be the basis of the root system $R$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let $x_\alpha\in\mathfrak{g}$ with $\alpha\in R$ be the root vectors. The simple roots $\alpha_1,\ldots,\alpha_l$ are labelled[a] as in Figure 1. We denote by $\alpha_1^\vee,\ldots,\alpha_l^\vee$ the corresponding simple coroots. Let $\lambda_1,\ldots,\lambda_l\in\mathfrak{h}^*$ be the fundamental weights, i.e. the weights such that $\langle\lambda_i,\alpha_j^\vee\rangle=\delta_{ij}$. Let $Q=\sum_{i=1}^{l}\mathbb{Z}\alpha_i$ and $P=\sum_{i=1}^{l}\mathbb{Z}\lambda_i$ be the root lattice and the weight lattice of $\mathfrak{g}$ respectively. We assume that the form $\langle\cdot,\cdot\rangle$ is normalized so that $\langle\alpha,\alpha\rangle=2$ for every long root $\alpha\in R$. Hence, in particular, we have $\langle\alpha_i,\alpha_i\rangle\in\{2/3,1,2\}$ for all $i=1,\ldots,l$. Denote by $R_+$ and $R_-$ the sets of positive and negative roots. Let

$$\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+,\qquad\text{where}\qquad \mathfrak{n}_\pm=\bigoplus_{\alpha\in R_\pm}\mathfrak{n}_\alpha\quad\text{and}\quad\mathfrak{n}_\alpha=\mathbb{C}x_\alpha\ \text{for all}\ \alpha\in R,$$

be the triangular decomposition of $\mathfrak{g}$; see [21] for more details on simple Lie algebras.




[a] In contrast with [21] and [23, Table Fin], we reverse the labels in the Dynkin diagram of type $C_l$ in Figure 1, so that the root lengths in the sequence $\alpha_1,\ldots,\alpha_l$ decrease for all types of $\mathfrak{g}$, thus getting a simpler formulation of Theorem 3.1.

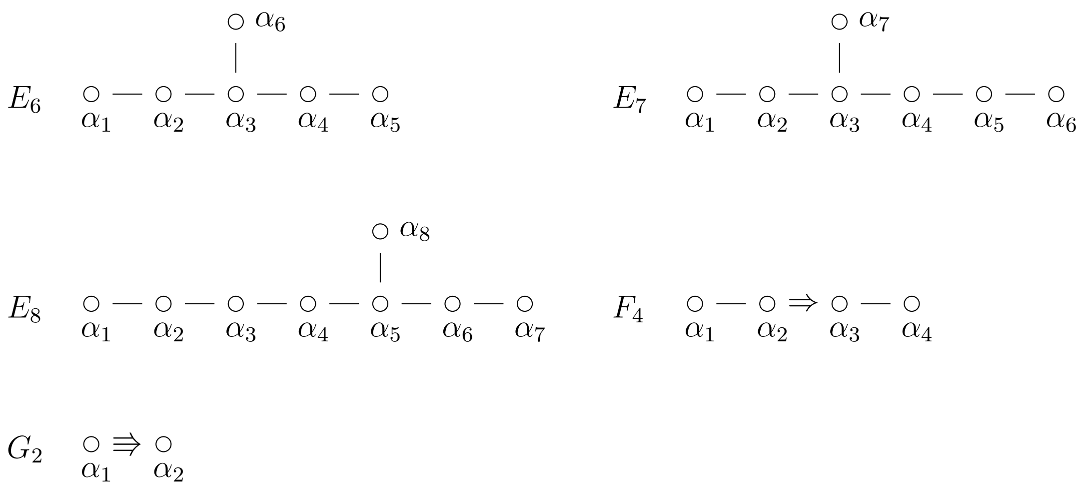



FIGURE 1. Finite Dynkin diagrams

The affine Kac–Moody Lie algebra $\widetilde{\mathfrak{g}}$ associated to $\mathfrak{g}$ is defined by

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where the elements $x(m) = x \otimes t^m$ for $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$ are subject to relations

$$[c, \widetilde{\mathfrak{g}}] = 0, \qquad [d, x(m)] = mx(m),$$
$$[x(m), y(n)] = [x, y]\,(m + n) + \langle x, y\rangle\, m\delta_{m+n\,0}\, c. \tag{2.1}$$

We denote by $\alpha_0, \alpha_1, \ldots, \alpha_l$ and $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee$ the simple roots and the simple coroots of $\widetilde{\mathfrak{g}}$. Let $\Lambda_i$ be the fundamental weights of $\widetilde{\mathfrak{g}}$, i.e. the weights such that $\Lambda_i(d) = 0$ and $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$ for all $i, j = 0, \ldots, l$. For more details on affine Lie algebras see [23].

Let $k_0, \ldots, k_l$ be nonnegative integers such that $k = k_0 + \ldots + k_l$ is positive and let $\lambda = k_1\lambda_1 + \ldots + k_l\lambda_l$. Denote by $U_\lambda$ the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight $\lambda$. The generalized Verma $\widetilde{\mathfrak{g}}$-module $N(\Lambda)$ of highest weight $\Lambda = k_0\Lambda_0 + \ldots + k_l\Lambda_l$ and of level $k$ is defined as the induced $\widetilde{\mathfrak{g}}$-module

$$N(\Lambda) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{g}}^{\geqslant 0})} U_\lambda,$$

where the action of the Lie algebra

$$\widetilde{\mathfrak{g}}^{\geqslant 0} = \bigoplus_{n \geqslant 0} (\mathfrak{g} \otimes t^n) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

on $U_\lambda$ is given by

$$\mathfrak{g} \otimes t^n \cdot u = 0 \text{ for all } n > 0, \quad c \cdot u = ku \quad \text{and} \quad d \cdot u = 0 \qquad \text{for all } u \in U_\lambda.$$

Denote by $L(\Lambda)$ the standard $\widetilde{\mathfrak{g}}$-module of highest weight $\Lambda$ and of level $k$, i.e. the integrable highest weight $\widetilde{\mathfrak{g}}$-module which equals the unique simple quotient of the generalized Verma module $N(\Lambda)$. In particular, for $\lambda = 0$ we obtain the generalized Verma $\widetilde{\mathfrak{g}}$-module $N(k\Lambda_0)$ of highest weight $k\Lambda_0$ and level $k = k_0$ which possesses a vertex operator algebra structure. Moreover, $L(k\Lambda_0)$ is a simple vertex operator algebra and the level $k$ standard

$\widetilde{\mathfrak{g}}$-modules are $L(k\Lambda_0)$-modules; see, e.g., [29, 33]. Finally, recall that Poincaré–Birkhoff–Witt theorem for the universal enveloping algebra implies the vector space isomorphism

$$N(k\Lambda_0) \cong U(\widetilde{\mathfrak{g}}^{<0}), \quad \text{where} \quad \widetilde{\mathfrak{g}}^{<0} = \bigoplus_{n<0} (\mathfrak{g} \otimes t^n).$$

For more details on the representation theory of affine Lie algebras see [23].

## 3. Quasi-particle bases of principal subspaces

In this section, we state our main results, Theorems 3.1 and 3.2.

### 3.1. Quasi-particles.

Introduce the following subalgebras of $\widetilde{\mathfrak{g}}$:

$$\widetilde{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}], \qquad \widetilde{\mathfrak{n}}_+^{\geqslant 0} = \mathfrak{n}_+ \otimes \mathbb{C}[t] \qquad \text{and} \qquad \widetilde{\mathfrak{n}}_+^{<0} = \mathfrak{n}_+ \otimes t^{-1}\mathbb{C}[t^{-1}].$$

Let $\Lambda$ be an arbitrary integral dominant weight of $\widetilde{\mathfrak{g}}$. Denote by $V$ the generalized Verma module $N(\Lambda)$ or the standard module $L(\Lambda)$ with a highest weight vector $v_V$. Following Feigin and Stoyanovsky [16], we define the *principal subspace* $W_V$ of $V$ by

$$W_V = U(\widetilde{\mathfrak{n}}_+) v_V.$$

Consider the vertex operators

$$x_{\alpha_i}(z) = \sum_{m \in \mathbb{Z}} x_{\alpha_i}(m) z^{-m-1} \in \operatorname{Hom}(V, V((z))) \subset (\operatorname{End} V)[[z^{\pm 1}]], \quad i = 1, \ldots, l.$$

Note that (2.1) implies $[x_{\alpha_i}(z_1), x_{\alpha_i}(z_2)] = 0$ so that

$$x_{n\alpha_i}(z) = \sum_{m \in \mathbb{Z}} x_{n\alpha_i}(m) z^{-m-n} = \underbrace{x_{\alpha_i}(z) \cdots x_{\alpha_i}(z)}_{n \text{ times}} = x_{\alpha_i}(z)^n \tag{3.1}$$

is a well-defined element of $\operatorname{Hom}(V, V((z)))$ for all $n \geqslant 1$. As in [20], define the *quasi-particle of color $i$, charge $n$ and energy $-m$* as the coefficient $x_{n\alpha_i}(m) \in \operatorname{End} V$ of (3.1).

Consider the quasi-particle monomial

$$b = \left( x_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l}) \ldots x_{n_{1,l}\alpha_l}(m_{1,l}) \right) \cdots \left( x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \ldots x_{n_{1,1}\alpha_1}(m_{1,1}) \right) \tag{m}$$

in $\operatorname{End} V$. Note that the quasi-particle colors in (m) are increasing from right to left and that the integers $r_j^{(1)} \geqslant 0$ with $j = 1, \ldots, l$ denote the parts of the conjugate partition of $n_j = n_{r_j^{(1)},j} + \cdots + n_{1,j}$; see [4–6, 20] for more details. It is convenient to write quasi-particle monomial (m) more briefly as

$$b = b_{\alpha_l} \cdots b_{\alpha_2} b_{\alpha_1}, \quad \text{where} \quad b_{\alpha_i} = x_{n_{r_i^{(1)},i}\alpha_i}(m_{r_i^{(1)},i}) \ldots x_{n_{1,i}\alpha_i}(m_{1,i}) \text{ for } i = 1, \ldots, l. \tag{3.2}$$

3.2. **Quasi-particle bases for $\Lambda = k\Lambda_0$.** Suppose that $\Lambda = k\Lambda_0$ for some positive integer $k$ so that $V$ denotes the generalized Verma module $N(k\Lambda_0)$ or the standard module $L(k\Lambda_0)$. We introduce certain difference conditions for energies and charges of quasi-particles in $(m)$. First, for the adjacent quasi-particles of the same color we require that

$$\text{for all} \quad i = 1,\ldots,l \quad \text{and} \quad p = 1,\ldots,r_i^{(1)} - 1$$
$$n_{p+1,i} \leqslant n_{p,i} \qquad \text{and} \qquad \text{if} \quad n_{p+1,i} = n_{p,i} \quad \text{then} \quad m_{p+1,i} \leqslant m_{p,i} - 2n_{p,i}. \qquad (c_1)$$

Next, we turn to the difference conditions which describe the interaction of two quasi-particles of adjacent colors. For all $i = 1,\ldots,l$ define

$$\nu_i = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \qquad \text{and} \qquad i' = \begin{cases} l-2, & \text{if } i = l \text{ and } \mathfrak{g} = D_l, \\ 3, & \text{if } i = l \text{ and } \mathfrak{g} = E_6, E_7, \\ 5, & \text{if } i = l \text{ and } \mathfrak{g} = E_8, \\ i-1, & \text{otherwise.} \end{cases} \qquad (3.3)$$

Introduce the following difference conditions:

$$\text{for all} \quad i = 1,\ldots,l \quad \text{and} \quad p = 1,\ldots,r_i^{(1)}$$
$$m_{p,i} \leqslant -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} \min\left\{ \tfrac{\nu_i}{\nu_{i'}} n_{q,i'}, n_{p,i} \right\} - 2(p-1)n_{p,i}, \qquad (c_2)$$

where we set $r_0^{(1)} = 0$ so that the sum in $(c_2)$ is zero for $i = 1$. In the end, we impose the following restrictions on the quasi-particle charges:

$$n_{p,i} \leqslant k\nu_i \quad \text{for all} \quad i = 1,\ldots,l \quad \text{and} \quad p = 1,\ldots,r_i^{(1)}. \qquad (c_3)$$

Let $B_{N(k\Lambda_0)}$ be the set of all monomials $(m)$, regarded as elements of $\operatorname{End} N(k\Lambda_0)$, satisfying conditions $(c_1)$ and $(c_2)$. Moreover, let $B_{L(k\Lambda_0)}$ be the set of all monomials $(m)$, regarded as elements of $\operatorname{End} L(k\Lambda_0)$, satisfying $(c_1)$, $(c_2)$ and $(c_3)$. Finally, let

$$\mathfrak{B}_V = \{ b v_V : b \in B_V \} \subset W_V \quad \text{for} \quad V = N(k\Lambda_0), L(k\Lambda_0).$$

**Theorem 3.1.** *For any positive integer $k$ the set $\mathfrak{B}_V$ forms a basis of the principal subspace $W_V$ of the $\widetilde{\mathfrak{g}}$-module $V = N(k\Lambda_0), L(k\Lambda_0)$.*

Even though Theorem 3.1 is formulated for an arbitrary untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$, we only give proof for $\mathfrak{g}$ of type $D$, $E$ and $F$; see Sections 5 and 6. The proofs for the remaining types can be found in [4–6, 20].

3.3. **Quasi-particle bases for rectangular weights in types $D$ and $E$.** Suppose that the affine Lie algebra $\widetilde{\mathfrak{g}}$ is of type $D_l^{(1)}$, $E_6^{(1)}$ or $E_7^{(1)}$. Let $\Lambda$ be the *rectangular weight*, i.e. the dominant integral weight of the form

$$\Lambda = k_0\Lambda_0 + k_j\Lambda_j, \qquad (3.4)$$

where $k_0, k_j$ are nonnegative integers and $\Lambda_j$ is the fundamental weight of level one; cf. [20]. Recall that $j = 1, l-1, l$ for $\widetilde{\mathfrak{g}} = D_l^{(1)}$, $j = 1, 6$ for $\widetilde{\mathfrak{g}} = E_6^{(1)}$ and $j = 1$ for $\widetilde{\mathfrak{g}} = E_7^{(1)}$; see [23]. Denote by $k = k_0 + k_j$ the level of $\Lambda$. Define

$$j_t = \begin{cases} 0, & \text{if} \quad 1 \leqslant t \leqslant k_0, \\ j, & \text{if} \quad k_0 < t \leqslant k_0 + k_j. \end{cases} \tag{3.5}$$

Introduce the following difference condition:

$$\text{for all} \quad i = 1, \ldots, l \quad \text{and} \quad p = 1, \ldots, r_i^{(1)}$$

$$m_{p,i} \leqslant -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} \min\left\{n_{q,i'}, n_{p,i}\right\} - 2(p-1)n_{p,i} - \sum_{t=1}^{n_{p,i}} \delta_{ij_t}. \tag{$c_2'$}$$

Note that this condition differs from ($c_2$) by a new term $\sum_{t=1}^{n_{p,i}} \delta_{ij_t}$. For a given rectangular weight $\Lambda$ denote by $B_{L(\Lambda)}$ be the set of all monomials ($m$), regarded as elements of $\operatorname{End} L(\Lambda)$, satisfying ($c_1$), ($c_2'$) and ($c_3$). Finally, let

$$\mathfrak{B}_{L(\Lambda)} = \left\{ b v_{L(\Lambda)} : b \in B_{L(\Lambda)} \right\} \subset W_{L(\Lambda)}.$$

**Theorem 3.2.** *Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra of type $D_l^{(1)}$, $E_6^{(1)}$ or $E_7^{(1)}$. For any rectangular weight $\Lambda$ the set $\mathfrak{B}_{L(\Lambda)}$ forms a basis of the principal subspace $W_{L(\Lambda)}$.*

The proof of Theorem 3.2 is given in Section 6.

