

ON THE QUANTUM AFFINE VERTEX ALGEBRA ASSOCIATED WITH TRIGONOMETRIC R -MATRIX

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ABSTRACT. We apply the theory of ϕ -coordinated modules, developed by H.-S. Li, to the Etingof–Kazhdan quantum affine vertex algebra associated with the trigonometric R -matrix of type A . We prove, for a certain associate ϕ of the one-dimensional additive formal group, that any ϕ -coordinated module for the level $c \in \mathbb{C}$ quantum affine vertex algebra is naturally equipped with a structure of restricted level c module for the quantum affine algebra in type A and vice versa. Moreover, we show that any ϕ -coordinated module is irreducible with respect to the action of the quantum affine vertex algebra if and only if it is irreducible with respect to the corresponding action of the quantum affine algebra. In the end, we discuss relation between the centers of the quantum affine algebra and the quantum affine vertex algebra.

INTRODUCTION

The notion of *vertex algebra*, originally introduced by Borcherds [2], presents a remarkable connection between mathematics and theoretical physics. The vertex algebra theory led to important breakthroughs in multiple areas such as automorphic forms, finite simple groups and soliton equations; see, e.g., the books by E. Frenkel and Ben-Zvi [15], I. Frenkel, Lepowsky and Meurman [18] and Kac [24]. Some of the most extensively studied examples of vertex algebras come from the theory of affine Kac–Moody Lie algebras; see the books by Kac [23] and Lepowsky and Li [28]. Motivated by a parallel between the development of the theories of affine Lie algebras and quantum affine algebras, as well as by further applications to two-dimensional statistical models and the quantum Yang–Baxter equation, I. Frenkel and Jing [17] formulated a fundamental problem of generalizing the vertex algebra theory to the quantum case.

The notion of *quantum vertex algebra* was introduced by Etingof and Kazhdan [11] based on the ideas of E. Frenkel and Reshetikhin [16]. The examples of quantum vertex algebras were constructed in [11] as quantizations of the quasiclassical structure on the universal affine vertex algebra in type A when the classical R -matrix is of rational, trigonometric and elliptic type. Recently, a structure theory of quantum vertex algebras was developed by De Sole, Gardini and Kac [5] and the Etingof–Kazhdan construction was generalized to the rational R -matrix in types B , C and D by Butorac, Jing and the author [3]. On the other hand, several more general related notions, in particular, of *h -adic nonlocal vertex algebra* and of its *module*, were introduced and extensively studied by Li [30]. They present analogues of the corresponding notions, coming from the Li nonlocal vertex algebra theory [29] and the Bakalov–Kac field algebra theory [1], which are defined over the commutative ring $\mathbb{C}[[h]]$, thus being compatible with Etingof–Kazhdan’s theory. Moreover, the notion of *h -adic nonlocal vertex algebra module*, which presents a generalization of vertex algebra module, appears to provide the right setting for the study of representations of double Yangians and of Etingof–Kazhdan’s quantum vertex

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2010 *Mathematics Subject Classification.* 17B37, 17B69, 81R50.

Key words and phrases. Quantum affine algebra, Quantum vertex algebra, ϕ -Coordinated module, Quantum current.

algebras associated with the rational R -matrix; see [30] and [27] respectively. However, Li's subsequent results [31] suggest that the solution of the original Frenkel–Jing problem of associating quantum vertex algebras to quantum affine algebras requires a new concept of ϕ -coordinated module. Following such an approach, Li, Tan and Wang [32] recently established a correspondence between restricted modules for the Ding–Iohara algebra of level 0 associated with the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ [8] and ϕ -coordinated modules for certain quantum vertex algebra.

The definition of a ϕ -coordinated module W for a quantum vertex algebra V , as given in [31], is characterized by a certain deformed version of the weak associativity property. Roughly speaking, it requires that the expressions

$$((z_1 - z_2)^p Y_W(u, z_1) Y_W(v, z_2)) \Big|_{z_1=\phi(z_2, z_0)} \quad \text{and} \quad (\phi(z_2, z_0) - z_2)^p Y_W(Y(u, z_0)v, z_2)$$

coincide for all $u, v \in V$, where $Y(z)$ is the vertex operator map on V , $Y_W(z)$ the ϕ -coordinated module map, $\phi(z_2, z_0) \in \mathbb{C}((z_2))[[z_0]]$ an associate of the one-dimensional additive formal group and $p \geq 0$ an integer depending on u, v . While setting $\phi(z_2, z_0) = z_2 + z_0$ leads to the usual weak associativity property, a different choice of the associate appears to be required in order to adapt the theory to quantum affine algebras; see [31].

Let $\mathfrak{g}_N = \mathfrak{gl}_N, \mathfrak{sl}_N$. In this paper, we consider the quantum affine vertex algebra $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ associated with the trigonometric R -matrix, as defined by Etingof and Kazhdan [11]. We should mention that $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ can be also regarded as an associative algebra over $\mathbb{C}[[h]]$, which is topologically generated by the coefficients of certain Taylor series organized into the matrix $T^+(u) \in \text{End } \mathbb{C}^N \otimes \overline{\mathcal{V}}_c(\mathfrak{g}_N)[[u]]$, subject to certain dual Yangian-type defining relations. As a quantum vertex algebra, its vertex operator map $Y(z)$ is given in the form of *quantum currents* $\mathcal{T}(u)$, which go back to Reshetikhin and Semenov-Tian-Shansky [35]. Furthermore, the \mathcal{S} -locality property for $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$, which is a quantum analogue of the locality in the corresponding affine vertex algebra, comes from the *quantum current commutation relation* which, in this particular setting, can be expressed as

$$\begin{aligned} \mathcal{T}_1(u_1) R_{21}(e^{-u_1+u_2-hc}) \mathcal{T}_2(u_2) R_{21}(e^{-u_1+u_2})^{-1} \\ = R_{12}(e^{-u_2+u_1})^{-1} \mathcal{T}_2(u_2) R_{12}(e^{-u_2+u_1-hc}) \mathcal{T}_1(u_1), \end{aligned} \quad (0.1)$$

where $R(x) = R_{12}(x)$ is the trigonometric R -matrix of type A .¹ On the other hand, the original quantum current commutation relation in [35] is given in the multiplicative form,

$$\begin{aligned} \mathcal{L}_1(x_1) R_{21}(x_2 e^{-hc}/x_1) \mathcal{L}_2(x_2) R_{21}(x_2/x_1)^{-1} \\ = R_{12}(x_1/x_2)^{-1} \mathcal{L}_2(x_2) R_{12}(x_1 e^{-hc}/x_2) \mathcal{L}_1(x_1). \end{aligned} \quad (0.2)$$

Its significance comes from Ding's quantum current realization of the quantum affine algebra $U_h(\widehat{\mathfrak{g}}_N)$ [6], which relies on the famous Ding–Frenkel isomorphism [7]. The algebra generators are given as coefficients of matrix entries of the *quantum current* $\mathcal{L}(x)$, so that $\mathcal{L}(x)$ belongs to $\text{End } \mathbb{C}^N \otimes U_h(\widehat{\mathfrak{g}}_N)[[x^{\pm 1}]]$, while the defining relations at the level $c \in \mathbb{C}$ are given by (0.2), along with one more family of relations in the $\mathfrak{g}_N = \mathfrak{sl}_N$ case. As in [31, Sect. 5], in this paper we consider the ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -modules for the associate $\phi(z_2, z_0) = z_2 e^{z_0}$, which connects commutation relations (0.1) and (0.2). More specifically, by applying the substitutions $x_i = z e^{u_i}$ with $i = 1, 2$, multiplicative relation (0.2) takes the additive form as in (0.1). It is worth noting that both additive and multiplicative forms of the trigonometric R -matrix naturally occur in the theories of quantum groups and exactly solvable models; see [12, 21, 34].

As with the rational R -matrix case [27], the multiple copies of quantum currents $\mathcal{L}(x_i)$ with $i = 1, \dots, n$ can be organized into the operators $\mathcal{L}_{[n]}(x_1, \dots, x_n)$ in the variables x_1, \dots, x_n which satisfy certain generalized version of commutation relation (0.2).

¹We explain the precise meaning of relations (0.1) and (0.2) in Subsection 1.2.

Roughly speaking, such operators take place of the normal-ordered products of n quantum currents. In particular, for any *restricted* $U_h(\widehat{\mathfrak{g}}_N)$ -module, i.e. for any module W such that $\mathcal{L}(x)w$ belongs to $\text{End } \mathbb{C}^N \otimes W((x))[[h]]$ for all $w \in W$, the series $\mathcal{L}_{[n]}(x_1, \dots, x_n)w$ possesses only finitely many negative powers of the variables x_1, \dots, x_n modulo h^k for all $k \geq 0$ and $w \in W$. By combining Ding's quantum current realization [6] with Li's theory of ϕ -coordinated modules [31] and Cherednik's fusion procedure for the trigonometric R -matrix [4] in the $\mathfrak{g}_N = \mathfrak{sl}_N$ case, we establish the following correspondence between restricted modules for the quantum affine algebra and ϕ -coordinated modules for the Etingof–Kazhdan quantum vertex algebra, which is the main result of this paper.

Main Theorem. *Let $\mathfrak{g}_N = \mathfrak{gl}_N, \mathfrak{sl}_N$. Let W be a restricted $U_h(\widehat{\mathfrak{g}}_N)$ -module of level $c \in \mathbb{C}$. There exists a unique structure of ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module on W , where $\phi(z_2, z_0) = z_2 e^{z_0}$, such that*

$$Y_W(T_{[n]}^+(u_1, \dots, u_n) \mathbf{1}, z) = \mathcal{L}_{[n]}(x_1, \dots, x_n) \Big|_{x_1=ze^{u_1}, \dots, x_n=ze^{u_n}} \quad \text{for all } n \geq 1. \quad (0.3)$$

Conversely, let (W, Y_W) be a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module, where $\phi(z_2, z_0) = z_2 e^{z_0}$. There exists a unique structure of restricted $U_h(\widehat{\mathfrak{g}}_N)$ -module of level c on W such that

$$\mathcal{L}(z) = Y_W(T^+(0) \mathbf{1}, z). \quad (0.4)$$

Moreover, a topologically free $\mathbb{C}[[h]]$ -submodule W_1 of W is a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -submodule of W if and only if W_1 is an $U_h(\widehat{\mathfrak{g}}_N)$ -submodule of W .

In order to establish this correspondence, some minor modifications had to be made to the definitions of quantum affine algebra and ϕ -coordinated module. More specifically, both notions were redefined over the ring $\mathbb{C}[[h]]$ and suitably completed, so that they are compatible with Etingof–Kazhdan's definition of quantum vertex algebra.