## 4. Presentations of the principal subspaces $W_{L(k\Lambda_0)}$

In this section, we give the presentations of the principal subspaces $W_{L(k\Lambda_0)}$ for an arbitrary untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$; see Theorem 4.1 below. Next, in Theorem 4.2, we give the presentations of $W_{L(\Lambda)}$ for all rectangular weights $\Lambda$ in types $D$ and $E$. As pointed out in Section 1, the presentations of the principal subspaces of certain standard $\widetilde{\mathfrak{g}}$-modules in types $A$, $D$ and $E$ were originally found and proved in [9–11,16,39,40] while their general form was conjectured in [40].

Let $\Lambda$ be an integral dominant highest weight. Consider the natural surjective map

$$\begin{aligned} f_{L(\Lambda)} : U(\widetilde{\mathfrak{n}}_+) &\to W_{L(\Lambda)} \\ a &\mapsto a \cdot v_{L(\Lambda)}. \end{aligned} \tag{4.1}$$

For any $i = 1, \ldots, l$ and integer $m \geqslant k\nu_i + 1$ define the elements $R_{\alpha_i}(-m) \in U(\widetilde{\mathfrak{n}}_+)$ by

$$R_{\alpha_i}(-m) = \sum_{\substack{m_1, \ldots, m_{k\nu_i+1} \leqslant -1 \\ m_1 + \ldots + m_{k\nu_i+1} = -m}} x_{\alpha_i}(m_1) \ldots x_{\alpha_i}(m_{k\nu_i+1}).$$

Let $I_{L(k\Lambda_0)}$ be the left ideal in the universal enveloping algebra $U(\widetilde{\mathfrak{n}}_+)$ defined by

$$I_{L(k\Lambda_0)} = U(\widetilde{\mathfrak{n}}_+)\, \widetilde{\mathfrak{n}}_+^{\geqslant 0} + \sum_{i=1}^{l} \sum_{m \geqslant k\nu_i + 1} U(\widetilde{\mathfrak{n}}_+) R_{\alpha_i}(-m). \tag{4.2}$$

We have the following natural presentations of the principal subspaces:

**Theorem 4.1.** *For all positive integers $k$ we have*

$$\ker f_{L(k\Lambda_0)} = I_{L(k\Lambda_0)} \qquad \text{or, equivalently,} \qquad W_{L(k\Lambda_0)} \cong U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}.$$

In Section 5, we employ the sets $\mathfrak{B}_{L(k\Lambda_0)}$ from Theorem 3.1 to prove Theorem 4.1 for the affine Lie algebra $\widetilde{\mathfrak{g}} = F_4^{(1)}$. We omit the proof for other types of $\widetilde{\mathfrak{g}}$ since it goes analogously, by using the corresponding quasi-particle bases.

Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra of type $D_l^{(1)}$, $E_6^{(1)}$ or $E_7^{(1)}$. As in [40], for a given rectangular weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ define the left ideal in the universal enveloping algebra $U(\widetilde{\mathfrak{n}}_+)$ by

$$I_{L(\Lambda)} = I_{L((k_0+k_j)\Lambda_0)} + U(\widetilde{\mathfrak{n}}_+)x_{\alpha_j}(-1)^{k_0+1}. \tag{4.3}$$

**Theorem 4.2.** *Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra of type $D_l^{(1)}$, $E_6^{(1)}$ or $E_7^{(1)}$. For a given rectangular weight $\Lambda$ we have*

$$\ker f_{L(\Lambda)} = I_{L(\Lambda)} \qquad \text{or, equivalently,} \qquad W_{L(\Lambda)} \cong U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)}.$$

The proof of Theorem 4.2 is given in Section 6.

**Remark 4.3.** The form of the elements $R_{\alpha_i}(-m)$ is motivated by the integrability condition

$$x_{(k\nu_i+1)\alpha_i}(z) = 0 \quad \text{on any level } k \text{ standard module,} \tag{4.4}$$

which is due to Lepowsky and Primc [31]. It implies quasi-particle charges constraint ($c_3$).

## 5. Proof of Theorems 3.1 and 4.1 in type $F$

In this section, we prove Theorems 3.1 and 4.1 in type $F$. The proof is divided into six steps, i.e. Sections 5.1–5.6. We consider the affine Lie algebra $\widetilde{\mathfrak{g}}$ of type $F_4^{(1)}$ so that $l = 4$ and the basis $\Pi$ of the root system $R$ for the corresponding simple Lie algebra $\mathfrak{g}$ consists of the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$; see [21, Chap. III]. The maximal root $\theta$ equals

$$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \quad \text{and satisfies} \quad \alpha_i(\theta^\vee) = \delta_{1i} \text{ for } i = 1,2,3,4. \tag{5.1}$$

5.1. **Linear order on quasi-particle monomials.** In this section, we briefly cover some basic concepts originated in [20] which are typically used to handle quasi-particle monomials. In particular, we introduce a certain linear order among such monomials which will come in useful in Section 5.5. Let

$$\begin{aligned} b = &\left(x_{n_{r_4^{(1)},4}\alpha_4}(m_{r_4^{(1)},4})\ldots x_{n_{1,4}\alpha_4}(m_{1,4})\right)\left(x_{n_{r_3^{(1)},3}\alpha_3}(m_{r_3^{(1)},3})\ldots x_{n_{1,3}\alpha_3}(m_{1,3})\right)\\ &\left(x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2})\ldots x_{n_{1,2}\alpha_2}(m_{1,2})\right)\left(x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\ldots x_{n_{1,1}\alpha_1}(m_{1,1})\right) \end{aligned} \tag{$m_{F_4}$}$$

be an element of $\operatorname{End} V$, where $V = N(k\Lambda_0)$ or $V = L(k\Lambda_0)$, such that

$$n_{r_i^{(1)},i} \leqslant \ldots \leqslant n_{1,i} \quad \text{and} \quad m_{r_i^{(1)},i} \leqslant \ldots \leqslant m_{1,i} \qquad \text{for all } i = 1,2,3,4. \tag{5.2}$$

Define the *charge-type* $\mathcal{C}$ and the *energy-type* $\mathcal{E}$ of $b$ by

$$\mathcal{C} = \left(n_{r_4^{(1)},4},\ldots,n_{1,4};\ n_{r_3^{(1)},3},\ldots,n_{1,3};\ n_{r_2^{(1)},2},\ldots,n_{1,2};\ n_{r_1^{(1)},1},\ldots,n_{1,1}\right), \tag{5.3}$$

$$\mathcal{E} = \left( m_{r_4^{(1)},4}, \ldots, m_{1,4};\, m_{r_3^{(1)},3}, \ldots, m_{1,3};\, m_{r_2^{(1)},2}, \ldots, m_{1,2};\, m_{r_1^{(1)},1}, \ldots, m_{1,1} \right).$$

Moreover, define the *color-type* of $b$ as the quadruple $(n_4, n_3, n_2, n_1)$ such that $n_j$ denotes the sum of charges of all color $j$ quasi-particles, i.e. such that $n_j = n_{r_j^{(1)},j} + \ldots + n_{1,j}$.

Let $b_1, b_2$ be any two quasi-particle monomials of the same color-type, expressed as in ($m_{F_4}$), such that their charges and energies satisfy (5.2). Denote their charge-types and energy-types by $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{E}_1, \mathcal{E}_2$ respectively. Define the strict linear order among quasi-particle monomials of the same color-type by

$$b_1 < b_2 \qquad \text{if} \qquad \mathcal{C}_1 < \mathcal{C}_2 \quad \text{or} \quad \mathcal{C}_1 = \mathcal{C}_2 \text{ and } \mathcal{E}_1 < \mathcal{E}_2, \tag{5.4}$$

where the order on (finite) sequences of integers is defined as follows:

$$\begin{aligned} &(x_p, \ldots, x_1) < (y_r, \ldots, y_1) \qquad \text{if there exists } s \text{ such that} \\ &x_1 = y_1, \ldots, x_{s-1} = y_{s-1} \qquad \text{and} \qquad s = p+1 \leqslant r \quad \text{or} \quad x_s < y_s. \end{aligned} \tag{5.5}$$

5.2. **Projection of the principal subspace.** As in [4], we now generalize Georgiev's projection [20] to type $F$. Consider quasi-particle monomial ($m_{F_4}$) as an element of $\operatorname{End} L(k\Lambda_0)$. Suppose that its charges and energies satisfy (5.2). Define its *dual charge-type* $\mathcal{D}$ as

$$\mathcal{D} = \left( r_4^{(1)}, \ldots, r_4^{(2k)};\, r_3^{(1)}, \ldots, r_3^{(2k)};\, r_2^{(1)}, \ldots, r_2^{(k)};\, r_1^{(1)}, \ldots, r_1^{(k)} \right), \tag{5.6}$$

where $r_i^{(n)}$ denotes the number of color $i$ quasi-particles of charge greater than or equal to $n$ in the monomial. Observe that, due to (4.4), the monomial does not posses any quasi-particles of color $i$ whose charge is strictly greater than $k\nu_i$.

The standard module $L(k\Lambda_0)$ can be regarded as a submodule of the tensor product module $L(\Lambda_0)^{\otimes k}$ generated by the highest weight vector $v_{L(k\Lambda_0)} = v_{L(\Lambda_0)}^{\otimes k}$. Let $\pi_{\mathcal{D}}$ be the projection of the principal subspace $W_{L(k\Lambda_0)}$ on the tensor product space

$$W_{(\mu_4^{(k)};\mu_3^{(k)};r_2^{(k)};r_1^{(k)})} \otimes \cdots \otimes W_{(\mu_4^{(1)};\mu_3^{(1)};r_2^{(1)};r_1^{(1)})} \subset W_{L(\Lambda_0)}^{\otimes k} \subset L(\Lambda_0)^{\otimes k}, \tag{5.7}$$

where $W_{(\mu_4^{(t)};\mu_3^{(t)};r_2^{(t)};r_1^{(t)})}$ denote the $\mathfrak{h}$-weight subspaces of the level 1 principal subspace $W_{L(\Lambda_0)}$ of weight $\mu_4^{(t)}\alpha_4 + \mu_3^{(t)}\alpha_3 + r_2^{(t)}\alpha_2 + r_1^{(t)}\alpha_1 \in R$ with

$$\mu_i^{(t)} = r_i^{(2t)} + r_i^{(2t-1)} \qquad \text{for} \qquad t = 1, \ldots, k \quad \text{and} \quad i = 3, 4. \tag{5.8}$$

Note that by (5.8) the $\mathfrak{h}$-weight of monomial ($m_{F_4}$) equals

$$\sum_{t=1}^{k} \left( \mu_4^{(t)}\alpha_4 + \mu_3^{(t)}\alpha_3 + r_2^{(t)}\alpha_2 + r_1^{(t)}\alpha_1 \right).$$

We denote by the same symbol $\pi_{\mathcal{D}}$ the generalization of the projection to the space of formal series with coefficients in $W_{L(k\Lambda_0)}$. Applying the generating function corresponding to ($m_{F_4}$) on the highest weight vector $v_{L(k\Lambda_0)} = v_{L(\Lambda_0)}^{\otimes k}$ we obtain

$$\begin{aligned} &\left( x_{n_{r_4^{(1)},4}\alpha_4}(z_{r_4^{(1)},4}) \cdots x_{n_{1,4}\alpha_4}(z_{1,4}) \right) \left( x_{n_{r_3^{(1)},3}\alpha_3}(z_{r_3^{(1)},3}) \cdots x_{n_{1,3}\alpha_3}(z_{1,3}) \right) \\ &\times \left( x_{n_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2}) \cdots\cdots x_{n_{1,2}\alpha_2}(z_{1,2}) \right) \left( x_{n_{r_1^{(1)},1}\alpha_1}(z_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(z_{1,1}) \right) v_{L(k\Lambda_0)}. \end{aligned} \tag{5.9}$$

Relations (4.4) imply that by applying the projection $\pi_{\mathcal{D}}$ on (5.9) we get

$$\begin{aligned}&\Big(x_{n^{(k)}_{r_4^{(2k-1)},4}\alpha_4}(z_{r_4^{(2k-1)},4})\cdots x_{n^{(k)}_{1,4}\alpha_4}(z_{1,4})\Big)\Big(x_{n^{(k)}_{r_3^{(2k-1)},3}\alpha_3}(z_{r_3^{(2k-1)},3})\cdots x_{n^{(k)}_{1,3}\alpha_3}(z_{1,3})\Big)\\ &\times\Big(x_{n^{(k)}_{r_2^{(k)},2}\alpha_2}(z_{r_2^{(1)},2})\cdots x_{n^{(k)}_{1,2}\alpha_2}(z_{1,2})\Big)\Big(x_{n^{(k)}_{r_1^{(k)},1}\alpha_1}(z_{r_1^{(k)},1})\cdots x_{n^{(k)}_{1,1}\alpha_1}(z_{1,1})\Big)\ v_{L(\Lambda_0)}\\ \otimes\cdots\otimes&\Big(x_{n^{(1)}_{r_4^{(1)},4}\alpha_4}(z_{r_4^{(1)},4})\cdots x_{n^{(1)}_{1,4}\alpha_4}(z_{1,4})\Big)\Big(x_{n^{(1)}_{r_3^{(1)},3}\alpha_3}(z_{r_3^{(1)},3})\cdots\cdots x_{n^{(1)}_{1,3}\alpha_3}(z_{1,3})\Big)\\ &\times\Big(x_{n^{(1)}_{r_2^{(1)},2}\alpha_2}(z_{r_2^{(1)},2})\cdots x_{n^{(1)}_{1,2}\alpha_2}(z_{1,2})\Big)\Big(x_{n^{(1)}_{r_1^{(1)},1}\alpha_1}(z_{r_1^{(1)},1})\cdots x_{n^{(1)}_{1,1}\alpha_1}(z_{1,1})\Big)v_{L(\Lambda_0)}\end{aligned}\tag{5.10}$$

multiplied by some nonzero scalar, where we set $x_{0\alpha_i}(z)=1$. Indeed, the form of expression (5.10) is uniquely determined by (4.4) and the requirement that the $\mathfrak{h}$-weights of the tensor factors in (5.7) and in (5.10) coincide. In particular, this requirement uniquely determines the integers $n_{p,i}^{(t)}$ in (5.10). They are found by

$$0\leqslant n_{p,i}^{(k)}\leqslant\ldots\leqslant n_{p,i}^{(2)}\leqslant n_{p,i}^{(1)}\leqslant\nu_i\quad\text{and}\quad n_{p,i}=\sum_{t=1}^{k}n_{p,i}^{(t)}\qquad\text{for all } i=1,2,3,4,$$

where for fixed $p$ and $i=3,4$ at most one $n_{p,i}^{(t)}$ equals 1. Clearly, if $n_{p,i}$, where $i=3,4$, is even, then all $n_{p,i}^{(t)}$ belong to $\{0,2\}$. Moreover, if $n_{p,i}$, where $i=3,4$, is odd, then there exists $1\leqslant t_0\leqslant k$ such that $n_{p,i}^{(t_0)}=1$ and $n_{p,i}^{(t)}\in\{0,2\}$ for $t\neq t_0$. Therefore, for every variable $z_{r,i}$, where $i=1,2,3,4$ and $r=1,\ldots,r_i^{(1)}$, the projection $\pi_{\mathcal{D}}$ places at most one generating function $x_{\alpha_i}(z_{r,i})$ if $i=1,2$ and at most two generating functions $x_{\alpha_i}(z_{r,i})$ if $i=3,4$ on each tensor factor of $W(\Lambda_0)^{\otimes k}$. Note that the inequalities $n_{p,i}^{(k)}\leqslant\ldots\leqslant n_{p,i}^{(2)}\leqslant n_{p,i}^{(1)}$ may not hold if the projection $\pi_{\mathcal{D}'}$ with $\mathcal{D}'\neq\mathcal{D}$ is applied on power series (5.9) of dual charge-type $\mathcal{D}$, as we demonstrate in Example 5.2.