In the end, we recollect that the universal affine vertex algebra, which governs the representation theory of the corresponding affine Lie algebra $\widehat{\mathfrak{g}}_N$, is constructed on the vacuum module over the universal enveloping algebra $U(\widehat{\mathfrak{g}}_N)$; see [20, 33]. In contrast, $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ is not the vacuum module over $U_h(\widehat{\mathfrak{g}}_N)$, although its quantum vertex algebra structure turns into the corresponding affine vertex algebra in the classical limit. Furthermore, it is not clear whether the *vacuum module* $\mathcal{V}_c(\mathfrak{g}_N)$ at the level c over the quantum affine algebra $U_h(\widehat{\mathfrak{g}}_N)$ possesses any natural quantum vertex algebra-like structure that governs the representation theory of $U_h(\widehat{\mathfrak{g}}_N)$. However, we have the following simple consequence of the [Main Theorem](#):

Corollary 0.1. *Let $\mathfrak{g}_N = \mathfrak{gl}_N, \mathfrak{sl}_N$. The vacuum module $\mathcal{V}_c(\mathfrak{g}_N)$ over the quantum affine algebra $U_h(\widehat{\mathfrak{g}}_N)$ is a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module. Moreover, $\mathcal{V}_c(\mathfrak{g}_N)$ is an irreducible $U_h(\widehat{\mathfrak{g}}_N)$ -module if and only if it is an irreducible ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module.*

The paper is organized as follows. In Sections 1 and 2, we introduce the notation and provide preliminary definitions and results on restricted $U_h(\widehat{\mathfrak{g}}_N)$ -modules and on ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -modules respectively. In Section 3, we prove the [Main Theorem](#). Finally, in Section 4, we discuss a connection between the families of central elements of the quantum affine algebra and the quantum affine vertex algebra established by ϕ -coordinated module map (0.3).

1. RESTRICTED MODULES FOR THE QUANTUM AFFINE ALGEBRA

In this section, we recall some basic properties of the trigonometric R -matrix of type A . Next, we recall Ding's quantum current realization of the quantum affine algebra in type A and the corresponding notion of restricted module. Also, we derive certain properties of the quantum currents which are required in the following sections. Finally, we demonstrate how the [Main Theorem](#) implies [Corollary 0.1](#).

1.1. **Trigonometric R -matrix.** We use the standard tensor notation, i.e. for any

$$A = \sum_{i,j,k,l=1}^N a_{ijkl} e_{ij} \otimes e_{kl} \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$$

and indices $r, s = 1, \dots, m$ such that $r \neq s$, where $m \geq 2$ and $e_{ij} \in \text{End } \mathbb{C}^N$ are the matrix units, we denote by A_{rs} the element of the algebra $(\text{End } \mathbb{C}^N)^{\otimes m}$,

$$A_{rs} = \sum_{i,j,k,l=1}^N a_{ijkl} (e_{ij})_r (e_{kl})_s, \quad \text{where } (e_{ij})_p = 1^{\otimes(p-1)} \otimes e_{ij} \otimes 1^{\otimes(m-p)}. \quad (1.1)$$

Let $N \geq 2$ be an integer and h a formal parameter. Introduce the trigonometric R -matrix of type A by

$$\begin{aligned} \bar{R}(x) = & \sum_{i=1}^N e_{ii} \otimes e_{ii} + e^{-h/2} \frac{1-x}{1-e^{-hx}} \sum_{\substack{i,j=1 \\ i \neq j}}^N e_{ii} \otimes e_{jj} \\ & + \frac{(1-e^{-h})x}{1-e^{-hx}} \sum_{\substack{i,j=1 \\ i > j}}^N e_{ij} \otimes e_{ji} + \frac{1-e^{-h}}{1-e^{-hx}} \sum_{\substack{i,j=1 \\ i < j}}^N e_{ij} \otimes e_{ji}. \end{aligned} \quad (1.2)$$

R -matrix (1.2) can be regarded as a rational function in the variables x and $e^{h/2}$, i.e. as an element of $(\text{End } \mathbb{C}^N)^{\otimes 2}(x, e^{h/2})$. It satisfies the *Yang-Baxter equation*

$$\bar{R}_{12}(x/y) \bar{R}_{13}(x) \bar{R}_{23}(y) = \bar{R}_{23}(y) \bar{R}_{13}(x) \bar{R}_{12}(x/y) \quad (1.3)$$

and it possesses the *unitarity property*

$$\bar{R}_{12}(x) \bar{R}_{21}(1/x) = 1, \quad (1.4)$$

where, in accordance with (1.1), the subscripts indicate the copies in the tensor product algebra $(\text{End } \mathbb{C}^N)^{\otimes m}$ with $m = 3$ in (1.3) and $m = 2$ in (1.4).

Recall the formal Taylor Theorem,

$$b(z + z_0) = e^{z_0 \frac{\partial}{\partial z}} b(z) = \sum_{k=0}^{\infty} \frac{z_0^k}{k!} \frac{\partial^k}{\partial z^k} b(z) \quad \text{for } b(z) \in V[[z^{\pm 1}]], \quad (1.5)$$

where V is a vector space. Due to (1.5), we can regard the R -matrix $\bar{R}(x)$ as an element of $(\text{End } \mathbb{C}^N)^{\otimes 2}[[x, h]]$ via the expansion

$$\frac{1}{1 - e^{ah}x} = \frac{1}{1 - (x + (e^{ah} - 1)x)} = \sum_{k=0}^{\infty} \frac{(e^{ah} - 1)^k x^k}{k!} \frac{\partial^k}{\partial x^k} \left(\frac{1}{1 - x} \right), \quad a \in \mathbb{C}, \quad (1.6)$$

where

$$e^{ah} = \sum_{k \geq 0} (ah)^k / k! \in \mathbb{C}[[h]] \quad \text{and} \quad (1 - x)^{-1} = \sum_{k \geq 0} x^k \in \mathbb{C}[[x]]. \quad (1.7)$$

Due to [19], there exists a unique series $f_q(x)$ in $\mathbb{C}(q)[[x]]$ such that

$$f_q(xq^{2N}) = f_q(x) \frac{(1 - xq^2)(1 - xq^{2N-2})}{(1 - x)(1 - xq^{2N})}. \quad (1.8)$$

As demonstrated in [26], the series $f_q(x)$ can be expressed as

$$f_q(x) = 1 + \sum_{k=1}^{\infty} f_{q,k} \left(\frac{x}{1-x} \right)^k, \quad (1.9)$$

where all $f_{q,k}(q-1)^{-k} \in \mathbb{C}(q)$ are regular at $q = 1$. Hence, applying the substitution $q = e^{h/2}$ to (1.9) and using the expansions in (1.7) we obtain

$$f(x) := 1 + \sum_{k=1}^{\infty} f_k \left(\frac{x}{1-x} \right)^k \in \mathbb{C}[[x, h]], \quad \text{where } f_k := (f_{q,k})|_{q=e^{h/2}} \in h^k \mathbb{C}[[h]]. \quad (1.10)$$

By [26, Equation (2.11)] series (1.10) satisfies

$$f(x)f(xe^h) \dots f(xe^{(N-1)h}) = \frac{1-x}{1-xe^{(N-1)h}}. \quad (1.11)$$

The normalized R -matrix

$$R(x) = f(x)\bar{R}(x) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N[[x, h]] \quad (1.12)$$

possesses the *crossing symmetry properties*

$$R(xe^{Nh})^{t_1} D_1 (R(x)^{-1})^{t_1} = D_1 \quad \text{and} \quad (R(x)^{-1})^{t_2} D_2 R(xe^{Nh})^{t_2} = D_2, \quad (1.13)$$

where D denotes the diagonal matrix

$$D = \text{diag} \left(e^{(N-1)h/2}, e^{(N-3)h/2}, \dots, e^{-(N-1)h/2} \right) \quad (1.14)$$

and t_i denotes the transposition applied on the tensor factor $i = 1, 2$; see [19].

Express the R -matrix $R(x)$ defined by (1.12) as

$$R(x) = g(x)R^+(x), \quad \text{where } g(x) = \frac{f(x)}{1-e^{-hx}}, \quad R^+(x) = (1-e^{-hx})\bar{R}(x). \quad (1.15)$$

Clearly, $R^+(x)$ is a polynomial with respect to the variable x , i.e. $R^+(x)$ belongs to $(\text{End } \mathbb{C}^N)^{\otimes 2}[[h]][x]$. On the other hand, as $(e^{ah} - 1)x \in xh\mathbb{C}[[h]]$, we conclude by (1.6) and (1.10) that $g(x)$ admits the presentation

$$g(x) = \sum_{k=0}^{\infty} g_k \frac{x^k}{(1-x)^{k+1}}, \quad \text{where } g_k \in h^k \mathbb{C}[[h]] \quad \text{and} \quad g_0 = 1. \quad (1.16)$$

Denote by $\mathbb{C}_*(z_1, \dots, z_n)$ the localization of the ring of Taylor series $\mathbb{C}[[z_1, \dots, z_n]]$ at $\mathbb{C}[z_1, \dots, z_n]^\times$. Consider the unique embedding $\mathbb{C}_*(z_1, \dots, z_n) \rightarrow \mathbb{C}((z_1)) \dots ((z_n))$. Extending the embedding to the h -adic completion of $\mathbb{C}_*(z_1, \dots, z_n)$ we obtain the map

$$\iota_{z_1, \dots, z_n} : \mathbb{C}_*(z_1, \dots, z_n)[[h]] \rightarrow \mathbb{C}((z_1)) \dots ((z_n))[[h]]. \quad (1.17)$$

As in [26], we now apply the substitution $x = e^u$ to the normalized R -matrix $R(x)$ given by (1.12). First, replacing the variable x by e^u in (1.16) we obtain

$$g(e^u) = \sum_{k=0}^{\infty} g_k \frac{e^{ku}}{(1-e^u)^{k+1}} = \sum_{k=0}^{\infty} g_k \frac{e^{ku} \left(\frac{u}{1-e^u} \right)^{k+1}}{u^{k+1}} \in \mathbb{C}_*(u)[[h]]$$

since all numerators $e^{ku}u^{k+1}(1-e^u)^{-k-1}$ belong to $\mathbb{C}[[u]]$ and $g_k \in h^k \mathbb{C}[[h]]$. By applying the embedding ι_u we get $\iota_u g(e^u) \in \mathbb{C}((u))[[h]]$. Next, as $R^+(x)$ is a polynomial with respect to the variable x , by applying the substitution $x = e^u$ we obtain $R^+(e^u)$, which belongs to $(\text{End } \mathbb{C}^N)^{\otimes 2}[[h, u]]$. Finally, there exists a unique $\psi \in 1 + h\mathbb{C}[[h]]$ such that the R -matrix

$$R(e^u) := \psi \iota_u g(e^u) R^+(e^u) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N((u))[[h]] \quad (1.18)$$

possesses the *unitarity property*

$$R_{12}(e^u) R_{21}(e^{-u}) = 1 \quad (1.19)$$

and the *crossing symmetry properties*

$$R(e^{u+Nh})^{t_1} D_1 (R(e^u)^{-1})^{t_1} = D_1 \quad \text{and} \quad (R(e^u)^{-1})^{t_2} D_2 R(e^{u+Nh})^{t_2} = D_2; \quad (1.20)$$

see [10, Prop. 1.2] and [26, Prop. 2.1]. Of course, R -matrix (1.18) also satisfies the *Yang–Baxter equation*

$$R_{12}(e^u)R_{13}(e^{u+v})R_{23}(e^v) = R_{23}(e^v)R_{13}(e^{u+v})R_{12}(e^u). \quad (1.21)$$

In what follows, whenever it is clear from the context, we omit the embedding symbol ι and write, e.g., $f(e^u)$ instead of $\iota_u f(e^u)$. Furthermore, in the multiple variable case, we employ the usual expansion convention where the choice of the embedding is determined by the order of the variables. For example, if σ is a permutation in the symmetric group \mathfrak{S}_n , then $f(e^{u_{\sigma_1} + \dots + u_{\sigma_n}})$ denotes $\iota_{u_{\sigma_1}, \dots, u_{\sigma_n}} f(e^{u_{\sigma_1} + \dots + u_{\sigma_n}}) \in \mathbb{C}((u_{\sigma_1})) \dots ((u_{\sigma_n}))[[h]]$. In particular, by $R_{13}(e^{u+v})$ in (1.21) is denoted $\iota_{u,v} g(e^{u+v})R_{13}^+(e^{u+v})$.

1.2. Quantum affine algebra. Ding’s quantum current realization of the quantum affine algebra of type A was given in [6, Prop. 3.1]. We slightly modify the original definition [6, Def. 3.1] in order to make the setting compatible with the quantum vertex algebra theory; see Remark 1.3 for more details. Our exposition starts in parallel with [27, Subsection 2.1], where a certain quantum current algebra associated with the suitably normalized Yang R -matrix was introduced. We omit some simple proofs as they present a straightforward generalization of the arguments from the aforementioned paper to the trigonometric case.

For any integer $N \geq 2$ denote by $F(N)$ the associative algebra over the ring $\mathbb{C}[[h]]$ generated by the elements $1, C$ and $\lambda_{ij}^{(r)}$, where $i, j = 1, \dots, N$ and $r \in \mathbb{Z}$, subject to the defining relations

$$C \cdot a = a \cdot C \quad \text{and} \quad 1 \cdot a = a \cdot 1 = a \quad \text{for all } a \in F(N),$$

i.e. 1 is the unit and C is a central element in $F(N)$. Introduce the Laurent series

$$\lambda_{ij}(x) = \delta_{ij} - h \sum_{r \in \mathbb{Z}} \lambda_{ij}^{(r)} x^{-r-1} \in F(N)[[x^{\pm 1}]], \quad \text{where } i, j = 1, \dots, N, \quad (1.22)$$

and arrange them into the matrix $\mathcal{L}(x) \in \text{End } \mathbb{C}^N \otimes F(N)[[x^{\pm 1}]]$,

$$\mathcal{L}(x) = \sum_{i,j=1}^N e_{ij} \otimes \lambda_{ij}(x). \quad (1.23)$$

We now introduce certain completion of the algebra $F(N)$ which is suitable for expressing the defining relations for the quantum affine algebra. For an integer $p \geq 1$ let $I_p(N)$ be the left ideal in $F(N)$ generated by all $\lambda_{ij}^{(r)}$, where $i, j = 1, \dots, N$ and $r \geq p-1$. Define the completion of $F(N)$ as the inverse limit

$$\tilde{F}(N) = \varprojlim F(N)/I_p(N).$$

The algebra $\tilde{F}(N)$ is naturally equipped with the h -adic topology and its h -adic completion is equal to $\tilde{F}(N)[[h]]$. For any integer $p \geq 1$ let $I_p^h(N)$ be the h -adically completed left ideal in $\tilde{F}(N)[[h]]$ generated by $I_p(N)$ and the element $h^p \cdot 1$.

We generalize the tensor notation from (1.1) to the matrix $\mathcal{L}(x)$ so that the subscript indicates the copy in the corresponding tensor product algebra,

$$\mathcal{L}_r(x) = \sum_{i,j=1}^N 1^{\otimes(r-1)} \otimes e_{ij} \otimes 1^{\otimes(m-r)} \otimes \lambda_{ij}(x) \in (\text{End } \mathbb{C}^N)^{\otimes m} \otimes F(N)[[x^{\pm 1}]]. \quad (1.24)$$

Employing such notation for $m = 2$ and $r = 1, 2$ we introduce the expressions

$$\begin{aligned} \mathcal{L}_{[2]}^{(1)}(x, y) &= \mathcal{L}_1(x)R_{21}(ye^{-hC}/x)\mathcal{L}_2(y)R_{21}(y/x)^{-1}, \\ \mathcal{L}_{[2]}^{(2)}(x, y) &= R_{12}(x/y)^{-1}\mathcal{L}_2(y)R_{12}(xe^{-hC}/y)\mathcal{L}_1(x). \end{aligned}$$

In accordance with the discussion in Subsection 1.1, the R -matrices $R_{21}(ye^{ahC}/x)^{\pm 1}$ and $R_{12}(xe^{ahC}/y)^{\pm 1}$ with $a \in \mathbb{C}$ are regarded as Taylor series with respect to y/x and x/y respectively. By arguing as in [27, Lemma 2.1], one can prove

Lemma 1.1. *The expressions $\mathcal{L}_{[2]}^{(1)}(x, y)$ and $\mathcal{L}_{[2]}^{(2)}(x, y)$ are well-defined elements of*

$$\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes \widetilde{\mathbb{F}}(N)[[x^{\pm 1}, y^{\pm 1}, h]].$$

Moreover, for any integer $p \geq 1$ both $\mathcal{L}_{[2]}^{(1)}(x, y)$ and $\mathcal{L}_{[2]}^{(2)}(y, x)^2$ modulo $\mathbb{I}_p^h(N)$ belong to

$$\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes \mathbb{F}(N)[[x^{\pm 1}]]((y)).$$

By Lemma 1.1, there exist elements $\lambda_{ijkl}^{(r,s;t)}$ in $\widetilde{\mathbb{F}}(N)[[h]]$ such that

$$\mathcal{L}_{[2]}^{(t)}(x, y) = \sum_{i,j,k,l=1}^N \sum_{r,s \in \mathbb{Z}} e_{ij} \otimes e_{kl} \otimes \lambda_{ijkl}^{(r,s;t)} x^{-r-1} y^{-s-1} \quad \text{for } t = 1, 2.$$

Let $J(N)$ be the ideal in the algebra $\widetilde{\mathbb{F}}(N)[[h]]$ generated by all elements

$$\lambda_{ijkl}^{(r,s;1)} - \lambda_{ijkl}^{(r,s;2)}, \quad \text{where } r, s \in \mathbb{Z} \quad \text{and } i, j, k, l = 1, \dots, N. \quad (1.25)$$

Introduce the completion of $J(N)$ as the inverse limit

$$\widetilde{J}(N) = \varprojlim J(N) / J(N) \cap \mathbb{I}_p(N).$$

Note that the h -adic completion $[\widetilde{J}(N)][[h]]$ of

$$[\widetilde{J}(N)] = \left\{ a \in \widetilde{\mathbb{F}}(N)[[h]] : h^n a \in \widetilde{J}(N) \text{ for some integer } n \geq 0 \right\}$$

is also an ideal in $\widetilde{\mathbb{F}}(N)[[h]]$. Following [6, Def. 3.1], we define the (*completed*) *quantum affine algebra* $U_h(\widehat{\mathfrak{gl}}_N)$ as the quotient of the algebra $\widetilde{\mathbb{F}}(N)[[h]]$ by the ideal $[\widetilde{J}(N)][[h]]$,

$$U_h(\widehat{\mathfrak{gl}}_N) = \widetilde{\mathbb{F}}(N)[[h]] / [\widetilde{J}(N)][[h]]. \quad (1.26)$$

Denote the images of the elements 1, C and $\lambda_{ij}^{(r)}$ in quotient (1.26) again by 1, C and $\lambda_{ij}^{(r)}$. Also, denote by $\lambda_{ij}(x)$ and $\mathcal{L}(x)$ the corresponding series in $U_h(\widehat{\mathfrak{gl}}_N)[[x^{\pm 1}]]$ and $\text{End } \mathbb{C}^N \otimes U_h(\widehat{\mathfrak{gl}}_N)[[x^{\pm 1}]]$ respectively. Defining relations (1.25) for the algebra $U_h(\widehat{\mathfrak{gl}}_N)$ can be expressed by the *quantum current commutation relation*

$$\mathcal{L}_1(x) R_{21}(ye^{-hC}/x) \mathcal{L}_2(y) R_{21}(y/x)^{-1} = R_{12}(x/y)^{-1} \mathcal{L}_2(y) R_{12}(xe^{-hC}/y) \mathcal{L}_1(x), \quad (1.27)$$

as given by Reshetikhin and Semenov-Tian-Shansky [35]. As the images of the elements $\lambda_{ijkl}^{(r,s;1)}$ and $\lambda_{ijkl}^{(r,s;2)}$ in quotient (1.26) coincide, we denote them by $\lambda_{ijkl}^{(r,s)}$. Also, we write

$$\mathcal{L}_{[2]}(x, y) = \sum_{i,j,k,l=1}^N \sum_{r,s \in \mathbb{Z}} e_{ij} \otimes e_{kl} \otimes \lambda_{ijkl}^{(r,s)} x^{-r-1} y^{-s-1}. \quad (1.28)$$

and $\mathcal{L}_{[1]}(x) = \mathcal{L}(x)$. Observe that the both sides of relation (1.27) coincide with $\mathcal{L}_{[2]}(x, y)$. Motivated by [35], we refer to the series $\mathcal{L}(x)$ as *quantum currents*. Our next goal is to derive a certain generalized version of (1.27) consisting of $n + m$ quantum currents.