**Example 5.1.** Consider the formal power series

$$x_{\alpha_4}(z_{2,4})x_{4\alpha_4}(z_{1,4})x_{2\alpha_3}(z_{2,3})x_{3\alpha_3}(z_{1,3})x_{\alpha_2}(z_{1,2})x_{2\alpha_1}(z_{1,1})v_{L(2\Lambda_0)}\tag{5.11}$$

with coefficients in $W_{L(2\Lambda_0)}$. Its dual charge-type is equal to

$$\mathcal{D}=(2,1,1,1;\ 2,2,1,0;\ 1,0;\ 1,1).$$

As before, we denote by $\pi_{\mathcal{D}}$ the generalization of the projection

$$W_{L(2\Lambda_0)}\to W_{(2;1;0;1)}\otimes W_{(3;4;1;1)}\subset W(\Lambda_0)^{\otimes 2}$$

to the space of formal power series with coefficients in $W_{L(2\Lambda_0)}$. By combining relations (4.4) and the fact that the $\mathfrak{h}$-weights of $W_{(2;1;0;1)}$ and $W_{(3;4;1;1)}$ are $\alpha_1+\alpha_3+2\alpha_4$ and $\alpha_1+\alpha_2+4\alpha_3+3\alpha_4$ respectively, one finds that the image of (5.11) with respect to the projection $\pi_{\mathcal{D}}$ equals, up to a nonzero scalar multiple,

$$x_{2\alpha_4}(z_{1,4})x_{\alpha_3}(z_{1,3})x_{\alpha_1}(z_{1,1})v_{L(\Lambda_0)}\tag{5.12}$$

$$\otimes\, x_{\alpha_4}(z_{2,4})x_{2\alpha_4}(z_{1,4})x_{2\alpha_3}(z_{2,3})x_{2\alpha_3}(z_{1,3})x_{\alpha_2}(z_{1,2})x_{\alpha_1}(z_{1,1})v_{L(\Lambda_0)}.\tag{5.13}$$

More specifically, the projection $\pi_{\mathcal{D}}$ applies every factor $x_{\alpha_1}(z_{1,1})$ of the vertex operator $x_{2\alpha_1}(z_{1,1})=x_{\alpha_1}(z_{1,1})^2$ on the different tensor factor, so that, using the notation as in

(5.10), we have $n_{1,1}^{(1)}=n_{1,1}^{(2)}=1$. Next, the vertex operator $x_{\alpha_2}(z_{1,2})$ is applied only on the rightmost tensor factor, so $n_{1,2}^{(1)}=1$ and $n_{1,2}^{(2)}=0$. As for the color $i=3$, the relation $x_{3\alpha_3}(z_{1,3})=x_{\alpha_3}(z_{1,3})^3=0$ on $L(\Lambda_0)$ ensures the projection $\pi_{\mathcal{D}}$ applies two vertex operators $x_{\alpha_3}(z_{1,3})$ on the rightmost tensor factor and one vertex operator $x_{\alpha_3}(z_{1,3})$ on the remaining tensor factor. Hence we have $n_{1,3}^{(1)}=2$ and $n_{1,3}^{(2)}=1$. Finally, the vertex operator $x_{2\alpha_3}(z_{2,3})$ is applied on the rightmost tensor factor, so that we have $n_{2,3}^{(1)}=2$ and $n_{2,3}^{(2)}=0$. As with the color $i=3$, in the $i=4$ case we have $x_{3\alpha_4}(z_{2,4})=x_{\alpha_4}(z_{2,4})^3=0$ on $L(\Lambda_0)$ so by arguing analogously we find $n_{1,4}^{(1)}=n_{1,4}^{(2)}=2$, $n_{2,4}^{(1)}=1$ and $n_{2,4}^{(2)}=0$.

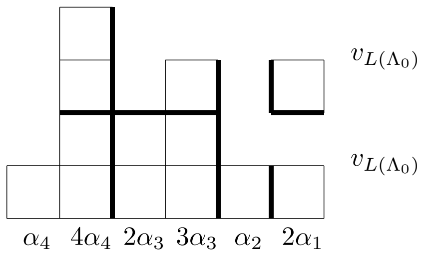



FIGURE 2. $\pi_{\mathcal{D}}\left(x_{\alpha_4}(z_{2,4})x_{4\alpha_4}(z_{1,4})x_{2\alpha_3}(z_{2,3})x_{3\alpha_3}(z_{1,3})x_{\alpha_2}(z_{1,2})x_{2\alpha_1}(z_{1,1})v_{L(2\Lambda_0)}\right)$

The image of (5.11) with respect to the projection $\pi_{\mathcal{D}}$ can be represented graphically as in Figure 2. The number of boxes in each column equals the corresponding quasi-particle charge in (5.11). The first two and the second two rows in the diagram correspond to the first and the second tensor factor, i.e. to (5.12) and (5.13) respectively. Hence the number of boxes in the first two and in the second two rows in the column corresponding to $x_{n_{p,i}\alpha_i}(z_{p,i})$ equals $n_{p,i}^{(2)}$ and $n_{p,i}^{(1)}$ respectively.

**Example 5.2.** In Example 5.1, we considered the image of formal power series (5.11) of dual charge-type $\mathcal{D}$ with respect to the projection $\pi_{\mathcal{D}}$. In this example, the same projection $\pi_{\mathcal{D}}$ is applied on the formal power series of dual charge-type $\mathcal{D}'\neq\mathcal{D}$.

The charge-type of

$$x_{\alpha_4}(z_{2,4})x_{4\alpha_4}(z_{1,4})x_{2\alpha_3}(z_{2,3})x_{3\alpha_3}(z_{1,3})x_{\alpha_2}(z_{1,2})x_{\alpha_1}(z_{2,1})x_{\alpha_1}(z_{1,1})v_{L(2\Lambda_0)} \tag{5.14}$$

is less than the charge-type of (5.11) with respect to linear order (5.5). However, the both expressions posses the same color-type. By arguing as in Example 5.1 we find that the image of (5.14) with respect to the projection $\pi_{\mathcal{D}}$ is a linear combination of two formal power series presented in Figure 3. Note that for the color $i=1$, in the first case we have $n_{1,i}^{(1)}=n_{2,i}^{(2)}=0$ and $n_{1,i}^{(2)}=n_{2,i}^{(1)}=1$ while in the second case $n_{1,i}^{(1)}=n_{2,i}^{(2)}=1$ and $n_{1,i}^{(2)}=n_{2,i}^{(1)}=0$.

The charge-type of

$$x_{\alpha_4}(z_{2,4})x_{4\alpha_4}(z_{1,4})x_{\alpha_3}(z_{2,3})x_{4\alpha_3}(z_{1,3})x_{\alpha_2}(z_{1,2})x_{2\alpha_1}(z_{1,1})v_{L(2\Lambda_0)} \tag{5.15}$$

is greater than the charge-type of (5.11) although the both expressions are of the same color-type. By arguing as before, we find that the image of (5.15) with respect to the projection $\pi_{\mathcal{D}}$ is zero. Indeed, this is caused by the term $x_{4\alpha_3}(z_{1,3})=x_{2\alpha_3}(z_{1,3})x_{2\alpha_3}(z_{1,3})$.

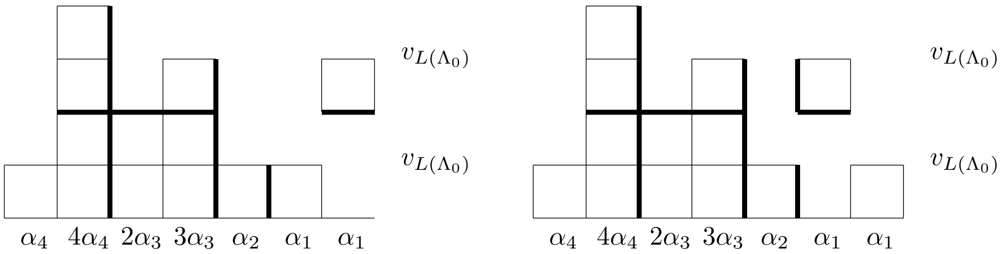



FIGURE 3. $\pi_{\mathcal{D}}\left(x_{\alpha_4}(z_{2,4})x_{4\alpha_4}(z_{1,4})x_{2\alpha_3}(z_{2,3})x_{3\alpha_3}(z_{1,3})x_{\alpha_2}(z_{1,2})x_{\alpha_1}(z_{2,1})x_{\alpha_1}(z_{1,1})v_{L(2\Lambda_0)}\right)$

More specifically, we have $x_{3\alpha_3}(z_{1,3}) = 0$ on $L(\Lambda_0)$. Hence, in order for the image to be nonzero, the projection would have to move one copy of $x_{2\alpha_3}(z_{1,3})$ to the first and another copy of $x_{2\alpha_3}(z_{1,3})$ to the second tensor factor. However, this is not possible as the weight of the first tensor factor is only $\alpha_1 + \alpha_3 + 2\alpha_4$.

5.3. **Operators $A_\theta$ and $e_\alpha$.** Let $b \in B_{L(k\Lambda_0)}$ be a quasi-particle monomial of charge-type $\mathcal{C}$ and dual charge-type $\mathcal{D}$. Denote the charges and the energies of its quasi-particles as in ($m_{F_4}$). In this section, generalizing the approach from [6], we demonstrate how to reduce $b$ to obtain a new monomial $b' \in B_{L(k\Lambda_0)}$ such that its charge-type $\mathcal{C}'$ satisfies $\mathcal{C}' < \mathcal{C}$ with respect to linear order (5.4). This will be a key step in the proof of linear independence of the set $\mathfrak{B}_{L(k\Lambda_0)}$ in Section 5.4.

Let $A_\theta$ be the constant term of the operator

$$x_\theta(z) = \sum_{r\in\mathbb{Z}} x_\theta(r) z^{-r-1} \in \operatorname{End} L(\Lambda_0)[[z^{\pm 1}]],$$

i.e. $A_\theta = x_\theta(-1)$, where $\theta$ is the maximal root; recall (5.1). Consider the image of the vector $\pi_{\mathcal{D}}\, b v_{L(k\Lambda_0)} \in W_{L(\Lambda_0)}^{\otimes k}$ with respect to the operator

$$(A_\theta)_s := \underbrace{1\otimes\cdots\otimes 1}_{k-s} \otimes A_\theta \otimes \underbrace{1\otimes\cdots\otimes 1}_{s-1} \qquad \text{for} \qquad s = n_{1,1}.$$

This image can be obtained as the coefficient of the variables

$$\overline{z} := z_{r_4^{(1)},4}^{-m_{r_4^{(1)},4}-n_{r_4^{(1)},4}} \cdots z_{2,1}^{-m_{2,1}-n_{2,1}} z_{1,1}^{-m_{1,1}-n_{1,1}} \tag{5.16}$$

in the expression

$$(A_\theta)_s\, \pi_{\mathcal{D}}\, x_{n_{r_4^{(1)},4}\alpha_4}(z_{r_4^{(1)},4}) \cdots x_{n_{2,1}\alpha_1}(z_{2,1}) x_{n_{1,1}\alpha_1}(z_{1,1}) v_{L(k\Lambda_0)}. \tag{5.17}$$

Using the commutator formula, see, e.g., [17, Eq. (2.3.13)], one can check that the operator $A_\theta$ commutes with the action of quasi-particles. Hence, using (5.10), we find that the $s$-th tensor factor (from the right) in (5.17) equals $F_s\, x_\theta(-1) v_{L(\Lambda_0)}$, where

$$\begin{aligned} F_s = &\left(x_{n^{(s)}_{r_4^{(2s-1)},4}\alpha_4}(z_{r_4^{(2s-1)},4}) \cdots x_{n^{(s)}_{1,4}\alpha_4}(z_{1,4})\right)\left(x_{n^{(s)}_{r_3^{(2s-1)},3}\alpha_3}(z_{r_3^{(2s-1)},3}) \cdots x_{n^{(s)}_{1,3}\alpha_3}(z_{1,3})\right) \\ &\times \left(x_{n^{(s)}_{r_2^{(s)},2}\alpha_2}(z_{r_2^{(s)},2}) \cdots x_{n^{(s)}_{1,2}\alpha_2}(z_{1,2})\right)\left(x_{n^{(s)}_{r_1^{(s)},1}\alpha_1}(z_{r_1^{(s)},1}) \cdots x_{n^{(s)}_{1,1}\alpha_1}(z_{1,1})\right). \end{aligned}$$

Consider the Weyl group translation operator $e_\alpha \in \operatorname{End} L(\Lambda_0)$ defined by

$$e_\alpha = \exp x_{-\alpha}(1) \exp(-x_\alpha(-1)) \exp x_{-\alpha}(1) \exp x_\alpha(0) \exp(-x_{-\alpha}(0)) \exp x_\alpha(0)$$

for $\alpha \in R$; see [23, Chap. 3]. It possesses the following properties:

$$e_\alpha v_{L(\Lambda_0)} = -x_\alpha(-1) v_{L(\Lambda_0)} \quad \text{for every long root } \alpha, \tag{5.18}$$

$$x_\beta(j) e_\alpha = e_\alpha x_\beta(j + \beta(\alpha^\vee)) \quad \text{for all } \alpha, \beta \in R \text{ and } j \in \mathbb{Z}. \tag{5.19}$$

Using (5.18) and (5.19) for $\alpha = \theta$ we rewrite the $s$-th tensor factor as

$$F_s\, x_\theta(-1) v_{L(\Lambda_0)} = -e_\theta\, F_s\, v_{L(\Lambda_0)}\, z_{r_1^{(s)},1} \cdots z_{2,1} z_{1,1}. \tag{5.20}$$