For integers $n, m \geq 1$ introduce the functions depending on the variable z and the families of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ with values in the space $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m}$ by

$$R_{nm}^{12}(zxe^{ah}/y) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\leftarrow} R_{ij}(zx_i e^{ah}/y_{j-n}), \quad (1.29)$$

²Notice the swapped variables in this term.

$$R_{nm}^{21}(ye^{ah}/zx) = \prod_{i=1, \dots, n}^{\leftarrow} \prod_{j=n+1, \dots, n+m}^{\rightarrow} R_{ji}(y_{j-n}e^{ah}/zx_i), \quad (1.30)$$

where $a \in \mathbb{C}$ and the arrows indicate the order of the factors. For example, we have

$$R_{22}^{12}(zx/y) = R_{14}R_{13}R_{24}R_{23} \quad \text{and} \quad R_{22}^{21}(y/xz) = R'_{32}R'_{42}R'_{31}R'_{41},$$

where $R_{ij} = R_{ij}(zx_i/y_{j-n})$ and $R'_{ji} = R_{ji}(y_{j-n}/zx_i)$. The corresponding functions associated with the R -matrix $R^+(x)$ given by (1.15), $R_{nm}^{+12}(zxe^{ah}/y)$ and $R_{nm}^{+21}(ye^{ah}/zx)$, can be defined analogously. Note that the evaluations of (1.29) and (1.30) at $z = 1$ are well-defined. We denote them by $R_{nm}^{12}(xe^{ah}/y)$ and $R_{nm}^{21}(ye^{ah}/x)$ respectively. Next, for any integer $n \geq 1$ and the family of variables $x = (x_1, \dots, x_n)$ define the functions with values in $(\text{End } \mathbb{C}^N)^{\otimes n}$ by

$$R_{[n,a]}(x) = \prod_{i=1, \dots, n-1}^{\rightarrow} \prod_{j=i+1, \dots, n}^{\rightarrow} R_{ji}(x_j e^{-ah}/x_i)^{-1}, \quad (1.31)$$

$$\tilde{R}_{[n,a]}(x) = \prod_{i=1, \dots, n-1}^{\leftarrow} \prod_{j=i+1, \dots, n}^{\leftarrow} R_{ji}(x_j e^{-ah}/x_i)^{-1}, \quad (1.32)$$

where $a \in \mathbb{C}$ and the arrows again indicate the order of the factors. For example, we have

$$R_{[n,a]}(x) = R_{21}R_{31}R_{41}R_{32}R_{42}R_{43} \quad \text{and} \quad \tilde{R}_{[n,a]}(x) = R_{43}R_{42}R_{32}R_{41}R_{31}R_{21},$$

where $R_{ji} = R_{ji}(x_j e^{-ah}/x_i)^{-1}$. If $a = 0$, we omit the second subscript and write

$$R_{[n]}(x) = R_{[n,0]}(x) \quad \text{and} \quad \tilde{R}_{[n]}(x) = \tilde{R}_{[n,0]}(x).$$

Finally, for any integer $n \geq 2$ we generalize $\mathcal{L}_{[2]}(x, y)$, as given by (1.28), by setting

$$\mathcal{L}_{[n]}(x) = \prod_{i=1, \dots, n}^{\rightarrow} (\mathcal{L}_i(x_i) R_{i+1i}(x_{i+1} e^{-hC}/x_i) \dots R_{ni}(x_n e^{-hC}/x_i)) \cdot \tilde{R}_{[n]}(x). \quad (1.33)$$

Denote by $I_p^h(\widehat{\mathfrak{gl}}_N)$, $I_p(\widehat{\mathfrak{gl}}_N)$ the images of the left ideals $I_p^h(N)$, $I_p(N) \subset \tilde{F}(N)[[h]]$ in the algebra $U_h(\widehat{\mathfrak{gl}}_N)$ with respect to the canonical map $\tilde{F}(N)[[h]] \rightarrow U_h(\widehat{\mathfrak{gl}}_N)$. In the next proposition, we use the superscripts 1, 2, 3 to indicate the following tensor factors:

$$\overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes m}}^2 \otimes \overbrace{U_h(\widehat{\mathfrak{gl}}_N)}^3.$$

The proposition can be proved by using Lemma 1.1, Yang–Baxter equation (1.3), quantum current commutation relation (1.27) and arguing as in [27, Prop. 2.4 and 2.5].

Proposition 1.2. *For any integers $n, m \geq 1$ and the families of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ we have:*

(1) *The expression $\mathcal{L}_{[n]}(x)$ is a well-defined element of*

$$(\text{End } \mathbb{C}^N)^{\otimes n} \otimes U_h(\widehat{\mathfrak{gl}}_N)[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]].$$

(2) *For any $p \geq 1$ the element $\mathcal{L}_{[n]}(x)$ modulo $I_p^h(\widehat{\mathfrak{gl}}_N)$ belongs to*

$$(\text{End } \mathbb{C}^N)^{\otimes n} \otimes U_h(\widehat{\mathfrak{gl}}_N)((x_1, \dots, x_n)).$$

(3) *The following quantum current commutation relation holds:*

$$\begin{aligned} \mathcal{L}_{[n]}^{13}(x) R_{nm}^{21}(ye^{-hC}/x) \mathcal{L}_{[m]}^{23}(y) R_{nm}^{21}(y/x)^{-1} \\ = R_{nm}^{12}(x/y)^{-1} \mathcal{L}_{[m]}^{23}(y) R_{nm}^{12}(xe^{-hC}/y) \mathcal{L}_{[n]}^{13}(x). \end{aligned} \quad (1.34)$$

Moreover, both sides of (1.34) coincide with $\mathcal{L}_{[n+m]}(x, y)$.

Generalizing (1.28) we denote the coefficients of the matrix entries in (1.33) as follows:

$$\mathcal{L}_{[n]}(x) = \sum_{i_1, j_1, \dots, i_n, j_n=1}^N \sum_{r_1, \dots, r_n \in \mathbb{Z}} e_{i_1 j_1} \otimes \dots \otimes e_{i_n j_n} \otimes \lambda_{i_1 j_1 \dots i_n j_n}^{(r_1, \dots, r_n)} x_1^{-r_1-1} \dots x_n^{-r_n-1}.$$

Our next goal is to introduce the quantum affine algebra associated with the affine Lie algebra $\widehat{\mathfrak{sl}}_N$. Let P^h be the h -permutation operator,

$$P^h = \sum_{i=1}^N e_{ii} \otimes e_{ii} + e^{h/2} \sum_{\substack{i, j=1 \\ i > j}}^N e_{ij} \otimes e_{ji} + e^{-h/2} \sum_{\substack{i, j=1 \\ i < j}}^N e_{ij} \otimes e_{ji} \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N[[h]].$$

Consider the action of the symmetric group \mathfrak{S}_n on the space $(\mathbb{C}^N)^{\otimes n}$ which is given by $\sigma_i \mapsto P_{\sigma_i}^h = P_{i i+1}^h$ for $i = 1, \dots, n-1$, where σ_i is the transposition $(i, i+1)$. For a reduced decomposition of a permutation $\sigma = \sigma_{i_1} \dots \sigma_{i_k} \in \mathfrak{S}_n$ set $P_{\sigma}^h = P_{\sigma_{i_1}}^h \dots P_{\sigma_{i_k}}^h$. Let $A^{(n)}$ be the image of the normalized anti-symmetrizer with respect to this action, so that

$$A^{(n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn } \sigma \cdot P_{\sigma}^h. \quad (1.35)$$

Define the *quantum determinant* of the matrix $\mathcal{L}(x)$ by

$$\text{qdet } \mathcal{L}(x) = \text{tr}_{1, \dots, N} A^{(N)} \mathcal{L}_{[N]}(x_1, \dots, x_N) \Big|_{x_1=x, \dots, x_N=x e^{-(N-1)h}} D_1 \dots D_N, \quad (1.36)$$

where the trace is taken over all N copies of $\text{End } \mathbb{C}^N$ and the matrix D is given by (1.14). The quantum determinant is a formal power series in the variable x with coefficients in the quantum affine algebra, i.e. $\text{qdet } \mathcal{L}(x)$ belongs to $U_h(\widehat{\mathfrak{gl}}_N)[[x^{\pm 1}]]$. Indeed, the substitution $x_1 = x, \dots, x_N = x e^{-(N-1)h}$ in (1.36) is well-defined due to the second assertion of Proposition 1.2. Furthermore, all coefficients d_r of the quantum determinant

$$\text{qdet } \mathcal{L}(x) = 1 - h \sum_{r \in \mathbb{Z}} d_r x^r \quad (1.37)$$

belong to the center of the quantum affine algebra at the level $c \in \mathbb{C}$; see Proposition 4.1.

Let I_{qdet} be the ideal in the algebra $U_h(\widehat{\mathfrak{gl}}_N)$ generated by the elements d_r , where $r \in \mathbb{Z}$. Introduce its completion as the inverse limit

$$\widetilde{I}_{\text{qdet}} = \varprojlim I_{\text{qdet}} / I_{\text{qdet}} \cap I_p(\mathfrak{gl}_N).$$

The h -adic completion $[\widetilde{I}_{\text{qdet}}][[h]]$ of

$$[\widetilde{I}_{\text{qdet}}] = \left\{ a \in U_h(\widehat{\mathfrak{gl}}_N) : h^n a \in \widetilde{I}_{\text{qdet}} \text{ for some integer } n \geq 0 \right\}$$

is also an ideal in $U_h(\widehat{\mathfrak{gl}}_N)$. Define the (*completed*) *quantum affine algebra* $U_h(\widehat{\mathfrak{sl}}_N)$ as the quotient of the algebra $U_h(\widehat{\mathfrak{gl}}_N)$ by the relation $\text{qdet } \mathcal{L}(x) = 1$, i.e.

$$U_h(\widehat{\mathfrak{sl}}_N) = U_h(\widehat{\mathfrak{gl}}_N) / [\widetilde{I}_{\text{qdet}}][[h]].$$

Remark 1.3. In Ding's definition [6, Def. 3.1], the quantum affine algebra is introduced as an associative algebra over the field $\mathbb{C}(q)$. However, as our goal is to study quantum vertex algebras associated to quantum affine algebras, we used the identification $q = e^{h/2}$ and introduced the quantum affine algebra as a suitably completed associative algebra over the commutative ring $\mathbb{C}[[h]]$. Thus we established the setting compatible with Etingof–Kazhdan's notion of quantum vertex algebra [11, Sect. 1.4], which, in particular, is required to be a topologically free $\mathbb{C}[[h]]$ -module; see also Li's notion of h -adic quantum vertex algebra [30, Def. 2.20]. Furthermore, in contrast with Ding's realization, we use normalized R -matrix (1.12) instead of (1.2). Such choice of the R -matrix enables

the constructions of certain large families of central elements of the quantum affine algebra at the critical level and of the topological generators of the quantum Feigin–Frenkel center, as demonstrated in [13] and [26] respectively; see also Section 4.

1.3. Restricted modules. Recall that a $\mathbb{C}[[h]]$ -module W is said to be *torsion-free* if $hw = 0$ for all nonzero $w \in W$ and that W is said to be *separable* if $\bigcap_{n \geq 1} h^n W = 0$. Moreover, W is said to be *topologically free* if it is separable, torsion-free and complete with respect to h -adic topology; see [25, Chapter XVI].