Recall (5.1) and notation (3.2). Taking the coefficient of variables (5.16) in (5.20) we find

$$(A_\theta)_s\, \pi_{\mathcal{D}}\, b v_{L(k\Lambda_0)} = -(e_\theta)_s\, \pi_{\mathcal{D}}\, b^+ v_{L(k\Lambda_0)},$$

where $(e_\theta)_s$ denotes the action of $e_\theta$ on the $s$-th tensor factor (from the right) and

$$b^+ = b_{\alpha_4}\, b_{\alpha_3}\, b_{\alpha_2}\, b_{\alpha_1}^{<s}\, b_{\alpha_1}^{s}, \quad \text{where} \quad b_{\alpha_1}^{<s} = x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{r_1^{(s)}+1,1}\alpha_1}(m_{r_1^{(s)}+1,1})$$

$$\text{and} \quad b_{\alpha_1}^{s} = x_{n_{r_1^{(s)},1}\alpha_1}(m_{r_1^{(s)},1} + 1) \cdots x_{n_{1,1}\alpha_1}(m_{1,1} + 1).$$

Therefore, by applying the above procedure we increased the energies of all quasi-particles of color 1 and charge $s = n_{1,1}$ in the monomial $b \in B_{L(k\Lambda_0)}$ by 1. Recall that by ($c_2$) we have $m_{1,1} \leqslant -n_{1,1} = -s$. We may continue to apply the same procedure, now starting with $b^+ v_{L(k\Lambda_0)}$, until we obtain the monomial

$$\widetilde{b} = b_{\alpha_4}\, b_{\alpha_3}\, b_{\alpha_2}\, \widetilde{b}_{\alpha_1}, \qquad \text{where} \qquad \widetilde{b}_{\alpha_1} = x_{n_{r_1^{(1)},1}\alpha_1}(\widetilde{m}_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(\widetilde{m}_{1,1}) \qquad \text{and}$$

$$(\widetilde{m}_{r_1^{(1)},1}, \ldots, \widetilde{m}_{r_1^{(s)}+1,1}, \widetilde{m}_{r_1^{(s)},1}, \ldots, \widetilde{m}_{1,1}) = (m_{r_1^{(1)},1}, \ldots, m_{r_1^{(s)}+1,1}, m_{r_1^{(s)},1} - m_{1,1} - s, \ldots, -s).$$

Since $b$ is an element of $B_{L(k\Lambda_0)}$, the quasi-particle monomial $\widetilde{b}$ belongs to $B_{L(k\Lambda_0)}$ as well. Moreover, the charge-type and the dual charge-type of $\widetilde{b}$ equal $\mathcal{C}$ and $\mathcal{D}$ respectively.

By (5.18) we have $x_{\alpha_1}(-1) v_{L(\Lambda_0)} = -e_{\alpha_1} v_{L(\Lambda_0)}$. Hence, the vector $\pi_{\mathcal{D}}\, \widetilde{b} v_{L(k\Lambda_0)}$, which belongs to $W_{L(\Lambda_0)}^{\otimes k}$, equals the coefficient of the variables

$$\overline{z} \left( z_{r_1^{(s)},1} \cdots z_{2,1} z_{1,1} \right)^{m_{1,1}+s} \tag{5.21}$$

in

$$(-1)^s\, \pi_{\mathcal{D}}\, x_{n_{r_4^{(1)},4}\alpha_4}(z_{r_4^{(1)},4}) \cdots x_{n_{2,1}\alpha_1}(z_{2,1})\, (1^{\otimes(k-s)} \otimes e_{\alpha_1}^{\otimes s})\, v_{L(\Lambda_0)}^{\otimes k}, \tag{5.22}$$

where $\overline{z}$ is given by (5.16). We now employ (5.19) to move $1^{\otimes(k-s)} \otimes e_{\alpha_1}^{\otimes s}$ all the way to the left in (5.22). Next, by dropping the invertible operator $(-1)^s(1^{\otimes(k-s)} \otimes e_{\alpha_1}^{\otimes s})$ and taking the coefficient of variables (5.21) we get $\pi_{\mathcal{D}'}\, b' v_{L(k\Lambda_0)}$, where the quasi-particle monomial $b'$ of charge-type $\mathcal{C}'$ and dual charge-type $\mathcal{D}'$ is given by

$$b' = b_{\alpha_4}\, b_{\alpha_3}\, b'_{\alpha_2}\, b'_{\alpha_1} \qquad \text{for} \qquad b'_{\alpha_1} = x_{n_{r_1^{(1)},1}\alpha_1}(\widetilde{m}_{r_1^{(1)},1} + 2n_{r_1^{(1)},1}) \cdots x_{n_{2,1}\alpha_1}(\widetilde{m}_{2,1} + 2n_{2,1}),$$

$$b'_{\alpha_2} = x_{n_{r_2^{(1)},2}\alpha_2}(m_{r_2^{(1)},2} - n^{(1)}_{r_2^{(1)},2} - \cdots - n^{(s)}_{r_2^{(1)},2}) \cdots x_{n_{1,2}\alpha_2}(m_{1,2} - n^{(1)}_{1,2} - \cdots - n^{(s)}_{1,2}).$$

Clearly, the energies of the quasi-particles in colors 3 and 4 did not change. Furthermore, if the dual charge-type $\mathcal{D}$ of $b$ equals

$$\mathcal{D} = \big(r_4^{(1)},\ldots,r_4^{(2k)};\, r_3^{(1)},\ldots,r_3^{(2k)};\, r_2^{(1)},\ldots,r_2^{(k)};\, r_1^{(1)},\ldots,r_1^{(n_{1,1})},\underbrace{0,\ldots,0}_{k-s}\big),$$

then the dual charge-type $\mathcal{D}'$ of $b'$ equals

$$\mathcal{D}' = \big(r_4^{(1)},\ldots,r_4^{(2k)};\, r_3^{(1)},\ldots,r_3^{(2k)};\, r_2^{(1)},\ldots,r_2^{(k)};\, r_1^{(1)}-1,\ldots,r_1^{(n_{1,1})}-1,\underbrace{0,\ldots,0}_{k-s}\big).$$

Finally, by arguing as in [5, Proposition 3.3.1] one can check that $b'$ belongs to $B_{L(k\Lambda_0)}$.

5.4. **Linear independence of the sets $\mathfrak{B}_V$.** In this section, we prove linear independence of the set $\mathfrak{B}_{L(k\Lambda_0)}$. Linear independence of $\mathfrak{B}_{N(k\Lambda_0)}$ can be verified by arguing as in [4, Sect. 3]. Suppose there exists a linear dependence relation among some elements $b^a v_{L(k\Lambda_0)} \in \mathfrak{B}_{L(k\Lambda_0)}$,

$$\sum_{a\in A} c_a\, b^a v_{L(k\Lambda_0)} = 0, \quad \text{where} \quad c_a \in \mathbb{C},\, c_a \neq 0 \text{ for all } a\in A \tag{5.23}$$

and $A$ is a finite nonempty set. As the principal subspace $W_{L(k\Lambda_0)}$ is a direct sum of its $\mathfrak{h}$-weight subspaces, we can assume that all $b^a \in B_{L(k\Lambda_0)}$ posses the same color-type.

Recall strict linear order (5.4) and choose $a_0 \in A$ such that $b^{a_0} < b^a$ for all $a \in A$, $a \neq a_0$. Suppose that the charge-type $\mathcal{C}$ and the dual charge-type $\mathcal{D}$ of $b^{a_0}$ are given by (5.3) and (5.6) respectively. Applying the projection $\pi_{\mathcal{D}}$ on (5.23) we obtain a linear combination of elements in

$$W_{(\mu_4^{(k)};\mu_3^{(k)};r_2^{(k)};0)} \otimes\cdots\otimes W_{(\mu_4^{(n_{1,1}+1)};\mu_3^{(n_{1,1}+1)};r_2^{(n_{1,1}+1)};0)}$$
$$\otimes W_{(\mu_4^{(n_{1,1})};\mu_3^{(n_{1,1})};r_2^{(n_{1,1})};r_1^{(n_{1,1})})} \otimes\cdots\otimes W_{(\mu_4^{(1)};\mu_3^{(1)};r_2^{(1)};r_1^{(1)})},$$

recall (5.8). The definition of the projection $\pi_{\mathcal{D}}$ implies that all $b^a v_{L(k\Lambda_0)}$ such that the charge-type of $b^a$ is strictly greater than $\mathcal{C}$ with respect to (5.5) are annihilated by $\pi_{\mathcal{D}}$. Therefore, we can assume that all $b^a$ posses the same charge-type $\mathcal{C}$ and, consequently, the same dual-charge-type $\mathcal{D}$.

As in (3.2), write the monomials $b^a$ as $b^a = b^a_{\alpha_4} b^a_{\alpha_3} b^a_{\alpha_2} b^a_{\alpha_1}$, where $b^a_{\alpha_j}$ consist of quasi-particles of color $j$. We now apply the procedure described in Section 5.3 on the linear combination

$$c_{a_0}\, \pi_{\mathcal{D}}\, b^{a_0} v_{L(k\Lambda_0)} + \sum_{a\in A,\, a\neq a_0} c_a\, \pi_{\mathcal{D}}\, b^a v_{L(k\Lambda_0)} = 0. \tag{5.24}$$

We repeat it until all quasi-particles of color 1 are removed from the first summand $c_{a_0}\pi_{\mathcal{D}}\, b^{a_0} v_{L(k\Lambda_0)}$. This also removes all quasi-particles of color 1 from other summands, so that (5.24) becomes

$$\widetilde{c}_{a_0}\, \pi_{\widetilde{\mathcal{D}}}\, b^{a_0}_{\alpha_4} b^{a_0}_{\alpha_3} \widetilde{b}^{a_0}_{\alpha_2} v_{L(k\Lambda_0)} + \sum_{\substack{a\in A,\, a\neq a_0\\ b^{a_0}_{\alpha_1} = b^a_{\alpha_1}}} \widetilde{c}_a\, \pi_{\widetilde{\mathcal{D}}}\, b^a_{\alpha_4} b^a_{\alpha_3} \widetilde{b}^a_{\alpha_2} v_{L(k\Lambda_0)} = 0 \tag{5.25}$$

for some quasi-particle monomials $\widetilde{b}^a_{\alpha_2}$ of color 2 and scalars $\widetilde{c}_a \neq 0$ such that $\widetilde{\mathcal{D}}$ is the dual charge-type of all quasi-particle monomials $b^a_{\alpha_4} b^a_{\alpha_3} \widetilde{b}^a_{\alpha_2}$ in (5.25). The summation in

(5.25) goes over all $a \neq a_0$ such that $b^a_{\alpha_1} = b^{a_0}_{\alpha_1}$ because the summands $\pi_{\mathcal{D}}\, b^a v_{L(k\Lambda_0)}$ such that $b^{a_0}_{\alpha_1} < b^a_{\alpha_1}$ get annihilated in the process.

The vectors $b^a_{\alpha_4} b^a_{\alpha_3} \widetilde{b}^a_{\alpha_2} v_{L(k\Lambda_0)}$ in (5.25) belong to $\mathfrak{B}_{L(k\Lambda_0)}$. Furthermore, they can be realized as elements of the principal subspace of the level $k$ standard module $L(k\Lambda_0)$ with the highest weight vector $v_{L(k\Lambda_0)}$ for the affine Lie algebra of type $C_3^{(1)}$. Moreover, their realizations belong to the corresponding basis in type $C_3^{(1)}$, as given by Theorem 3.1 (for a detailed proof in type $C_l^{(1)}$ see [5]). This implies $\widetilde{c}_{a_0} = 0$ and, consequently, $c_{a_0} = 0$, thus contradicting (5.23). Finally, we conclude that the set $\mathfrak{B}_{L(k\Lambda_0)}$ is linearly independent.

5.5. **Small spanning sets $\bar{\mathfrak{B}}_V$.** In this section, we construct certain small spanning sets $\bar{\mathfrak{B}}_{N(k\Lambda_0)}$ and $\bar{\mathfrak{B}}_{L(k\Lambda_0)}$ for the quotients $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$ and $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$ of the algebra $U(\widetilde{\mathfrak{n}}_+)$ over its left ideals $I_{N(k\Lambda_0)} = U(\widetilde{\mathfrak{n}}_+)\widetilde{\mathfrak{n}}_+^{\geqslant 0}$ and $I_{L(k\Lambda_0)}$ defined by (4.2). We denote by $\bar{x}$ the image of the element $x \in U(\widetilde{\mathfrak{n}}_+)$ in these quotients with respect to the corresponding canonical epimorphisms. First, we consider $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$. By Poincaré–Birkhoff–Witt theorem for the universal enveloping algebra we have

$$U(\widetilde{\mathfrak{n}}_+) = U(\widetilde{\mathfrak{n}}_{\alpha_4})U(\widetilde{\mathfrak{n}}_{\alpha_3})U(\widetilde{\mathfrak{n}}_{\alpha_2})U(\widetilde{\mathfrak{n}}_{\alpha_1}), \quad \text{where} \quad \widetilde{\mathfrak{n}}_{\alpha_i} = \mathfrak{n}_{\alpha_i} \otimes \mathbb{C}[t,t^{-1}] \text{ and } \mathfrak{n}_{\alpha_i} = \mathbb{C}x_{\alpha_i}.$$

By (2.1) quasi-particles of the same color commute, so all monomials

$$\bar{b} = \left(\bar{x}_{n_{r_4^{(1)},4}\alpha_4}(m_{r_4^{(1)},4}) \ldots \bar{x}_{n_{1,4}\alpha_4}(m_{1,4})\right) \ldots \left(\bar{x}_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \ldots \bar{x}_{n_{1,1}\alpha_1}(m_{1,1})\right) \qquad (\bar{m}_{F_4})$$

such that their charges and energies satisfy (5.2) form a spanning set for $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$. Denote this set by $\bar{\mathfrak{S}}_{N(k\Lambda_0)}$. We now list two families of quasi-particle relations which we will use to reduce $\bar{\mathfrak{S}}_{N(k\Lambda_0)}$, i.e. to obtain a smaller spanning set for $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$.

**Lemma 5.3.** (a) *For any color $i = 1,2,3,4$ and charges $n_1$ and $n_2$ such that $n_2 \leqslant n_1$ the following relation holds for operators on $N(k\Lambda_0)$:*

$$\left(\frac{d^p}{dz^p}x_{n_2\alpha_i}(z)\right) x_{n_1\alpha_i}(z) = A_p(z)x_{(n_1+1)\alpha_i}(z) + B_p(z)\frac{d^p}{dz^p}x_{(n_1+1)\alpha_i}(z), \qquad (r_1)$$

*where $p = 0,1,\ldots,2n_2-1$ and $A_p(z), B_p(z)$ are some formal series with coefficients in the set of polynomials of color $i$ quasi-particles.*
(b) *For any color $i = 2,3,4$ and charges $n_{i-1}$ and $n_i$ the following relation holds for operators on $N(k\Lambda_0)$:*

$$(z_1 - z_2)^{M_i} x_{n_{i-1}\alpha_{i-1}}(z_1)x_{n_i\alpha_i}(z_2) = (z_1 - z_2)^{M_i} x_{n_i\alpha_i}(z_2)x_{n_{i-1}\alpha_{i-1}}(z_1), \qquad (r_2)$$

*where $M_i = \min\left\{\frac{\nu_i}{\nu_{i-1}}n_{i-1}, n_i\right\}$.*

*Proof.* Relations $(r_1)$ are verified by arguing as in the proof of [22, Lemma 4.2] and relations $(r_2)$ follow by a direct computation which employs commutation relations for vertex operators [29, Eq. (6.2.8)], or, alternatively, commutator formula [17, Eq. (2.3.13)]. □

In the next two lemmas we establish techniques which we will use to reduce the spanning set $\bar{\mathfrak{S}}_{N(k\Lambda_0)}$. The former relies on relations $(r_1)$ and the latter on $(r_2)$.

**Lemma 5.4.** *For any color $i=1,2,3,4$, energies $m_1,m_2$ and charges $n_1,n_2$ such that $n_2\leqslant n_1$ the monomials*

$x_{n_2\alpha_i}(m_2)x_{n_1\alpha_i}(m_1),\ x_{n_2\alpha_i}(m_2-1)x_{n_1\alpha_i}(m_1+1),\ \ldots,x_{n_2\alpha_i}(m_2-2n_2+1)x_{n_1\alpha_i}(m_1+2n_2-1)$

*of operators on $N(k\Lambda_0)$ can be expressed as a linear combination of monomials*

$x_{n_2\alpha_i}(p_2)x_{n_1\alpha_i}(p_1)$ *such that* $p_2\leqslant m_2-2n_2,\quad p_1\geqslant m_1+2n_2$ *and* $p_1+p_2=m_1+m_2$

*and monomials which contain a quasi-particle of color $i$ and charge $n_1+1$. Moreover, for $n_2=n_1$ the monomials*

$$x_{n_2\alpha_i}(m_2)x_{n_2\alpha_i}(m_1)\quad \text{with}\ \ m_1-2n_2<m_2\leqslant m_1$$

*can be expressed as a linear combination of monomials*

$$x_{n_2\alpha_i}(p_2)x_{n_2\alpha_i}(p_1)\quad \text{such that}\quad p_2\leqslant p_1-2n_2\quad \text{and}\quad p_1+p_2=m_1+m_2$$

*and monomials which contain a quasi-particle of color $i$ and charge $n_2+1$.*

*Proof.* The first statement of the lemma follows by repeating the arguments from [22, Remark 4.6] which rely on ($r_1$); see also [4, Lemma 2.2.1]. Moreover, it implies the second statement; see [4, Corollary 2.2.1]. □

Consider monomial ($\bar{m}_{F_4}$) in $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$ satisfying (5.2). Clearly, the monomial coincides with the coefficient of the variables

$$z_{r_4^{(1)},4}^{-m_{r_4^{(1)},4}-n_{r_4^{(1)},4}}\cdots z_{j,i}^{-m_{j,i}-n_{j,i}}\cdots z_{2,1}^{-m_{2,1}-n_{2,1}}z_{1,1}^{-m_{1,1}-n_{1,1}} \tag{5.26}$$

in the generating function

$$\bar{X}=\bar{x}_{n_{r_4^{(1)},4}\alpha_4}(z_{r_4^{(1)},4})\cdots\bar{x}_{n_{j,i}\alpha_i}(z_{j,i})\cdots\bar{x}_{n_{2,1}\alpha_1}(z_{2,1})\bar{x}_{n_{1,1}\alpha_1}(z_{1,1}). \tag{5.27}$$

Introduce the Laurent polynomial

$$P=\prod_{i=2}^{4}\prod_{q=1}^{r_{i-1}^{(1)}}\prod_{p=1}^{r_i^{(1)}}\left(1-\frac{z_{q,i-1}}{z_{p,i}}\right)^{\min\left\{\frac{\nu_i}{\nu_{i-1}}n_{q,i-1},\,n_{p,i}\right\}}. \tag{5.28}$$

**Lemma 5.5.** *The product $P\bar{X}$ belongs to*

$$\prod_{i=2}^{4}\prod_{p=1}^{r_i^{(1)}}z_{p,i}^{-\sum_{q=1}^{r_{i-1}^{(1)}}\min\left\{\frac{\nu_i}{\nu_{i-1}}n_{q,i-1},n_{p,i}\right\}}(U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)})[[z_{r_4^{(1)},4},\ldots,z_{1,1}]].$$

*Proof.* Every vertex operator $\bar{x}_{n\alpha_i}(z)$ in the product $P\bar{X}$ can be moved all the way to the right by using ($r_2$). Furthermore, the right hand side of

$$x_{n\alpha_i}(m)=\sum_{m_1+\ldots+m_n=m}x_{\alpha_i}(m_1)\cdots x_{\alpha_i}(m_n)$$

contains at least one quasi-particle $x_{\alpha_i}(m_j)$ with energy $m_j\geqslant 0$ if $m>-n$. Therefore, $x_{n\alpha_i}(m)$ belongs to $I_{N(k\Lambda_0)}$ for $m>-n$. This implies the statement of the lemma as the negative powers of the variables $z_{p,i}$ in $P\bar{X}$ come only from $P$. □

Let $\bar{\mathfrak{B}}_{N(k\Lambda_0)}$ be the set of all monomials ($\bar{m}_{F_4}$) satisfying difference conditions ($c_1$) and ($c_2$) (with $l = 4$ and $i' = i - 1$ for all $i = 1, 2, 3, 4$). Now we are ready to show that $\bar{\mathfrak{B}}_{N(k\Lambda_0)}$ spans $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$. We only briefly outline the proof, as it goes in parallel with the proofs for other types; see, e.g., the proof of [20, Theorem 5.1].

Suppose that a monomial $\bar{b}_1 \in \bar{\mathfrak{S}}_{N(k\Lambda_0)}$ given by ($\bar{m}_{F_4}$) contains a quasi-particle $\bar{x}_{n_{p,i}\alpha_i}(m_{p,i})$ which does not satisfy

$$m_{p,i} \leqslant -n_{p,i} + \sum_{q=1}^{r_{i'}^{(1)}} \min\left\{\tfrac{\nu_i}{\nu_{i'}} n_{q,i'}, n_{p,i}\right\}. \tag{5.29}$$

By using Lemma 5.5, one can express $\bar{b}_1$ as a linear combination of monomials $\bar{b}_2$ of the same charge-type and of the same total energy $m_{r_4^{(1)},4} + \ldots + m_{1,1}$ such that $\bar{b}_1 < \bar{b}_2$ with respect to linear order (5.4). However, there are only finitely many such monomials $\bar{b}_2$ which are nonzero. Hence, by repeating this procedure for an appropriate number of times, now starting with these new monomials, we can express $\bar{b}_1$ as a linear combination of monomials whose quasi-particles satisfy (5.29).

Next, suppose that all quasi-particles in a monomial $\bar{b}_1 \in \bar{\mathfrak{S}}_{N(k\Lambda_0)}$ given by ($\bar{m}_{F_4}$) satisfy (5.29) and suppose that some quasi-particle $\bar{x}_{n_{p,i}\alpha_i}(m_{p,i})$ in $\bar{b}_1$ does not satisfy ($c_2$). By repeating the arguments from the proof of [20, Theorem 5.1], which now rely on the first statement of Lemma 5.4, we can express $\bar{b}_1$ as a linear combination of two families of monomials. The first family consists of monomials of the same charge-type and of the same total energy as $\bar{b}_1$, such that the energies $m'_{p,i}$ of their quasi-particles $\bar{x}_{n_{p,i}\alpha_i}(m'_{p,i})$ now satisfy ($c_2$). The monomials in the second family posses strictly greater charge-type than $\bar{b}_1$ and the same total charge $n_{r_4^{(1)},4} + \ldots + n_{1,1}$. However, for a fixed charge-type $\mathcal{C}$, there are only finitely many charge-types $\mathcal{C}'$ of the same total charge such that $\mathcal{C} < \mathcal{C}'$. Therefore, after repeating this procedure for a sufficient number of times, the monomial $\bar{b}_1$ is expressed as a linear combination of monomials whose quasi-particles satisfy ($c_2$).

Remaining constraint ($c_1$) is established in parallel with the preceding discussion, by employing the second statement of Lemma 5.4. Finally, we conclude that $\bar{\mathfrak{B}}_{N(k\Lambda_0)}$ forms a spanning set for $U(\widetilde{\mathfrak{n}}_+)/I_{N(k\Lambda_0)}$.

**Remark 5.6.** Due to ($r_2$), the quasi-particles of colors 1 and 2 and the quasi-particles of colors 3 and 4 interact as the quasi-particles of colors 1 and 2 for the affine Lie algebra $A_2^{(1)}$ while the quasi-particles of colors 2 and 3 interact as the quasi-particles of colors 1 and 2 for the affine Lie algebra $B_2^{(1)}$.

We now consider $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$. It is clear that all monomials ($\bar{m}_{F_4}$), regarded as elements of $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$ and satisfying difference conditions ($c_1$) and ($c_2$), form a spanning set for the quotient $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$. However, by (4.2) we have

$$x_{n\alpha_i}(m) \in I_{L(k\Lambda_0)} \qquad \text{for all} \quad n \geqslant k\nu_i + 1, \quad m \in \mathbb{Z}, \quad \text{and} \quad i = 1, 2, 3, 4, \tag{$r_3$}$$

so we can obtain a smaller spanning set, as follows.

Suppose that monomial ($\bar{m}_{F_4}$) in $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$ satisfies difference conditions ($c_1$) and ($c_2$) and contains a quasi-particle $\bar{x}_{n_{p,i}\alpha_i}(m_{p,i})$ of charge $n_{p,i} \geqslant k\nu_i + 1$ and color $1 \leqslant i \leqslant 4$.

Clearly, such monomial coincides with the coefficient of variables (5.26) in the generating function $\bar{X}$ given by (5.27). As before, we consider the product $P\bar{X}$, where the Laurent polynomial $P$ is defined by (5.28).

By combining relations ($r_2$) and ($r_3$) we find $P\bar{X} = 0$. Indeed, the operator $\bar{x}_{n_{p,i}\alpha_i}(z_{p,i})$ in $P\bar{X}$ can be moved all the way to the right, thus annihilating the expression. By taking the coefficient of variables (5.26) in $P\bar{X} = 0$ we express ($\bar{m}_{F_4}$) as a linear combination of some quasi-particle monomials of the same charge-type and of the same total energy $m_{r_4^{(1)},4} + \ldots + m_{1,1}$, which are greater than ($\bar{m}_{F_4}$) with respect to linear order (5.4). However, there exists only finitely many such quasi-particle monomials which are nonzero. Hence, by repeating the same procedure for an appropriate number of times, now starting with these new monomials, we find, after finitely many steps, that ($\bar{m}_{F_4}$) equals zero. Therefore, we conclude that the set $\bar{\mathfrak{B}}_{L(k\Lambda_0)}$ of all monomials ($\bar{m}_{F_4}$) in $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$ which satisfy difference conditions ($c_1$), ($c_2$) and ($c_3$) forms a spanning set for $U(\widetilde{\mathfrak{n}}_+)/I_{L(k\Lambda_0)}$.

5.6. **Proof of Theorems 3.1 and 4.1.** In Section 5.4, we established the linear independence of the sets $\mathfrak{B}_{N(k\Lambda_0)}$ and $\mathfrak{B}_{L(k\Lambda_0)}$. We now prove that they span the principal subspaces $W_{L(k\Lambda_0)}$ and $W_{N(k\Lambda_0)}$, thus finishing the proof of Theorem 3.1. Moreover, as a consequence of the proof, we obtain the presentations of the principal subspace $W_{L(k\Lambda_0)}$ given by Theorem 4.1. Introduce the natural surjective map

$$\begin{aligned} f_{N(k\Lambda_0)} \colon U(\widetilde{\mathfrak{n}}_+) &\to W_{N(k\Lambda_0)} \\ a &\mapsto a \cdot v_{N(k\Lambda_0)}, \end{aligned}$$

so that we can consider the cases $V = L(k\Lambda_0)$ and $V = N(k\Lambda_0)$ simultaneously. Recall that the surjective map $f_{L(k\Lambda_0)}$ is given by (4.1), the left ideal $I_{L(k\Lambda_0)}$ is defined by (4.2) and $I_{N(k\Lambda_0)} = U(\widetilde{\mathfrak{n}}_+)\widetilde{\mathfrak{n}}_+^{\geqslant 0}$ .

Let $V$ be $N(k\Lambda_0)$ or $L(k\Lambda_0)$. It is clear that the left ideal $I_V$ belongs to the kernel of $f_V$. Hence, there exists a unique map

$$\bar{f}_V \colon U(\widetilde{\mathfrak{n}}_+)/I_V \to W_V \quad \text{such that} \quad f_V = \bar{f}_V\, \pi_V, \tag{5.30}$$

where $\pi_V$ is the canonical epimorphism $U(\widetilde{\mathfrak{n}}_+) \to U(\widetilde{\mathfrak{n}}_+)/I_V$. The map $\bar{f}_V$ is surjective as $f_V$ is surjective and, furthermore, it maps bijectively $\bar{\mathfrak{B}}_V$ to $\mathfrak{B}_V$. Therefore, the linearly independent set $\mathfrak{B}_V$ spans the principal subspace $W_V$ and so it forms a basis of $W_V$, which proves Theorem 3.1. This implies that the map (5.30) is a vector space isomorphism, so, in particular, we conclude that $\ker f_{L(k\Lambda_0)} = I_{L(k\Lambda_0)}$, thus proving Theorem 4.1.

## 6. Proof of Theorems 3.1, 3.2 and 4.2 in types $D$ and $E$

In this section, unless stated otherwise, we denote by $\widetilde{\mathfrak{g}}$ the affine Lie algebra of type $D_l^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$. First, we give an outline of the proof of Theorem 3.1 for $\widetilde{\mathfrak{g}}$. As the generalization of the arguments from Section 5.5 is straightforward, we only discuss the proof of linear independence. It relies on the coefficients of certain level 1 intertwining operators and on the vertex operator algebra construction of basic modules, thus resembling the corresponding proofs in types $A_l^{(1)}$, $B_l^{(1)}$ and $C_l^{(1)}$; see [4,5,20]. In Section 6.1 we recall the aforementioned construction while in Section 6.2 we demonstrate how to use the corresponding operators to complete the proof of Theorem 3.1. Next, in Section 6.3 we add some details as compared to Sections 5 and 6.2 to take care of the modifications needed to carry out the argument for rectangular weights, i.e. to prove Theorems 3.2 and 4.2. Finally, in Section 6.4 we construct different quasi-particle bases in type $E$, such that their linear independence can be verified by employing the operator $A_\theta$ associated with the maximal root $\theta$, thus resembling the corresponding proof in type $F$ from Section 5.