Let $\mathfrak{g}_N = \mathfrak{gl}_N, \mathfrak{sl}_N$. By arguing as in [27, Prop. 2.2] one can show that the algebra $U_h(\widehat{\mathfrak{g}}_N)$ is topologically free. Define a *restricted* $U_h(\widehat{\mathfrak{g}}_N)$ -module W as a topologically free $\mathbb{C}[[h]]$ -module such that

$$\mathcal{L}(x)w \in \text{End } \mathbb{C}^N \otimes W((x))[[h]] \quad \text{for all } w \in W. \quad (1.38)$$

Proposition 1.4. *Let W be a restricted $U_h(\widehat{\mathfrak{g}}_N)$ -module. Then for any $n \geq 1$ and the variables $x = (x_1, \dots, x_n)$ we have*

$$\mathcal{L}_{[n]}(x)w \in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes W((x_1, \dots, x_n))[[h]] \quad \text{for all } w \in W. \quad (1.39)$$

Proof. Apply quantum current commutation relation (1.27) on an arbitrary element of some restricted module. For every integer $k \geq 0$ the left hand side contains finitely many negative powers of the variable y modulo h^k while the right hand side contains finitely many negative powers of the variable x modulo h^k . Hence the statement of the proposition holds for $n = 2$. The case $n > 2$ is proved by induction on n which relies on (1.34). \square

Remark 1.5. Note that (1.39) implies $\mathcal{L}_{[n]}(x) \in \text{End } \mathbb{C}^N \otimes \text{Hom}(W, W((x_1, \dots, x_n))[[h]])$ for all $n \geq 1$. Hence we can apply the substitutions $x_1 = ze^{u_1}, \dots, x_n = ze^{u_n}$, thus getting

$$\mathcal{L}_{[n]}(x_1, \dots, x_n) \Big|_{x_1=ze^{u_1}, \dots, x_n=ze^{u_n}} \in \text{End } \mathbb{C}^N \otimes \text{Hom}(W, W((z))[[h, u_1, \dots, u_n]]). \quad (1.40)$$

We will often denote the expression in (1.40) more briefly by $\mathcal{L}_{[n]}(x) \Big|_{x_i=ze^{u_i}}$.

As usual, an $U_h(\widehat{\mathfrak{g}}_N)$ -module W is said to be of *level* c if the central element $C \in U_h(\widehat{\mathfrak{g}}_N)$ acts on W as a scalar multiplication by some $c \in \mathbb{C}$. Denote by $U_h(\widehat{\mathfrak{g}}_N)_c$ the *quantum affine algebra at the level* c , i.e. the quotient of $U_h(\widehat{\mathfrak{g}}_N)$ by the ideal generated by the element $C - c$. Let K_c be the left ideal in the algebra $U_h(\widehat{\mathfrak{g}}_N)_c$ generated by all elements

$$\lambda_{i_1 j_1 \dots i_n j_n}^{(r_1, \dots, r_n)} \quad \text{such that } r_k \geq 0 \text{ for at least one integer } k = 1, \dots, n,$$

where $n \geq 1, i_1, \dots, i_n, j_1, \dots, j_n = 1, \dots, N$ and $r_1, \dots, r_n \in \mathbb{Z}$. Introduce the completion of K_c as the inverse limit

$$\widetilde{K}_c = \varprojlim K_c / K_c \cap I_p(\widehat{\mathfrak{g}}_N).$$

Then the h -adic completion $[\widetilde{K}_c][[h]]$ of

$$[\widetilde{K}_c] = \left\{ a \in U_h(\widehat{\mathfrak{g}}_N)_c : h^n a \in \widetilde{K}_c \text{ for some } n \geq 0 \right\}$$

is also a left ideal in $U_h(\widehat{\mathfrak{g}}_N)_c$. Define the *vacuum module* $\mathcal{V}_c(\mathfrak{g}_N)$ at the level c over the quantum affine algebra $U_h(\widehat{\mathfrak{g}}_N)$ as the quotient of $U_h(\widehat{\mathfrak{g}}_N)_c$ by its left ideal $[\widetilde{K}_c][[h]]$,

$$\mathcal{V}_c(\mathfrak{g}_N) = U_h(\widehat{\mathfrak{g}}_N)_c / [\widetilde{K}_c][[h]]. \quad (1.41)$$

Observe that the canonical map $U_h(\widehat{\mathfrak{g}}_N)_c \rightarrow \mathcal{V}_c(\mathfrak{g}_N)$ maps the left ideal $I_p^h(\widehat{\mathfrak{g}}_N)$ to $h^p \mathcal{V}_c(\mathfrak{g}_N)$. Denote by $\mathbf{1}$ the image of the unit $1 \in U_h(\widehat{\mathfrak{g}}_N)$ with respect to this map.

Proposition 1.6. *The vacuum module $\mathcal{V}_c(\mathfrak{g}_N)$ is a topologically free $\mathbb{C}[[h]]$ -module. Moreover, it is a restricted $U_h(\widehat{\mathfrak{g}}_N)$ -module.*

Proof. The first assertion is verified by arguing as in [27, Prop. 2.2]. As for the second assertion, we first observe that all elements

$$\lambda_{i_1 j_1 \dots i_n j_n}^{(r_1, \dots, r_n)} \mathbf{1} \quad \text{such that} \quad n \geq 0 \quad \text{and} \quad r_k < 0 \quad \text{for all } k = 1, \dots, n \quad (1.42)$$

span an h -adically dense $\mathbb{C}[[h]]$ -submodule of $\mathcal{V}_c(\mathfrak{g}_N)$. Indeed, this follows from the fact that each monomial $\lambda_{k_1 l_1}^{(s_1)} \dots \lambda_{k_m l_m}^{(s_m)} \mathbf{1} \in \mathcal{V}_c(\mathfrak{g}_N)$ can be expressed using elements (1.42). This is done by employing crossing symmetry properties (1.13) and invertibility of the trigonometric R -matrix to move all R -matrices which appear on the right hand side of

$$\mathcal{L}_{[a+b]}(x, y) \mathbf{1} = \mathcal{L}_{[a]}^{13}(x) R_{ab}^{21}(ye^{-hC}/x) \mathcal{L}_{[b]}^{23}(y) R_{ab}^{21}(y/x)^{-1} \mathbf{1}, \quad (1.43)$$

where $a + b = m$, $x = (x_1, \dots, x_a)$ and $y = (y_1, \dots, y_b)$, to the left hand side (for more details see Remark 2.4), and then taking the coefficient of $x_1^{-s_1-1} \dots x_a^{-s_a-1} y_1^{-s_{a+1}-1} \dots y_b^{-s_m-1}$ at the matrix entry $e_{k_1 l_1} \otimes \dots \otimes e_{k_m l_m}$. Note that (1.43) follows from Proposition 1.2.

Therefore, it is sufficient to check that $\mathcal{L}(z)w$ belongs to $\text{End } \mathbb{C}^N \otimes \mathcal{V}_c(\mathfrak{g}_N)((z))[[h]]$ for all $w \in \mathcal{V}_c(\mathfrak{g}_N)$ of the form as in (1.42). However, as (1.43) contains only nonnegative powers of the variables $x_1, \dots, x_a, y_1, \dots, y_b$, this follows by setting $a = 1$ and $b = n$ in (1.43), then moving $R_{1n}^{21}(ye^{-hC}/x)$ and $R_{1n}^{21}(y/x)^{-1}$ to the left hand side and, finally, by taking the coefficient of $y_1^{-r_1-1} \dots y_n^{-r_n-1}$ at the matrix entries $e_{ij} \otimes e_{i_1 j_1} \otimes \dots \otimes e_{i_n j_n}$ for $i, j = 1, \dots, N$. \square

Observe that Proposition 1.6 and the Main Theorem imply Corollary 0.1.

2. ϕ -COORDINATED MODULES FOR THE QUANTUM AFFINE VERTEX ALGEBRA

In this section, we recall Etingof–Kazhdan’s construction of the quantum affine vertex algebra associated with trigonometric R -matrix in type A . Next, we suitably modify Li’s definition of ϕ -coordinated module, thus establishing the setting for the Main Theorem.

2.1. Quantum affine vertex algebra. We follow [9, 10] to introduce the R -matrix algebras $\overline{U}_h^+(\widehat{\mathfrak{gl}}_N)$; see also [12, 35]. Let $\overline{U}_h^+(\widehat{\mathfrak{gl}}_N)$ be the associative algebra over the ring $\mathbb{C}[[h]]$ generated by elements $t_{ij}^{(-r)}$, where $i, j = 1, \dots, N$ and $r = 1, 2, \dots$, subject to the defining relations

$$R(e^{u-v})T_1^+(u)T_2^+(v) = T_2^+(v)T_1^+(u)R(e^{u-v}), \quad (2.1)$$

where $T^+(u) \in \text{End } \mathbb{C}^N \otimes \overline{U}_h^+(\widehat{\mathfrak{gl}}_N)[[u]]$ is given by

$$T^+(u) = \sum_{i,j=1}^N e_{ij} \otimes t_{ij}^+(u) \quad \text{for} \quad t_{ij}^+(u) = \delta_{ij} - h \sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1} \in \overline{U}_h^+(\widehat{\mathfrak{gl}}_N)[[u]].$$

As in (1.24), we use subscripts in (2.1) to indicate copies in the tensor product algebra $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes \overline{U}_h^+(\widehat{\mathfrak{gl}}_N)$. Note that the R -matrix $R(e^{u-v})$ in defining relation (2.1) can be replaced by $R^+(e^{u-v})$.

Define the *quantum determinant* of the matrix $T^+(u)$ by

$$\text{qdet } T^+(u) = \text{tr}_{1, \dots, N} A^{(N)} T_1^+(u) \dots T_N^+(u - (N-1)h) D_1 \dots D_N, \quad (2.2)$$

where the trace is taken over all N copies of $\text{End } \mathbb{C}^N$ and the matrix D is given by (1.14). The quantum determinant $\text{qdet } T^+(u)$ belongs to $\overline{U}_h^+(\widehat{\mathfrak{gl}}_N)[[u]]$. Moreover, its coefficients δ_r , which are given by

$$\text{qdet } T^+(u) = 1 - h \sum_{r \geq 0} \delta_r u^r, \quad (2.3)$$

belong to the center of the algebra $\overline{U}_h^+(\widehat{\mathfrak{gl}}_N)$; see proof of [26, Prop. 3.10]. Define the algebra $\overline{U}_h^+(\widehat{\mathfrak{sl}}_N)$ as the quotient of $\overline{U}_h^+(\widehat{\mathfrak{gl}}_N)$ over the h -adically completed ideal generated by the elements $\delta_0, \delta_1, \dots$. Hence we have the following relation in $\overline{U}_h^+(\widehat{\mathfrak{sl}}_N)$:

$$\text{qdet} T^+(u) = 1. \quad (2.4)$$

Let $\mathfrak{g}_N = \mathfrak{gl}_N, \mathfrak{sl}_N$. For positive integers n and m we extend the notation in (1.29) and (1.30) by introducing the functions depending on the variable z and the families of variables $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ with values in the space $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m}$ by

$$R_{nm}^{12}(e^{z+u-v+ah}) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\leftarrow} R_{ij}(e^{z+u_i-v_{j-n}+ah}), \quad (2.5)$$

$$R_{nm}^{21}(e^{z+u-v+ah}) = \prod_{i=1, \dots, n}^{\leftarrow} \prod_{j=n+1, \dots, n+m}^{\rightarrow} R_{ji}(e^{z+u_i-v_{j-n}+ah}), \quad (2.6)$$

where $a \in \mathbb{C}$. Note that the expansion convention, as introduced at the end of Subsection 1.1, is applied on every factor on the right hand side, i.e.

$$R_{ij}(e^{z+u_i-v_{j-n}+ah}) = \psi_{\iota_{z, u_i, v_{j-n}}} g(e^{z+u_i-v_{j-n}+ah}) R_{ij}^+(e^{z+u_i-v_{j-n}+ah}).$$