6.1. **Vertex operator algebra construction of basic modules.** We follow [19,29] to review the vertex operator algebra construction of the basic modules $L(\Lambda_i)$ [18,41]. Set

$$\widehat{\mathfrak{h}}_* = \bigoplus_{m\in\mathbb{Z}\setminus\{0\}} (\mathfrak{h}\otimes t^m)\oplus\mathbb{C}c \quad \text{and} \quad \widehat{\mathfrak{h}}^{<0} = \bigoplus_{m<0}\mathfrak{h}\otimes t^m.$$

Let

$$M(1) = U(\widehat{\mathfrak{h}}_*)\otimes_{U(\mathfrak{h}\otimes t\mathbb{C}[t]\oplus\mathbb{C}c)}\mathbb{C}$$

be the Fock space for the Heisenberg algebra $\widehat{\mathfrak{h}}_*$, i.e. the induced $\widehat{\mathfrak{h}}_*$-module, where $U(\mathfrak{h}\otimes t\mathbb{C}[t])$ acts trivially and $c$ acts as the identity on the one-dimensional module $\mathbb{C}$. Clearly, we have a vector space isomorphism $M(1)\cong S(\widehat{\mathfrak{h}}^{<0})$. Consider the tensor products

$$V_P = M(1)\otimes\mathbb{C}\left[P\right] \quad \text{and} \quad V_Q = M(1)\otimes\mathbb{C}\left[Q\right],$$

where $\mathbb{C}\left[P\right]$ and $\mathbb{C}\left[Q\right]$ denote the group algebras of the weight lattice $P$ and of the root lattice $Q$ with respective bases $\{e^\lambda : \lambda\in P\}$ and $\{e^\alpha : \alpha\in Q\}$. We use the identification of group elements $e^\lambda = 1\otimes e^\lambda\in V_P$.

Let $e_\lambda\colon V_P\to V_P$ be the linear isomorphism defined by

$$e_\lambda e^\mu = \epsilon(\lambda,\mu)e^{\mu+\lambda} \quad \text{for all } \lambda,\mu\in P, \tag{6.1}$$

where $\epsilon$ is a certain map $P\times P\to\mathbb{C}^\times$ satisfying $\epsilon(\lambda,0)=\epsilon(0,\lambda)=1$ for all $\lambda\in P$; see [19, 29] for more details. The space $V_Q$ is equipped with a structure of a vertex operator algebra, with $V_P$ being a $V_Q$-module, by

$$Y(e^\lambda,z) = E^-(-\lambda,z)E^+(-\lambda,z)e_\lambda z^\lambda, \quad \text{where} \quad E^\pm(-\lambda,z) = \exp\left(\sum_{n\leqslant 1}\lambda(\pm n)\frac{z^{\mp n}}{\pm n}\right)$$

and $z^\lambda = 1\otimes z^\lambda$ acts by $z^\lambda e^\mu = z^{\langle\lambda,\mu\rangle}e^\mu$ for all $\lambda,\mu\in P$. Moreover, the space $V_P$ acquires a structure of level one $\widetilde{\mathfrak{g}}$-module via

$$x_\alpha(m) = \operatorname*{Res}_z z^m Y(e^\alpha,z) \qquad \text{for } \alpha\in R \text{ and } m\in\mathbb{Z}.$$

With respect to this action, the space $V_Q$ is identified with the standard module $L(\Lambda_0)$ while the irreducible $V_Q$-submodules $V_Q e^{\lambda_i}$ of $V_P$ are identified with the standard modules $L(\Lambda_i)$ for all $i$ such that the weight $\Lambda_i$ is of level one. The corresponding highest weight vectors are $v_{L(\Lambda_0)} = 1$ and $v_{L(\Lambda_i)} = e^{\lambda_i}$.

6.2. **Operators $A_{\lambda_i}$ and proof of Theorem 3.1.** Let $b \in B_{L(k\Lambda_0)}$ be a quasi-particle monomial as in $(m)$, of charge-type $\mathcal{C}$ and dual charge-type

$$\mathcal{D} = \left(r_l^{(1)}, \ldots, r_l^{(k)}; \ \ldots \ r_2^{(1)}, \ldots, r_2^{(k)}; \ r_1^{(1)}, \ldots, r_1^{(k)}\right). \tag{6.2}$$

We now demonstrate how to carry out the procedure from Section 5.3, i.e. how to reduce $b$ to obtain a new monomial $b' \in B_{L(k\Lambda_0)}$ such that its charge-type $\mathcal{C}'$ satisfies $\mathcal{C}' < \mathcal{C}$ with respect to linear order (5.4). Denote by $I(\cdot, z)$ the intertwining operator of type $\binom{V_P}{V_P \ V_Q}$,

$$I(w,z)v = \exp(zL(-1))Y(v,-z)w, \quad \text{where} \quad w \in V_P, v \in V_Q,$$

see [17, Sect. 5.4]. For $i = 1, \ldots, l$ let $A_{\lambda_i}$ be the constant term of $I(e^{\lambda_i}, z)$, that is

$$A_{\lambda_i} = \operatorname*{Res}_z z^{-1} I(e^{\lambda_i}, z).$$

We have

$$A_{\lambda_i} v_{L(\Lambda_0)} = e^{\lambda_i} \quad \text{for all } i = 1, \ldots, l. \tag{6.3}$$

In contrast with Section 5.3, which relies on the application of the operators $A_\theta$ and $e_\theta$, we here make use of $A_{\lambda_i}$ and $e_{\lambda_i}$ in a similar fashion. In particular, we employ the following property of $e_{\lambda_i}$:

$$e_{\lambda_i} x_{\alpha_j}(m) = (-1)^{\delta_{ij}} x_{\alpha_j}(m - \delta_{ij}) e_{\lambda_i} \quad \text{for all } i,j = 1, \ldots, l \text{ and } m \in \mathbb{Z}, \tag{6.4}$$

see [11] for more details. Moreover, we use the fact that the operators $A_{\lambda_i}$ commute with the action of $x_\alpha(z)$ for all $\alpha \in R$, which comes as a consequence of the commutator formula for $x_\alpha(z)$ and $I(e^{\lambda_i}, z)$; see [17, Sect. 5.4].

As in Section 5.2, denote by $\pi_{\mathcal{D}}$ the projection of the principal subspace $W_{L(k\Lambda_0)}$ on

$$W_{(r_l^{(k)}; r_{l-1}^{(k)}; \ldots; r_2^{(k)}; r_1^{(k)})} \otimes \cdots \otimes W_{(r_l^{(1)}; r_{l-1}^{(1)}; \ldots; r_2^{(1)}; r_1^{(1)})} \subset W_{L(\Lambda_0)}^{\otimes k} \subset L(\Lambda_0)^{\otimes k},$$

where $W_{(r_l^{(t)}; \ldots; r_2^{(t)}; r_1^{(t)})}$ denote the $\mathfrak{h}$-weight subspaces of the level 1 principal subspace $W_{L(\Lambda_0)}$ of the weight $r_l^{(t)}\alpha_l + \cdots + r_2^{(t)}\alpha_2 + r_1^{(t)}\alpha_1 \in R$. Arguing as in Section 5.3, we conclude that the image of $\pi_{\mathcal{D}} b v_{L(k\Lambda_0)} \in W_{L(k\Lambda_0)} \subset W_{L(\Lambda_0)}^{\otimes k}$ with respect to the operator

$$(A_{\lambda_1})_s \coloneqq \underbrace{1 \otimes \cdots \otimes 1}_{k-s} \otimes A_{\lambda_1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{s-1}, \qquad \text{where} \qquad s = n_{1,1},$$

equals the coefficient of the variables

$$z_{r_l^{(1)},l}^{-m_{r_l^{(1)},l} - n_{r_l^{(1)},l}} \cdots z_{2,1}^{-m_{2,1}-n_{2,1}} z_{1,1}^{-m_{1,1}-n_{1,1}} \tag{6.5}$$

in the expression

$$(A_{\lambda_1})_s \, \pi_{\mathcal{D}} \, x_{n_{r_l^{(1)},l}\alpha_l}(z_{r_l^{(1)},l}) \cdots x_{n_{2,1}\alpha_1}(z_{2,1}) x_{n_{1,1}\alpha_1}(z_{1,1}) v_{L(k\Lambda_0)}. \tag{6.6}$$

Moreover, the $s$-th tensor factor in (6.6) (from the right) equals

$$F_s = \left(x_{n^{(s)}_{r_l^{(s)},l}\alpha_l}(z_{r_l^{(s)},l}) \cdots x_{n^{(s)}_{1,l}\alpha_l}(z_{1,l})\right) \cdots \left(x_{n^{(s)}_{r_1^{(s)},1}\alpha_1}(z_{r_1^{(s)},1}) \cdots x_{n^{(s)}_{1,1}\alpha_1}(z_{1,1})\right) e^{\lambda_1},$$

where the integers $n_{p,i}^{(t)}$ are given by

$$0 \leqslant n_{p,i}^{(k)} \leqslant \ldots \leqslant n_{p,i}^{(2)} \leqslant n_{p,i}^{(1)} \leqslant 1 \quad \text{and} \quad n_{p,i} = \sum_{t=1}^{k} n_{p,i}^{(t)} \qquad \text{for all } i = 1, \ldots, l.$$

By combining (6.1) and (6.4) we get

$$F_s = (-1)^{r_1^{(s)}} e_{\lambda_1} F_s \, z_{r_1^{(s)},1} \cdots z_{2,1} z_{1,1}. \tag{6.7}$$

Recall the notation from (3.2). By taking the coefficient of variables (6.5) in (6.7) we have

$$(A_{\lambda_1})_s \pi_{\mathcal{D}} \, b v_{L(k\Lambda_0)} = (-1)^{r_1^{(s)}} (e_{\lambda_1})_s \, \pi_{\mathcal{D}} \, b^+ v_{L(k\Lambda_0)},$$

where $(e_{\lambda_1})_s$ denotes the action of $e_{\lambda_1}$ on the $s$-th tensor factor (from the right) and

$$b^+ = b_{\alpha_l} \cdots b_{\alpha_2} b_{\alpha_1}^{<s} b_{\alpha_1}^{s} \quad \text{with} \quad b_{\alpha_1}^{<s} = x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{r_1^{(s)}+1,1}\alpha_1}(m_{r_1^{(s)}+1,1})$$
$$\text{and} \quad b_{\alpha_1}^{s} = x_{n_{r_1^{(s)},1}\alpha_1}(m_{r_1^{(s)},1}+1) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}+1).$$

Note that the monomial $b^+$ belongs to $B_{L(k\Lambda_0)}$.

As in Section 5.3, we can now continue to apply this procedure until we obtain a monomial $b' \in B_{L(k\Lambda_0)}$ of charge-type $\mathcal{C}' < \mathcal{C}$. Finally, by repeating the arguments from Section 5.4 almost verbatim, we can prove the linear independence of the set $\mathfrak{B}_{L(k\Lambda_0)}$. However, in contrast with Section 5.4, where the quasi-particle basis in type $F_4^{(1)}$ was reduced to a basis in type $C_3^{(1)}$, the quasi-particle basis in type $D_l^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$ is reduced, after sufficient number of steps, to a basis in type $A_m^{(1)}$ for some $m$ from Theorem 3.1. Note that such a modification of the argument is possible because we have the operators $A_{\lambda_i}$ and $e^{\lambda_i}$ satisfying (6.3) and (6.4) at our disposal; cf. corresponding properties (5.18) and (5.19) for $\alpha = \theta$.

6.3. **Proof of Theorems 3.2 and 4.2.** Let $\widetilde{\mathfrak{g}}$ be the affine Lie algebra of type $D_l^{(1)}$, $E_6^{(1)}$ or $E_7^{(1)}$ and let $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ be an arbitrary rectangular weight, as defined in Section 3.3. First, we prove that the set $\mathfrak{B}_{L(\Lambda)}$ is linearly independent. As in Section 5.2, we regard the standard module $L(\Lambda)$ as the submodule of $L(\Lambda_j)^{\otimes k_j} \otimes L(\Lambda_0)^{\otimes k_0}$ generated by the highest weight vector $v_{L(\Lambda)} = v_{L(\Lambda_j)}^{\otimes k_j} \otimes v_{L(\Lambda_0)}^{\otimes k_0}$. Suppose that

$$\sum_{a \in A} c_a \, b^a v_{L(\Lambda)} = 0, \quad \text{where} \quad c_a \in \mathbb{C}, \, c_a \neq 0 \text{ for all } a \in A, \tag{6.8}$$

$A$ is a finite nonempty set and all $b^a \in B_{L(\Lambda)}$ posses the same color-type. Let $b^{a_0}$ be a monomial of dual charge-type $\mathcal{D}$ given by (6.2), such that $b^{a_0} < b^a$ for all $a \in A$, $a \neq a_0$, with respect to linear order (5.4). Consider the projection

$$\pi_{\mathcal{D}} \colon W_{L(\Lambda_j)}^{\otimes k_j} \otimes W_{L(\Lambda_0)}^{\otimes k_0} \to W_{(r_l^{(k)};r_{l-1}^{(k)};\ldots;r_2^{(k)};r_1^{(k)})} \otimes \cdots \otimes W_{(r_l^{(1)};r_{l-1}^{(1)};\ldots;r_2^{(1)};r_1^{(1)})},$$

which is defined in parallel with Section 6.2, so that $W_{(r_l^{(t)};r_{l-1}^{(t)};\ldots;r_2^{(t)};r_1^{(t)})}$ is the $\mathfrak{h}$-weight subspace of $W(\Lambda_{j_t})$ of weight $\lambda_{j_t} + r_l^{(t)}\alpha_l + \ldots r_1^{(t)}\alpha_1$; recall (3.5). By applying $\pi_{\mathcal{D}}$ on linear combination (6.8), we obtain

$$\sum_{a \in A} c_a \, \pi_{\mathcal{D}} \, b^a v_{L(\Lambda)} = 0. \tag{6.9}$$

By Section 6.1, the highest weight vector $v_{L(\Lambda)}$ is identified with $(e^{\lambda_j})^{\otimes k_j} \otimes 1^{\otimes k_0}$, so that, due to (6.1), we have

$$v_{L(\Lambda)} = (e^{\lambda_j})^{\otimes k_j} \otimes 1^{\otimes k_0} = (e_{\lambda_j}^{\otimes k_j} \otimes 1^{\otimes k_0})\, 1^{\otimes k} = (e_{\lambda_j}^{\otimes k_j} \otimes 1^{\otimes k_0}) v_{L(\Lambda_0)}^{\otimes k} \quad \text{for } k = k_0 + k_j.$$

Therefore, linear combination (6.9) can be expressed as

$$\sum_{a\in A} c_a\, \pi_{\mathcal{D}}\, b^a (e_{\lambda_j}^{\otimes k_j} \otimes 1^{\otimes k_0}) v_{L(\Lambda_0)}^{\otimes k} = 0.$$

By employing (6.4) to move $e_{\lambda_j}^{\otimes k_j} \otimes 1^{\otimes k_0}$ all the way to the left and then dropping the invertible operator, we get

$$\sum_{a\in A} \tilde{c}_a\, \pi_{\mathcal{D}}\, \tilde{b}^a v_{L(\Lambda_0)}^{\otimes k} = 0$$

for some quasi-particle monomials $\tilde{b}^a$ and $\tilde{c}_a = \pm c_a$. Note that every $\tilde{b}^a$ is obtained by increasing the energy of each quasi-particle of color $j$ and charge $n$ in $b^a$ by $\sum_{t=1}^{n} \delta_{jj_t}$. Indeed, the operator $e_{\lambda_j}$ is applied on the tensor factors $1, \ldots, k_j$, while the projection $\pi_{\mathcal{D}}$ places $n$ quasi-particles of color $j$ and charge 1 on the tensor factors $k_0 + k_j - n + 1, \ldots, k_0 + k_j$. Hence the interactions between $e_{\lambda_j}$ and the quasi-particle of color $j$ and charge $n$ occur on the tensor factors $k_0 + k_j - n + 1, \ldots, k_j$ for $n > k_0$. There are $\sum_{t=1}^{n} \delta_{jj_t}$ such factors and, by (6.4), each interaction increases the energy of the aforementioned quasi-particle of charge $n$ and color $j$ by 1.

Finally, as the original monomials $b^a$ belong to $B_{L(\Lambda)}$, by comparing the difference conditions ($c_2$) and ($c_2'$) we see that the monomials $\tilde{b}^a$ belong to $B_{L(k\Lambda_0)}$. Therefore, due to the identification $v_{L(\Lambda_0)}^{\otimes k} = v_{L(k\Lambda_0)}$, the linear independence of the set $\mathfrak{B}_{L(\Lambda)}$ now follows from Theorem 3.1.

We now proceed as in Section 5.5 and construct a spanning set for $U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)}$. We denote the image of the element $x \in U(\widetilde{\mathfrak{n}}_+)$ in the quotient $U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)}$ by $\bar{x}$. Let $\bar{\mathfrak{B}}_{L(\Lambda)}$ be the set of all monomials

$$\bar{b} = \left(\bar{x}_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l}) \ldots \bar{x}_{n_{1,l}\alpha_l}(m_{1,l})\right) \ldots \left(\bar{x}_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \ldots \bar{x}_{n_{1,1}\alpha_1}(m_{1,1})\right) \tag{$\bar{m}$}$$

in $U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)}$ such that their charges and energies satisfy

$$n_{r_i^{(1)},i} \leqslant \ldots \leqslant n_{1,i} \quad \text{and} \quad m_{r_i^{(1)},i} \leqslant \ldots \leqslant m_{1,i} \qquad \text{for all } i = 1, \ldots, l \tag{6.10}$$

and difference conditions ($c_1$), ($c_2'$) and ($c_3$). It is clear from Theorem 3.1 that the set of all monomials $\bar{b}$ as in ($\bar{m}$) satisfying (6.10) and difference conditions ($c_1$), ($c_2$) and ($c_3$) spans the quotient $U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)}$. Suppose that such a monomial $\bar{b}$ does not satisfy the more restrictive condition ($c_2'$). Introduce the generating function

$$\bar{X} = \bar{x}_{n_{r_l^{(1)},l}\alpha_l}(z_{r_l^{(1)},l}) \cdots \bar{x}_{n_{2,1}\alpha_1}(z_{2,1}) \bar{x}_{n_{1,1}\alpha_1}(z_{1,1}).$$

Clearly, $\bar{b}$ equals the coefficient of the variables

$$z_{r_l^{(1)},l}^{-m_{r_l^{(1)},l} - n_{r_l^{(1)},l}} \cdots z_{2,1}^{-m_{2,1}-n_{2,1}} z_{1,1}^{-m_{1,1}-n_{1,1}}$$

in $\bar{X}$. By (4.3) we have $U(\widetilde{\mathfrak{n}}_+)\widetilde{\mathfrak{n}}_+^{\geqslant 0} \subset I_{L(\Lambda)}$. Therefore, due to commutation relations

$$(z_{p,i} - z_{q,i'})^{M_i} x_{n_{q,i'}\alpha_{i'}}(z_{q,i'}) x_{n_{p,i}\alpha_i}(z_{p,i}) = (z_{p,i} - z_{q,i'})^{M_i} x_{n_{p,i}\alpha_i}(z_{p,i}) x_{n_{q,i'}\alpha_{i'}}(z_{q,i'})$$

with $M_i = \min\{n_{q,i'}, n_{p,i}\}$, the product $P\bar{X}$, where $P$ is the Laurent polynomial

$$P = \prod_{i=2}^{l} \prod_{q=1}^{r_{i'}^{(1)}} \prod_{p=1}^{r_i^{(1)}} \left(1 - \frac{z_{q,i'}}{z_{p,i}}\right)^{\min\left\{n_{q,i'}, n_{p,i}\right\}},$$

belongs to

$$\prod_{i=1}^{l} \prod_{p=1}^{r_i^{(1)}} z_{p,i}^{-\sum_{q=1}^{r_{i'}^{(1)}} \min\left\{n_{q,i'}, n_{p,i}\right\}} (U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)})[[z_{r_l^{(1)},l}, \ldots, z_{1,1}]]. \tag{6.11}$$

However, every vertex operator $\bar{x}_{n\alpha_i}(z)$ in the product $P\bar{X}$ can be moved all the way to the right. By (4.3) we have $x_{\alpha_j}(-1)^{k_0+1} \in I_{L(\Lambda)}$, so that each $\bar{x}_{n\alpha_i}(z)$ increases the power of its variable $z$ in (6.11) by $\sum_{t=1}^{n} \delta_{ij_t}$. Therefore, we have

$$P\bar{X} \in \prod_{i=1}^{l} \prod_{p=1}^{r_i^{(1)}} z_{p,i}^{\sum_{t=1}^{n_{p,i}} \delta_{ij_t} - \sum_{q=1}^{r_{i'}^{(1)}} \min\left\{n_{q,i'}, n_{p,i}\right\}} (U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)})[[z_{r_l^{(1)},l}, \ldots, z_{1,1}]]. \tag{6.12}$$

By comparing the coefficients of powers of the variables $z_{p,i}$ in (6.12), the monomial $\bar{b}$ can be expressed as a linear combination of elements of $\bar{\mathfrak{B}}_{L(\Lambda)}$. Hence we conclude that the set $\bar{\mathfrak{B}}_{L(\Lambda)}$ spans the quotient $U(\widetilde{\mathfrak{n}}_+)/I_{L(\Lambda)}$.

Since the ideal $I_{L(\Lambda)}$ belongs to the kernel of the map $f_{L(\Lambda)}$ defined by (4.1), Theorems 3.2 and 4.2 can be now verified by arguing as in Section 5.6.

6.4. **Operator $A_\theta$ revisited.** As with type $G$ in [6], the linear independence proof in type $F$ employs certain operator $A_\theta = x_\theta(-1)$; see Sections 5.3 and 5.4. In this section we show that the operator $A_\theta$ associated with the maximal root $\theta$ in type $E$ can be also used to verify the linear independence, but of different bases. First, for $\mathfrak{g} = E_l$ set

$$(i_1, \ldots, i_l; i_3'', \ldots, i_l'') = \begin{cases} (1,7,2,3,4,5,6,8;1,2,3,4,5,5), & \text{if } l = 8, \\ (1,6,5,4,3,2,7;6,5,4,3,3), & \text{if } l = 7, \\ (6,5,4,3,2,1;5,4,3,2), & \text{if } l = 6. \end{cases}$$

Introduce the following families of difference conditions:

$$m_{p,i_j} \leqslant -n_{p,i_j} - 2(p-1)n_{p,i_j} \quad \text{for} \quad p = 1, \ldots, r_{i_j}^{(1)} \quad \text{and} \quad j = 1, 2; \tag{$c_2^0$}$$

$$m_{p,i_j} \leqslant -n_{p,i_j} + \sum_{q=1}^{r_{i_j''}^{(1)}} \min\left\{n_{q,i_j''}, n_{p,i_j}\right\} - 2(p-1)n_{p,i_j} \quad \text{for} \quad p = 1, \ldots, r_{i_j}^{(1)}; \tag{$c_2^j$}$$

$$m_{p,i_j} \leqslant -n_{p,i_j} + \sum_{s=i_j'', i_k} \sum_{q=1}^{r_s^{(1)}} \min\left\{n_{q,s}, n_{p,i_j}\right\} - 2(p-1)n_{p,i_j} \quad \text{for} \quad p = 1, \ldots, r_{i_j}^{(1)}. \tag{$c_2^{j,k}$}$$

Let $B_{L(k\Lambda_0)}^{E_l}$ be the set all monomials $(m)$ which satisfy (6.10) and the following difference conditions:

- $(c_1)$, $(c_3)$, $(c_2^0)$, $(c_2^j)$ for $j = 3,4,5,6,8$ and $(c_2^{j,k})$ for $(j,k) = (7,2)$ if $l = 8$;
- $(c_1)$, $(c_3)$, $(c_2^0)$, $(c_2^j)$ for $j = 3,4,5,7$ and $(c_2^{j,k})$ for $(j,k) = (6,1)$ if $l = 7$;
- $(c_1)$, $(c_3)$, $(c_2^0)$, $(c_2^j)$ for $j = 3,5,6$ and $(c_2^{j,k})$ for $(j,k) = (4,1)$ if $l = 6$.

**Proposition 6.1.** *For any positive integer $k$ the set*

$$\mathfrak{B}^{E_l}_{L(k\Lambda_0)} = \left\{ b v_{L(k\Lambda_0)} : b \in B^{E_l}_{L(k\Lambda_0)} \right\} \subset W_{L(k\Lambda_0)}$$

*forms a basis of the principal subspace $W_{L(k\Lambda_0)}$ of the standard module $L(k\Lambda_0)$ for the affine Lie algebra in type $E_l^{(1)}$.*

*Proof.* The maximal root $\theta$ in type $E$ satisfies

$$\alpha_i(\theta^\vee) = \delta_{6i} \quad \text{for } \mathfrak{g} = E_6 \qquad \text{and} \qquad \alpha_i(\theta^\vee) = \delta_{1i} \quad \text{for } \mathfrak{g} = E_7, E_8. \tag{6.13}$$

Therefore, as described in Section 5.4, by applying the procedure from Section 5.3 on an arbitrary linear combination of elements of $\mathfrak{B}^{E_8}_{L(k\Lambda_0)}$, one can remove all quasi-particles of color 1 from the corresponding quasi-particle monomials. The resulting linear combination can be identified as a linear combination of elements of $\mathfrak{B}^{E_7}_{L(k\Lambda_0)}$; see Figure 1. Due to (6.13), by applying the same procedure once again, one can remove all quasi-particles of color 1[b] from the corresponding quasi-particle monomials, thus obtaining the expression which can be identified as a linear combination of elements of the basis $\mathfrak{B}_{L(k\Lambda_0)}$ from Theorem 3.1 for $\mathfrak{g} = D_6$; see Figure 1. As for type $E_6$, due to (6.13), by applying the procedure from Section 5.3 on an arbitrary linear combination of elements of $\mathfrak{B}^{E_6}_{L(k\Lambda_0)}$, one can remove all quasi-particles of color 6 from the corresponding quasi-particle monomials. The resulting expression can be identified as a linear combination of elements of the basis $\mathfrak{B}_{L(k\Lambda_0)}$ from Theorem 3.1 for $\mathfrak{g} = A_5$; see Figure 1. Therefore, the proposition follows from Theorem 3.1 and the fact that the characters of the corresponding bases coincide which is verified by arguing as in Section 7. □

## 7. Character formulae and combinatorial identities

Let $\delta = \sum_{i=0}^{l} a_i \alpha_i$ be the imaginary root as in [23, Chap. 5], where the integers $a_i$ denote the labels in the Dynkin diagram [23, Table Aff] for $\widetilde{\mathfrak{g}}$. As before, let $V$ denote a standard module or a generalized Verma module. Define the character $\operatorname{ch} W_V$ of the corresponding principal subspace $W_V$ by

$$\operatorname{ch} W_V = \sum_{m,n_1,\ldots,n_l \geqslant 0} \dim (W_V)_{-m\delta + n_1\alpha_1 + \ldots + n_l\alpha_l} \, q^m y_1^{n_1} \cdots y_l^{n_l},$$

where $q, y_1, \ldots, y_l$ are formal variables and $(W_V)_{-m\delta + n_1\alpha_1 + \ldots + n_l\alpha_l}$ denote the weight subspaces of $W_V$ of weight $-m\delta + n_1\alpha_1 + \cdots + n_l\alpha_l$ with respect to

$$\widetilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

[b] Note that the quasi-particles of color 1 in type $E_7$ correspond, with respect to the aforementioned identification, to the quasi-particles of color 7 in type $E_8$; see Figure 1.

In order to simplify our notation, we set $\mu_i = \nu_i/\nu_{i'}$ for $i = 2, \ldots, l$; recall (3.3). Also, we write

$$(a;q)_r = \prod_{i=1}^{r}(1-aq^{i-1}) \quad \text{for} \quad r \geqslant 0 \qquad \text{and} \qquad (a;q)_\infty = \prod_{i\geqslant 1}(1-aq^{i-1}).$$

Theorem 3.1 implies the following character formulae:

**Theorem 7.1.** *Set $n_i = \sum_{t=1}^{\nu_i k} r_i^{(t)}$ for $i = 1, \ldots, l$. For any integer $k \geqslant 1$ we have*

$$\operatorname{ch} W_{L(k\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geqslant \cdots \geqslant r_1^{(\nu_1 k)}\geqslant 0\\ \vdots \\ r_l^{(1)}\geqslant \cdots \geqslant r_l^{(\nu_l k)}\geqslant 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t=1}^{\nu_i k} r_i^{(t)^2} - \sum_{i=2}^{l}\sum_{t=1}^{k}\sum_{p=0}^{\mu_i-1} r_{i'}^{(t)} r_i^{(\mu_i t - p)}}}{\prod_{i=1}^{l}(q;q)_{r_i^{(1)}-r_i^{(2)}}\cdots(q;q)_{r_i^{(\nu_i k)}}} \prod_{i=1}^{l} y_i^{n_i}.$$

*Proof.* We give the proof of this theorem for the case $F_4^{(1)}$, since the proof for the cases $D_l^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ goes analogously. The proof for other types can be found in [4–6, 20]. In order to determine the character of $W_{L(k\Lambda_0)}$, we write conditions on energies of quasi-particles of the set $B_{W_{L(k\Lambda_0)}}$ in terms of $r_i^{(s)}$. Fix a color-type $(n_4, n_3, n_2, n_1)$, charge-type

$$\mathcal{C} = \left(n_{r_4^{(1)},4}, \ldots, n_{1,4};\, n_{r_3^{(1)},3}, \ldots, n_{1,3};\, n_{r_2^{(1)},2}, \ldots, n_{1,2};\, n_{r_1^{(1)},1}, \ldots, n_{1,1}\right)$$

and dual-charge-type

$$\mathcal{D} = \left(r_4^{(1)}, \ldots, r_4^{(2k)};\, r_3^{(1)}, \ldots, r_3^{(2k)};\, r_2^{(1)}, \ldots, r_2^{(k)};\, r_1^{(1)}, \ldots, r_1^{(k)}\right)$$