If the variable z is omitted in (2.5) or (2.6), the embeddings $\iota_{u_i, v_{j-n}}$ are applied on the corresponding normalizing functions $g(e^{u_i-v_{j-n}+ah})$ instead. The functions $R_{nm}^{+12}(e^{z+u-v+ah})$ and $R_{nm}^{+21}(e^{z+u-v+ah})$ corresponding to the R -matrix $R^+(x)$ given by (1.15) can be defined analogously. Denote by $\mathbf{1}$ the unit in the algebra $\overline{U}_h^+(\widehat{\mathfrak{g}}_N)$. We recall [11, Lemma 2.1]:

Lemma 2.1. *For any $c \in \mathbb{C}$ there exists a unique operator series*

$$T^*(u) \in \text{End } \mathbb{C}^N \otimes \text{Hom}(\overline{U}_h^+(\widehat{\mathfrak{g}}_N)[[h]], \overline{U}_h^+(\widehat{\mathfrak{g}}_N)((u))[[h]])$$

such that for all $n \geq 0$ we have

$$\begin{aligned} T_1^*(u) T_2^+(v_1) \dots T_{n+1}^+(v_n) \mathbf{1} \\ = R_{1n}^{12}(e^{u-v+hc/2})^{-1} T_2^+(v_1) \dots T_{n+1}^+(v_n) R_{1n}^{12}(e^{u-v-hc/2}) \mathbf{1}. \end{aligned} \quad (2.7)$$

In order to indicate action (2.7), which is uniquely determined by the scalar $c \in \mathbb{C}$, we denote the topologically free $\mathbb{C}[[h]]$ -module $\overline{U}_h^+(\widehat{\mathfrak{g}}_N)[[h]]$ by $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$. Following [11], we introduce the operators on $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes \overline{\mathcal{V}}_c(\mathfrak{g}_N)$ by

$$T_{[n]}^+(u|z) = T_1^+(z+u_1) \dots T_n^+(z+u_n) \quad \text{and} \quad T_{[n]}^*(u|z) = T_1^*(z+u_1) \dots T_n^*(z+u_n).$$

By the expansion convention from Subsection 1.1, the operator $T_{[n]}^*(u|z)$ contains only nonnegative powers of the variables u_1, \dots, u_n as the embeddings ι_{z, u_i} are applied on its corresponding factors. If the variable z is omitted, we write

$$T_{[n]}^+(u) = T_1^+(u_1) \dots T_n^+(u_n) \quad \text{and} \quad T_{[n]}^*(u) = T_1^*(u_1) \dots T_n^*(u_n). \quad (2.8)$$

The next proposition, as given in [11, Prop. 2.2], is verified using (2.1) and (2.7). In relations (2.9)–(2.11), the superscripts 1, 2, 3 indicate the tensor factors as follows:

$$\overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes m}}^2 \otimes \overbrace{\overline{\mathcal{V}}_c(\mathfrak{g}_N)}^3.$$

For example, the superscripts 1, 3 in $T_{[n]}^{*13}(u|z_1)$ indicate that the operator $T_{[n]}^*(u|z_1)$ is applied on the tensor factors 1, \dots, n and $n+m+1$.

Proposition 2.2. For any integers $n, m \geq 1$ and the families of variables $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ the following equalities hold on $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$:

$$R_{nm}^{12}(e^{z_1-z_2+u-v})T_{[n]}^{*13}(u|z_1)T_{[m]}^{*23}(v|z_2) = T_{[m]}^{*23}(v|z_2)T_{[n]}^{*13}(u|z_1)R_{nm}^{12}(e^{z_1-z_2+u-v}), \quad (2.9)$$

$$R_{nm}^{12}(e^{z_1-z_2+u-v})T_{[n]}^{+13}(u|z_1)T_{[m]}^{+23}(v|z_2) = T_{[m]}^{+23}(v|z_2)T_{[n]}^{+13}(u|z_1)R_{nm}^{12}(e^{z_1-z_2+u-v}), \quad (2.10)$$

$$R_{nm}^{12}(e^{z_1-z_2+u-v+hc/2})T_{[n]}^{*13}(u|z_1)T_{[m]}^{+23}(v|z_2) = T_{[m]}^{+23}(v|z_2)T_{[n]}^{*13}(u|z_1)R_{nm}^{12}(e^{z_1-z_2+u-v-hc/2}). \quad (2.11)$$

From now on, the tensor products are understood as h -adically completed. The notion of *quantum vertex algebra* was introduced by Etingof and Kazhdan [11]. It is defined as a quadruple $(V, Y, \mathbf{1}, \mathcal{S})$ such that

1. V is a topologically free $\mathbb{C}[[h]]$ -module.
2. $Y = Y(z)$ is the *vertex operator map*, i.e. a $\mathbb{C}[[h]]$ -module map

$$Y: V \otimes V \rightarrow V((z))[[h]]$$

$$u \otimes v \mapsto Y(z)(u \otimes v) = Y(u, z)v = \sum_{r \in \mathbb{Z}} u_r v z^{-r-1}$$

which satisfies the *weak associativity*: for any $u, v, w \in V$ and $n \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that

$$(z_0 + z_2)^p Y(u, z_0 + z_2)Y(v, z_2)w - (z_0 + z_2)^p Y(Y(u, z_0)v, z_2)w \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]]. \quad (2.12)$$

3. $\mathbf{1}$ is the *vacuum vector*, i.e. a distinct element of V satisfying

$$Y(\mathbf{1}, z)v = v, \quad Y(v, z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z \rightarrow 0} Y(v, z)\mathbf{1} = v \quad \text{for all } v \in V, \quad (2.13)$$

4. $\mathcal{S} = \mathcal{S}(z)$ is the *braiding map*, i.e. a $\mathbb{C}[[h]]$ -module map $V \otimes V \rightarrow V \otimes V \otimes \mathbb{C}((z))[[h]]$ which satisfies the *\mathcal{S} -locality*: for any $u, v \in V$ and $n \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that for all $w \in V$

$$(z_1 - z_2)^p Y(z_1)(1 \otimes Y(z_2))(\mathcal{S}(z_1 - z_2)(u \otimes v) \otimes w) - (z_1 - z_2)^p Y(z_2)(1 \otimes Y(z_1))(v \otimes u \otimes w) \in h^n V[[z_1^{\pm 1}, z_2^{\pm 1}]]. \quad (2.14)$$

The given data should possess several other properties which we omit as they are not used in this paper; for a complete definition see [11, Sect. 1.4]. Finally, we recall Etingof–Kazhdan’s construction [11, Thm. 2.3] in the trigonometric R -matrix case:

Theorem 2.3. For any $c \in \mathbb{C}$ there exists a unique quantum vertex algebra structure on $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ such that the vertex operator map Y is given by

$$Y(T_{[n]}^+(u)\mathbf{1}, z) = T_{[n]}^+(u|z)T_{[n]}^*(u|z + hc/2)^{-1}, \quad (2.15)$$

the vacuum vector is $\mathbf{1} \in \overline{\mathcal{V}}_c(\mathfrak{g}_N)$ and the braiding map $\mathcal{S}(z)$ is defined by the relation

$$\begin{aligned} \mathcal{S}(z)(R_{nm}^{12}(e^{z+u-v})^{-1}T_{[m]}^{+24}(v)R_{nm}^{12}(e^{z+u-v-hc})T_{[n]}^{+13}(u)(\mathbf{1} \otimes \mathbf{1})) \\ = T_{[n]}^{+13}(u)R_{nm}^{12}(e^{z+u-v+hc})^{-1}T_{[m]}^{+24}(v)R_{nm}^{12}(e^{z+u-v})(\mathbf{1} \otimes \mathbf{1}) \end{aligned} \quad (2.16)$$

for operators on $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \overline{\mathcal{V}}_c(\mathfrak{g}_N) \otimes \overline{\mathcal{V}}_c(\mathfrak{g}_N)$.

Remark 2.4. Crossing symmetry properties (1.20) of R -matrix (1.18) can be expressed using the ordered product notation as

$$(D_1 R(e^{u+hN})D_1^{-1})_{\text{RL}} \cdot R(e^u)^{-1} = 1 \quad \text{and} \quad (D_2 R(e^u)^{-1}D_2^{-1})_{\text{LR}} \cdot R(e^{u+hN}) = 1, \quad (2.17)$$

where the subscript RL (LR) indicates that the first tensor factor of $D_i R(e^u)^{-1}D_i^{-1}$, $i = 1, 2$, is applied from the right (left) while the second tensor factor is applied from the

left (right). Such notation naturally extends to the products of multiple R -matrices such as (2.5) and (2.6). For example, by (2.17), we have

$$(D_{[m]}^2 R_{nm}^{12} (e^{z+u-v-h(N+c)})^{-1} (D_{[m]}^2)^{-1})_{\text{LR}} \cdot R_{nm}^{12} (e^{z+u-v-hc}) = 1, \quad (2.18)$$

where $D_{[m]}^2 = 1^{\otimes n} \otimes D^{\otimes m}$ and the subscript LR now indicates that the tensor factors $1, \dots, n$ ($n+1, \dots, n+m$) are applied from the left (right). As with (2.17), one can write crossing symmetry properties (1.13) of R -matrix (1.12) using the ordered product notation. As before, the notation naturally extends to the multiple R -matrix products such as (1.29)–(1.32).

Combining (2.16) and (2.18) we find the explicit formula for the action of the braiding,

$$\begin{aligned} \mathcal{S}(z) (T_{[n]}^{+13}(u) T_{[m]}^{+24}(v) (\mathbf{1} \otimes \mathbf{1})) &= (D_{[m]}^2 R_{nm}^{12} (e^{z+u-v-h(N+c)})^{-1} (D_{[m]}^2)^{-1}) \\ &\cdot (R_{nm}^{12} (e^{z+u-v}) T_{[n]}^{+13}(u) R_{nm}^{12} (e^{z+u-v+hc})^{-1} T_{[m]}^{+24}(v) R_{nm}^{12} (e^{z+u-v}) (\mathbf{1} \otimes \mathbf{1})). \end{aligned} \quad (2.19)$$

Remark 2.5. As with (1.6), by formal Taylor Theorem (1.5) we have

$$\frac{1}{1 - x e^{u-v+ah}} = \sum_{k=0}^{\infty} \frac{(e^{u-v+ah} - 1)^k x^k}{k!} \frac{\partial^k}{\partial x^k} \left(\frac{1}{1-x} \right).$$

Therefore, due to (1.15), we can regard the R -matrix $R(xe^{u-v+ah})$ as an element of $(\text{End } \mathbb{C}^N)^{\otimes 2}(x)[[u, v, h]]$ for any $a \in \mathbb{C}$, i.e. as a rational function in the variable x . Clearly, applying the embedding $\iota_{x,u,v}$ we obtain an element of $(\text{End } \mathbb{C}^N)^{\otimes 2}((x)[[u, v, h]])$.