The following identities are well-known, see, e.g., [20, Section 5] and [4, Section 4], and they can be verified by a direct calculation:

$$\sum_{p=1}^{r_i^{(1)}}(2(p-1)n_{p,i} + n_{p,i}) = \sum_{t=1}^{k} r_i^{(t)^2} \quad \text{for } i = 1, 2, \tag{7.1}$$

$$\sum_{p=1}^{r_i^{(1)}}((2(p-1)n_{p,i} + n_{p,i}) = \sum_{t=1}^{2k} r_i^{(t)^2} \quad \text{for } i = 3, 4, \tag{7.2}$$

$$\sum_{p=1}^{r_2^{(1)}}\sum_{q=1}^{r_1^{(1)}} \min\{n_{p,2}, n_{q,1}\} = \sum_{t=1}^{k} r_1^{(t)} r_2^{(t)}, \qquad \sum_{p=1}^{r_4^{(1)}}\sum_{q=1}^{r_3^{(1)}} \min\{n_{p,4}, n_{q,3}\} = \sum_{t=1}^{2k} r_3^{(t)} r_4^{(t)}, \tag{7.3}$$

$$\sum_{p=1}^{r_3^{(1)}}\sum_{q=1}^{r_2^{(1)}} \min\{n_{p,3}, 2n_{q,2}\} = \sum_{t=1}^{k} r_2^{(t)}(r_3^{(2t-1)} + r_3^{(2t)}). \tag{7.4}$$

By combining (7.1)–(7.4), difference conditions ($c_1$)–($c_3$) and the formula

$$\frac{1}{(q)_r} = \sum_{j\geqslant 0} p_r(j) q^j,$$

where $p_r(j)$ denotes the number of partitions of $j$ with at most $r$ parts, we get

$$\mathrm{ch}\ W_{L(k\Lambda_0)} = \sum_{\substack{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(k)}\geqslant 0\\ r_2^{(1)}\geqslant\cdots\geqslant r_2^{(k)}\geqslant 0}} \frac{q^{\sum_{i=1}^{2}\sum_{t=1}^{k} {r_i^{(t)}}^2 - \sum_{t=1}^{k} r_1^{(t)} r_2^{(t)}}}{\prod_{i=1}^{2} (q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots (q;q)_{r_i^{(k)}}} \prod_{i=1}^{2} y_i^{n_i}$$

$$\times \sum_{\substack{r_3^{(1)}\geqslant\cdots\geqslant r_3^{(2k)}\geqslant 0\\ r_4^{(1)}\geqslant\ldots\geqslant r_4^{(2k)}\geqslant 0}} \frac{q^{\sum_{i=3}^{4}\sum_{t=1}^{2k} {r_i^{(t)}}^2 - \sum_{t=1}^{2k} r_3^{(t)} r_4^{(t)} - \sum_{t=1}^{k} r_2^{(t)} (r_3^{(2t-1)} + r_3^{(2t)})}}{\prod_{i=3}^{4} (q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots (q;q)_{r_i^{(2k)}}} \prod_{i=3}^{4} y_i^{n_i},$$

where $n_i = \sum_{t=1}^{k} r_i^{(t)}$ for $i = 1,2$ and $n_i = \sum_{t=1}^{2k} r_i^{(t)}$ for $i = 3,4$, as required. The character formula for the generalized Verma module is verified analogously. □

Theorem 3.2 implies the following character formulae in types $D_l^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$ while the case $A_l^{(1)}$ is due to [20].

**Theorem 7.2.** *Set $n_i = r_i^{(1)} + \cdots + r_i^{(k)}$ for $i = 1,\ldots,l$. For any rectangular weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ of level $k = k_0 + k_j$ we have*

$$\mathrm{ch}\, W_{L(\Lambda)} = \sum_{\substack{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(k)}\geqslant 0\\ \vdots \\ r_l^{(1)}\geqslant\cdots\geqslant r_l^{(k)}\geqslant 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t=1}^{k} {r_i^{(t)}}^2 - \sum_{i=2}^{l}\sum_{t=1}^{k} r_{i'}^{(t)} r_i^{(t)} + \sum_{i=1}^{l}\sum_{t=1}^{k} r_i^{(t)} \delta_{i j_t}}}{\prod_{i=1}^{l} (q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots (q;q)_{r_i^{(k)}}} \prod_{i=1}^{l} y_i^{n_i}.$$

Note that from (5.30) we have an isomorphism of $\widetilde{\mathfrak{n}}_+$-modules $W_{N(k\Lambda_0)}$ and $U(\widetilde{\mathfrak{n}}_+^{<0})$, so we can obtain character formula of $W_{N(k\Lambda_0)}$ by using Poincaré–Birkhoff–Witt basis of $U(\widetilde{\mathfrak{n}}_+^{<0})$ as well. For example, in the case $F_4^{(1)}$, we get

$$\mathrm{ch}\, W_{N(k\Lambda_0)} = \frac{1}{(qy_1, qy_1y_2, qy_1y_2y_3, qy_1y_2y_3y_4, qy_2, qy_2y_3, qy_2y_3y_4, qy_2y_3^2; q)_\infty} \tag{7.5}$$
$$\times \frac{1}{(qy_1y_2y_3^2, qy_1y_2y_3^2y_4, qy_1y_2y_3^2y_4^2, qy_1y_2^2y_3^2, qy_1y_2^2y_3^2y_4, qy_1y_2^2y_3^2y_4^2, qy_3, qy_2y_3^2y_4; q)_\infty}$$
$$\times \frac{1}{(qy_1y_2^2y_3^3y_4, qy_1y_2^2y_3^3y_4^2, qy_1y_2^2y_3^4y_4^2, qy_1y_2^3y_3^4y_4^2, qy_1^2y_2^3y_3^4y_4^2, qy_3y_4, qy_2y_3^2y_4^2, qy_4; q)_\infty},$$

where

$$(a_1,\ldots,a_n;q)_\infty := (a_1;q)_\infty \cdots (a_n;q)_\infty.$$

For any positive root $\alpha = a_1\alpha_1 + \cdots a_l\alpha_l \in R_+$ we introduce the following notation

$$(\alpha;q)_\infty = (qy_1^{a_1}y_2^{a_2}\ldots y_l^{a_l};q)_\infty,$$

so that for an arbitrary affine Lie algebra $\widetilde{\mathfrak{g}}$ character formula (7.5) generalizes to

$$\mathrm{ch}\, W_{N(k\Lambda_0)} = \frac{1}{\prod_{\alpha\in R_+} (\alpha;q)_\infty}. \tag{7.6}$$

On the other hand, by comparing the sets $B_{W_{N(k\Lambda_0)}}$ and $B_{W_{L(k\Lambda_0)}}$, we conclude that the character formula $\mathrm{ch}\, W_{N(k\Lambda_0)}$ is obtained from $\mathrm{ch}\, W_{L(k\Lambda_0)}$ by removing the quasi-particle charges constraints coming from ($c_3$). Therefore, Theorem 7.1 and (7.6) imply the following generalization of Euler–Cauchy theorem; cf. [1].

**Theorem 7.3.** *For any untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$ we have*

$$\frac{1}{\prod_{\alpha\in R_+}(\alpha;q)_\infty} = \sum_{\substack{r_1^{(1)}\geqslant\cdots\geqslant r_1^{(m)}\geqslant\cdots\geqslant 0\\ \vdots \\ r_l^{(1)}\geqslant\cdots\geqslant r_l^{(m)}\geqslant\cdots\geqslant 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t\geqslant 1} r_i^{(t)^2} - \sum_{i=2}^{l}\sum_{t\geqslant 1}\sum_{p=0}^{\mu_i-1} r_{i'}^{(t)} r_i^{(\mu_i t-p)}}}{\prod_{i=1}^{l}\prod_{j\geqslant 1}(q;q)_{r_i^{(j)}-r_i^{(j+1)}}} \prod_{i=1}^{l} y_i^{n_i},$$

*where $n_i = \sum_{t\geqslant 1} r_i^{(t)}$ for $i = 1,\ldots,l$ and the sum on the right hand side goes over all descending infinite sequences of nonnegative integers with finite support.*

In particular, the theorem produces three new families of combinatorial identities which correspond to types $D$, $E$ and $F$.

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## References

[1] G. E. Andrews, *Partitions and Durfee dissection*, Amer. J. Math. **101**, no. 3, (1979), 735–742.

[2] E. Ardonne, R. Kedem, M. Stone, *Fermionic Characters and Arbitrary Highest-Weight Integrable $\widehat{\mathfrak{sl}}_{r+1}$-Modules*, Comm. Math. Phys. **264** (2006), 427–464.

[3] I. Baranović, M. Primc, G. Trupčević, *Bases of Feigin–Stoyanovsky's type subspaces for $C_l^{(1)}$*, Ramanujan J. **45** (2018), 265–289.

[4] M. Butorac, *Combinatorial bases of principal subspaces for the affine Lie algebra of type $B_2^{(1)}$*, J. Pure Appl. Algebra **218** (2014), 424–447.

[5] M. Butorac, *Quasi-particle bases of principal subspaces for the affine Lie algebras of types $B_l^{(1)}$ and $C_l^{(1)}$*, Glas. Mat. Ser. III **51** (2016), 59–108.

[6] M. Butorac, *Quasi-particle bases of principal subspaces for the affine Lie algebra of type $G_2^{(1)}$*, Glas. Mat. Ser. III **52** (2017), 79–98.

[7] M. Butorac, C. Sadowski, *Combinatorial bases of principal subspaces of modules for twisted affine Lie algebras of type $A_{2l-1}^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$*, New York J. Math. **25** (2019), 71–106.

[8] W. Cai, N. Jing, *On vertex operator realizations of Jack functions*, J. Algebraic Combin. **32** (2010), 579–595.

[9] C. Calinescu, J. Lepowsky, A. Milas, *Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$–modules, I: level one case*, Internat. J. Math. **19** (2008), 71–92.

[10] C. Calinescu, J. Lepowsky, A. Milas, *Vertex–algebraic structure of the principal subspaces of certain $A_1^{(1)}$–modules, II: higher level case*, J. Pure Appl. Algebra **212** (2008), 1928–1950.

[11] C. Calinescu, J. Lepowsky, A. Milas, *Vertex–algebraic structure of the principal subspaces of level one modules for the untwisted affine Lie algebras of types A, D, E*, J. Algebra **323** (2010), 167–192.

[12] C. Calinescu, M Penn, C. Sadowski, *Presentations of Principal Subspaces of Higher Level Standard $A_2^{(2)}$-Modules*, Algebr. Represent. Theory (2018) https://doi.org/10.1007/s10468-018-9828-y.

[13] S. Capparelli, J. Lepowsky, A. Milas, *The Rogers-Ramanujan recursion and intertwining operators*, Commun. Contemp. Math. **5** (2003), 947–966.

[14] S. Capparelli, J. Lepowsky, A. Milas, *The Rogers-Selberg recursions, the Gordon-Andrews identities and intertwining operators*, Ramanujan J. **12** (2006), 379–397.

[15] C. Dong and J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math. 112, Birkhäuser, Boston, 1993.

[16] A. V. Stoyanovsky, B. L. Feigin, *Functional models of the representations of current algebras and semi-infinite Schubert cells*, Funktsional Anal. i Prilozhen. **28** (1) (1994), 68–90, 96 (in Russian); translation in Funct. Anal. Appl. **28** (1) (1994), 55–72; preprint B. L. Feigin, A. V. Stoyanovsky, *Quasi-particles models for the representations of Lie algebras and geometry of flag manifold*, arXiv:hep-th/9308079.

[17] I. Frenkel, Y.-Z. Huang, J. Lepowsky, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, Mem. Amer. Math. Soc. **104**, No. 494 (1993), 64 pages.

[18] I. Frenkel, V. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23–66.

[19] I. Frenkel, J. Lepowsky, A. Meurman, Vertex operator algebras and the monster, Pure and Appl. Math., Academic Press, Boston, 1988.

[20] G. Georgiev, *Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace*, J. Pure Appl. Algebra **112** (1996), 247–286.

[21] J. Humphreys, Introduction to Lie Algebras and Their Representations, Graduated Texts in Mathematics, Springer-Verlag, New York, 1972.

[22] M. Jerković, M. Primc, *Quasi-particle fermionic formulas for* $(k, 3)$*-admissible configurations*, Cent. Eur. J. Math. **10** (2012), 703–721.

[23] V. G. Kac, Infinite Dimensional Lie Algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.

[24] S. Kanade, *On a Koszul complex related to the principal subspace of the basic vacuum module for* $A_1^{(1)}$, J. Pure Appl. Algebra **222** (2018), 323–339.

[25] K. Kawasetsu, *The Free Generalized Vertex Algebras and Generalized Principal Subspaces*, J. Algebra **444** (2015), 20–51.

[26] S. Kožić, *Principal subspaces for quantum affine algebra* $U_q(A_n^{(1)})$, J. Pure Appl. Algebra **218** (2014), 2119–2148.

[27] S. Kožić, *Higher level vertex operators for* $U_q(\widehat{\mathfrak{sl}}_2)$, Selecta Math. (N.S.) **23** (2017), 2397–2436.

[28] S. Kožić, *Commutative operators for double Yangian* $DY(\mathfrak{sl}_n)$, Glas. Mat. Ser. III Vol. **53**, No.1 (2018), 97–113.

[29] J. Lepowsky and H.-S. Li, Introduction to Vertex Operator Algebras and Their Representations, Progress in Math. Vol. **227**, Birkhäuser, Boston, 2003.

[30] J. Lepowsky, S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. Math. **29** (1978), 15–59.

[31] J. Lepowsky, M. Primc, *Structure of the standard modules for the affine Lie algebra* $A_1^{(1)}$, Contemp. Math. **46**, Amer. Math. Soc., Providence, 1985.

[32] J. Lepowsky, R. L. Wilson, *The structure of standard modules,* ***I****: Universal algebras and the Rogers-Ramanujan identities*, Invent. Math. **77** (1984), 199–290; ***II****, The case* $A_1$*, principal gradation*, Invent. Math. **79** (1985), 417–442.

[33] A. Meurman, M. Primc, *Annihilating ideals of standard modules of* $\mathfrak{sl}(2, \mathbb{C})\tilde{}$ *and combinatorial identities*, Adv. Math. **64** (1987), 177–240.

[34] A. Milas, M. Penn, *Lattice vertex algebras and combinatorial bases: general case and W-algebras*, New York J. Math. **18** (2012), 621–650.

[35] K. C. Misra, *Structure of Certain Standard Modules for* $A_n^{(1)}$ *and the Rogers–Ramanujan Identities*, J. Algebra **88** (1984), 196–227.

[36] M. Penn, C. Sadowski, *Vertex-algebraic structure of basic* $D_4^{(3)}$ *modules*, Ramanujan J. **43** (2017), 571–617.

[37] M. Penn, C. Sadowski, *Vertex-algebraic structure of principal subspaces of the basic modules for twisted Affine Kac-Moody Lie algebras of type* $A_{2n-1}^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$, J. Algebra **496** (2018), 242–291.

[38] M. Primc, *Vertex operator construction of standard modules for* $A_n^{(1)}$, Pacific J. Math. **162** (1994), 143–187.

[39] C. Sadowski, *Presentations of the principal subspaces of the higher-level standard* $\widehat{\mathfrak{sl}(3)}$*–modules*, J. Pure Appl. Algebra **219** (2015), 2300–2345.

[40] C. Sadowski, *Principal subspaces of higher-level standard* $\widehat{\mathfrak{sl}(n)}$*-modules*, Internat. J. Math. **26** 1550063 (2015).

[41] G. Segal, *Unitary representations of some infinite-dimensional groups*, Comm. Math. Phys. **80** (1981), 301–342.