We now extend the notation (2.5) and (2.6) by introducing the functions depending on the variable x and the families of variables $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ with values in the space $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m}$ by

$$R_{nm}^{12}(x e^{u-v+ah}) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\leftarrow} R_{ij}(x e^{u_i - v_j - n + ah}), \quad (2.20)$$

$$R_{nm}^{21}(x e^{u-v+ah}) = \prod_{i=1, \dots, n}^{\leftarrow} \prod_{j=n+1, \dots, n+m}^{\rightarrow} R_{ji}(x e^{u_i - v_j - n + ah}), \quad (2.21)$$

where $a \in \mathbb{C}$. In accordance with Remark 2.5, the R -matrices in (2.20) and (2.21) are regarded as rational functions in the variable x . We use the map given by the following lemma in Definition 2.7 below, to introduce the notion of ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module.

Lemma 2.6. *There exists a unique $\mathbb{C}[[h]]$ -module map*

$$\widehat{\mathcal{S}}(x): \overline{\mathcal{V}}_c(\mathfrak{g}_N) \otimes \overline{\mathcal{V}}_c(\mathfrak{g}_N) \rightarrow \overline{\mathcal{V}}_c(\mathfrak{g}_N) \otimes \overline{\mathcal{V}}_c(\mathfrak{g}_N)(x)[[h]]$$

such that

$$\begin{aligned} \widehat{\mathcal{S}}(x) (T_{[n]}^{+13}(u) T_{[m]}^{+24}(v) (\mathbf{1} \otimes \mathbf{1})) &= (D_{[m]}^2 R_{nm}^{12} (x e^{u-v-h(N+c)})^{-1} (D_{[m]}^2)^{-1}) \\ &\cdot (R_{nm}^{12} (x e^{u-v}) T_{[n]}^{+13}(u) R_{nm}^{12} (x e^{u-v+hc})^{-1} T_{[m]}^{+24}(v) R_{nm}^{12} (x e^{u-v}) (\mathbf{1} \otimes \mathbf{1})). \end{aligned} \quad (2.22)$$

Moreover, the map $\widehat{\mathcal{S}}(x)$ satisfies

$$\begin{aligned} \widehat{\mathcal{S}}(x) (R_{nm}^{12} (x e^{u-v})^{-1} T_{[m]}^{+24}(v) R_{nm}^{12} (x e^{u-v-hc}) T_{[n]}^{+13}(u) (\mathbf{1} \otimes \mathbf{1})) \\ = T_{[n]}^{+13}(u) R_{nm}^{12} (x e^{u-v+hc})^{-1} T_{[m]}^{+24}(v) R_{nm}^{12} (x e^{u-v}) (\mathbf{1} \otimes \mathbf{1}). \end{aligned} \quad (2.23)$$

Proof. The map $\widehat{\mathcal{S}}(x)$ is well-defined by (2.22), i.e. it maps the ideal of relations (2.1), and (2.4) in the $\mathfrak{g}_N = \mathfrak{sl}_N$ case, to itself. Indeed, this follows by a straightforward calculation which relies on the identity

$$R_{12}(e^{u-v}) D_1 D_2 = D_2 D_1 R_{12}(e^{u-v})$$

and the following version of Yang–Baxter equation (1.3):

$$R_{12}(e^{u-v})R_{13}(xe^{u+ah})R_{23}(xe^{v+ah}) = R_{23}(xe^{v+ah})R_{13}(xe^{u+ah})R_{12}(e^{u-v}), \quad \alpha \in \mathbb{C}.$$

Moreover, the proof in the $\mathfrak{g}_N = \mathfrak{sl}_N$ case employs identity (1.11) and some properties of the anti-symmetrizer $A^{(N)}$, which are given by (3.64), (3.65) and

$A^{(N)}T_1^+(u)T_2^+(u-h)\dots T_N^+(u-(N-1)h) = T_N^+(u-(N-1)h)\dots T_2^+(u-h)T_1^+(u)A^{(N)}$, see [26, Equality (3.12)]. As for relation (2.23), it follows from (2.22) and the equality

$$(D_{[m]}^2 R_{nm}^{12}(xe^{u-v-h(N+c)})^{-1} (D_{[m]}^2)^{-1}) \cdot R_{nm}^{12}(xe^{u-v-hc}) = 1,$$

which is verified by using crossing symmetry properties (1.13); recall Remark 2.4. \square

2.2. ϕ -Coordinated modules. The notion of ϕ -coordinated module, where ϕ is an associate of the one-dimensional additive formal group, was introduced by Li [31]. As in [31, Sect. 5], throughout this paper we consider the associate

$$\phi(z_2, z_0) = z_2 e^{z_0}. \quad (2.24)$$

Before we proceed to the definition of ϕ -coordinated module, we introduce some notation. Let V be a topologically free $\mathbb{C}[[h]]$ -module and $a_1, \dots, a_n, k > 0$ arbitrary integers. Suppose that some element A of $\text{Hom}(V, V[[z_1^{\pm 1}, z_2^{\pm 1}, u_1, \dots, u_n]])$ can be expressed as

$$A = B + u_1^{a_1} C_1 + \dots + u_n^{a_n} C_n + h^k C_{n+1} \quad \text{for some} \quad (2.25)$$

$B \in \text{Hom}(V, V((z_1, z_2))[[u_1, \dots, u_n, h]])$, $C_1, \dots, C_{n+1} \in \text{Hom}(V, V[[z_1^{\pm 1}, z_2^{\pm 1}, u_1, \dots, u_n]])$.

To indicate the fact that A possesses a decomposition as in (2.25), we write

$$A \in \text{Hom}(V, V((z_1, z_2))[[u_1, \dots, u_n]]) \pmod{u_1^{a_1}, \dots, u_n^{a_n}, h^k}. \quad (2.26)$$

Note that the substitution

$$B|_{z_1=\phi(z_2, z_0)} = \iota_{z_2, z_0, u_1, \dots, u_n} \left(B(z_1, z_2, u_1, \dots, u_n)|_{z_1=\phi(z_2, z_0)} \right) \quad (2.27)$$

is well-defined even though the substitution $A|_{z_1=\phi(z_2, z_0)}$ does not exist in general. In what follows, the substitution $z_1 = \phi(z_2, z_0)$ is always understood as in (2.27), i.e. the given expression is expanded in nonnegative powers of the variable z_0 . In order to simplify our notation, we denote (2.27) as

$$A|_{z_1=\phi(z_2, z_0)} \pmod{u_1^{a_1}, \dots, u_n^{a_n}, h^k} = A(z_1, z_2, u_1, \dots, u_n)|_{z_1=\phi(z_2, z_0)} \pmod{u_1^{a_1}, \dots, u_n^{a_n}, h^k}. \quad (2.28)$$

The element B as in (2.25) is clearly unique modulo

$$\sum_{i=1}^n u_i^{a_i} \text{Hom}(V, V[[z_1^{\pm 1}, z_2^{\pm 1}, u_1, \dots, u_n]]) + h^k \text{Hom}(V, V[[z_1^{\pm 1}, z_2^{\pm 1}, u_1, \dots, u_n]]).$$

Let $\mathfrak{g}_N = \mathfrak{gl}_N, \mathfrak{sl}_N$. The following definition of ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module is based on [31, Def. 3.4], which we slightly modify in order to make it compatible with Etingof–Kazhdan’s quantum vertex algebra theory; see Remark 2.9 for more details.

Definition 2.7. A ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module is a pair (W, Y_W) such that W is a topologically free $\mathbb{C}[[h]]$ -module and $Y_W = Y_W(z)$ is a $\mathbb{C}[[h]]$ -module map

$$Y_W: \overline{\mathcal{V}}_c(\mathfrak{g}_N) \otimes W \rightarrow W((z))[[h]]$$

$$u \otimes w \mapsto Y_W(z)(u \otimes w) = Y_W(u, z)w = \sum_{r \in \mathbb{Z}} u_r w z^{-r-1}$$

which satisfies $Y_W(\mathbf{1}, z)w = w$ for all $w \in W$; the *weak associativity*: for any $u, v \in \overline{\mathcal{V}}_c(\mathfrak{g}_N)$ and $k \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that

$$(z_1 - z_2)^p Y_W(u, z_1) Y_W(v, z_2) \in \text{Hom}(W, W((z_1, z_2))) \pmod{h^k} \quad \text{and} \quad (2.29)$$

$$\begin{aligned} & \left((z_1 - z_2)^p Y_W(u, z_1) Y_W(v, z_2) \right) \Big|_{z_1 = \phi(z_2, z_0)}^{\pmod{h^k}} \\ & - (\phi(z_2, z_0) - z_2)^p Y_W(Y(u, z_0)v, z_2) \in h^k \text{Hom}(W, W[[z_0^{\pm 1}, z_2^{\pm 1}]]); \end{aligned} \quad (2.30)$$

and the $\widehat{\mathcal{S}}$ -locality: for any $u, v \in \overline{\mathcal{V}}_c(\mathfrak{g}_N)$ and $k \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that

$$\begin{aligned} & (z_1 - z_2)^p Y_W(z_1)(1 \otimes Y_W(z_2)) \iota_{z_1, z_2}(\widehat{\mathcal{S}}(z_1/z_2)(u \otimes v) \otimes w) \\ & - (z_1 - z_2)^p Y_W(z_2)(1 \otimes Y_W(z_1))(v \otimes u \otimes w) \in h^k W[[z_1^{\pm 1}, z_2^{\pm 1}]] \quad \text{for all } w \in W. \end{aligned} \quad (2.31)$$

Let W_1 be a topologically free $\mathbb{C}[[h]]$ -submodule of W . A pair (W_1, Y_{W_1}) is said to be a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -submodule of W if $Y_W(v, z)w$ belongs to W_1 for all $v \in \overline{\mathcal{V}}_c(\mathfrak{g}_N)$ and $w \in W_1$, where Y_{W_1} denotes the restriction and corestriction of Y_W ,

$$Y_{W_1}(z) = Y_W(z) \Big|_{\overline{\mathcal{V}}_c(\mathfrak{g}_N) \otimes W_1}^{W_1((z))[[h]]} : \overline{\mathcal{V}}_c(\mathfrak{g}_N) \otimes W_1 \rightarrow W_1((z))[[h]].$$

Remark 2.8. Regarding the weak associativity, note that (2.29) and (2.30) employ the notation introduced in (2.26) and (2.28) for $n = 0$, i.e. there are no variables u_1, \dots, u_n . Next, observe that the $\widehat{\mathcal{S}}$ -locality already implies that there exists $p \in \mathbb{Z}_{\geq 0}$ such that (2.29) holds. However, we still include this requirement in the definition as it ensures that the integer p is large enough so that the substitution $z_1 = \phi(z_2, z_0)$ in (2.30) is well-defined. Finally, the motivation for expressing the weak associativity in the form as in (2.29) and (2.30) is given in [31, Rem. 3.2].

Remark 2.9. As with the quantum affine algebra in the previous section, we introduce the notion of ϕ -coordinated module over the ring $\mathbb{C}[[h]]$ instead of a field in order to make it compatible with the Etingof–Kazhdan quantum vertex algebra theory; cf. original definition [31, Def. 3.4]. Furthermore, unlike the original definition, we require that the ϕ -coordinated module map $Y_W(z)$ possesses $\widehat{\mathcal{S}}$ -locality property (2.31). The general theory developed by Li suggests that (2.31) might be omitted from the definition, due to the fact that the vertex operator map $Y(z)$ already possesses \mathcal{S} -locality property (2.14); see [31, Prop. 5.6]. However, we include the $\widehat{\mathcal{S}}$ -locality in the definition in order to emphasize the importance of quantum current commutation relation (1.27). More specifically, in the proof of the [Main Theorem](#), we derive the $\widehat{\mathcal{S}}$ -locality property directly from the quantum current commutation relation; see Lemma 3.8.

Introduce the series

$$\delta(z) = \sum_{k \in \mathbb{Z}} z^k \in \mathbb{C}[[z^{\pm 1}]] \quad \text{and} \quad \log(1+z) = - \sum_{k=1}^{\infty} \frac{(-z)^k}{k} \in z\mathbb{C}[[z]].$$

The following Jacobi-type identity was established in [31, Prop. 5.9]. Although, in contrast with [31], we consider quantum vertex algebras and ϕ -coordinated modules defined over the ring $\mathbb{C}[[h]]$, the next proposition can be proved by arguing as in the proofs of [31, Lemma 5.8] and [31, Prop. 5.9].

Proposition 2.10. *Let W be a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{g}_N)$ -module, where $\phi(z_2, z_0) = z_2 e^{z_0}$. For any $u, v \in \overline{\mathcal{V}}_c(\mathfrak{g}_N)$ we have*

$$(z_2 z)^{-1} \delta \left(\frac{z_1 - z_2}{z_2 z} \right) Y_W(z_1)(1 \otimes Y_W(z_2))(u \otimes v) \quad (2.32)$$

$$- (z_2 z)^{-1} \delta \left(\frac{z_2 - z_1}{-z_2 z} \right) Y_W(z_2)(1 \otimes Y_W(z_1)) \iota_{z_2, z_1} \widehat{\mathcal{S}}(z_2/z_1)(v \otimes u) \quad (2.33)$$

$$= z_1^{-1} \delta \left(\frac{z_2(1+z)}{z_1} \right) Y_W(Y(u, \log(1+z))v, z_2). \quad (2.34)$$

3. PROOF OF THE MAIN THEOREM

In this section we prove the [Main Theorem](#). The proof is divided into four parts, Subsections 3.1–3.4. In Subsection 3.1, we obtain some properties of the normalizing functions for the trigonometric R -matrix which are required in the later stages of the proof; see Lemmas 3.1–3.4. In Subsection 3.2, we demonstrate how to establish the ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -module structure on a restricted module of level c for the quantum affine algebra $U_h(\widehat{\mathfrak{gl}}_N)$; see Lemmas 3.5–3.8. The key ingredient in this part of the proof is Ding’s quantum current realization and, in particular, the fact that quantum current commutation relation (1.27) resembles $\widehat{\mathcal{S}}$ -locality property (2.31). In Subsection 3.3, we use Li’s Jacobi-type identity, as given in Proposition 2.10, to establish the structure of restricted module of level c for the quantum affine algebra $U_h(\widehat{\mathfrak{gl}}_N)$ on a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -module; see Lemma 3.9. Finally, we finish the proof in the $\mathfrak{g}_N = \mathfrak{gl}_N$ case by showing that the $\mathbb{C}[[h]]$ -submodules invariant with respect to the action of the quantum affine algebra and with respect to the corresponding action of the quantum vertex algebra coincide; see Lemma 3.10. In Subsection 3.4, we use the fusion procedure for the two-parameter trigonometric R -matrix to extend the results to the $\mathfrak{g}_N = \mathfrak{sl}_N$ case, thus completing the proof of the [Main Theorem](#); see Lemmas 3.11–3.15.

3.1. Normalizing functions. Introduce the function $r(x)$ by

$$r(x) = -x e^h (1 - e^h x)^{-1} f(x)^{-1}, \quad (3.1)$$

where $f(x)$ is given by (1.10).

Lemma 3.1. *The function $r(x) \in \mathbb{C}[[x, h]]$ satisfies*

$$R_{21}(x)^{-1} = r(x) R_{12}^+(1/x). \quad (3.2)$$

Moreover, it admits the presentation

$$r(x) = \sum_{k=0}^{\infty} r_k \frac{x^{k+1}}{(1-x)^{k+1}} \quad \text{such that} \quad r_k \in h^k \mathbb{C}[[h]] \quad \text{and} \quad r_0 = -e^h. \quad (3.3)$$

Proof. By combining unitarity property (1.4) and (1.15) we obtain

$$\begin{aligned} R_{21}(x)^{-1} &= (f(x) \overline{R}_{21}(x))^{-1} = f(x)^{-1} \overline{R}_{21}(x)^{-1} = f(x)^{-1} \overline{R}_{12}(1/x) \\ &= f(x)^{-1} (1 - e^{-h} x^{-1})^{-1} R_{12}^+(1/x) = -x e^h (1 - e^h x)^{-1} f(x)^{-1} R_{12}^+(1/x) = r(x) R_{12}^+(1/x), \end{aligned}$$

as required. Next, by using (1.10) we find

$$f(x)^{-1} = \sum_{l=0}^{\infty} \left(- \sum_{k=1}^{\infty} f_k \left(\frac{x}{1-x} \right)^k \right)^l = 1 + \sum_{k=1}^{\infty} \beta_k \left(\frac{x}{1-x} \right)^k \quad (3.4)$$

for some $\beta_k \in h^k \mathbb{C}[[h]]$. It is clear that the product of (1.6) for $a = 1$, (3.4) and $-x e^h$ is equal to $r(x)$ and, furthermore, that it admits presentation (3.3). \square

We use the following lemma in the proofs of weak associativity and $\widehat{\mathcal{S}}$ -locality of the ϕ -coordinated module map, as well as to establish the restricted module structure on a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -module; see Lemmas 3.7, 3.8 and 3.9 respectively.

Lemma 3.2. *Let $F = g^{\pm 1}$ or $F = r^{\pm 1}$. For any integers $a_1, a_2, k > 0$ and $\alpha \in \mathbb{C}$ there exists an integer $p \geq 0$ such that the coefficients of all monomials*

$$u_1^{a'_1} u_2^{a'_2} h^{k'}, \quad \text{where} \quad 0 \leq a'_1 < a_1, \quad 0 \leq a'_2 < a_2 \quad \text{and} \quad 0 \leq k' < k, \quad (3.5)$$

in $(z_1 - z_2)^p F(z_1 e^{u_1 - u_2 + \alpha h} / z_2)$ belong to $\mathbb{C}[z_1, z_2^{\pm 1}]$ and such that the coefficients of all monomials (3.5) in

$$\left((z_1 - z_2)^p F(z_1 e^{u_1 - u_2 + \alpha h} / z_2) \right) \Big|_{z_1 = z_2 e^{z_0}}^{\text{mod } u_1^{a_1}, u_2^{a_2}, h^k} \quad \text{and} \quad z_2^p (e^{z_0} - 1)^p F(e^{z_0 + u_1 - u_2 + \alpha h}) \quad (3.6)$$

coincide.

Proof. Set $\delta = 0$ for $F = g$ and $\delta = 1$ for $F = r$, i.e. $\delta = \delta_{F,r}$, so that we can consider both cases simultaneously. Let $U = \mathbb{C}[[x, x_0, h]]$. Recall (1.16) and (3.3). As the map ι_x commutes with partial differential operator $\partial/\partial x$, by using Taylor Theorem (1.5) we find

$$\iota_{x, x_0} F(x + x_0) = \sum_{s=0}^{\infty} \frac{x_0^s}{s!} \frac{\partial^s}{\partial x^s} \iota_x F(x) = \sum_{l,s=0}^{\infty} \frac{F_s x_0^l}{l!} \iota_x \frac{\partial^l}{\partial x^l} \left(\frac{x^{s+\delta}}{(1-x)^{s+1}} \right) \in U,$$

where $F_s = g_s$ for $F = g$ and $F_s = r_s$ for $F = r$. By (1.16) and (3.3), every F_s belongs to $h^s \mathbb{C}[[h]]$, so all summands with $s \geq k$ are trivial modulo $h^k U$. Hence the given expression modulo $U_0 := x_0^{a_1 + a_2 + k - 2} U + h^k U$ contains only finitely many nonzero summands and, consequently, only finitely many terms $(1-x)^{s+1}$ in the denominator. Therefore, there exists an integer $p \geq 0$ such that

$$\iota_{x, x_0} (1-x)^p F(x + x_0) = \sum_{l,s=0}^{\infty} \frac{F_s x_0^l}{l!} \iota_x (1-x)^p \frac{\partial^l}{\partial x^l} \left(\frac{x^{s+\delta}}{(1-x)^{s+1}} \right) \in \mathbb{C}[x, x_0, h] \quad \text{mod } U_0,$$

where the equality holds modulo U_0 and the map ι_x can be omitted on the right hand side as p can be chosen so that $(1-x)^p$ cancels all negative powers of $(1-x)$ modulo U_0 . By applying the substitution $(x, x_0) = (z_1/z_2, z_1(e^{u_1 - u_2 + \alpha h} - 1)/z_2)$ to

$$\iota_{x, x_0} (1-x)^p F(x + x_0) \quad \text{mod } U_0 \quad (3.7)$$

and then multiplying the resulting expression by $(-z_2)^p$ we get

$$(z_1 - z_2)^p F(z_1 e^{u_1 - u_2 + \alpha h} / z_2) \in \mathbb{C}[z_1, z_2^{\pm 1}, u_1, u_2, h] \quad \text{mod } V_0 \quad (3.8)$$

for $V_0 = u_1^{a_1} V + u_2^{a_2} V + h^k V$ and $V = \mathbb{C}[[z_1, z_2^{\pm 1}, u_1, u_2, h]]$, thus proving the first assertion of the lemma.

Set $W_0 = u_1^{a_1} W + u_2^{a_2} W + h^k W$ for $W = \mathbb{C}[[z_0, z_2, u_1, u_2, h]]$. As (3.7) is a polynomial in the variables x and x_0 , by applying the substitution $(x, x_0) = (e^{z_0}, e^{z_0}(e^{u_1 - u_2 + \alpha h} - 1))$ to (3.7) and then multiplying the resulting expression by $(-z_2)^p$ we get

$$z_2^p (e^{z_0} - 1)^p F(e^{z_0 + u_1 - u_2 + \alpha h}) \quad \text{mod } W_0, \quad (3.9)$$

where, by the expansion convention from Subsection 1.1, $F(e^{z_0 + u_1 - u_2 + \alpha h})$ stands for $\iota_{z_0, u_1, u_2} F(e^{z_0 + u_1 - u_2 + \alpha h})$.³ Finally, as (3.8) modulo V_0 is a polynomial with respect to the variables z_1/z_2 and z_2 , by applying the substitution $z_1 = z_2 e^{z_0}$ we again obtain (3.9), thus proving the second assertion of the lemma.

If $F = g^{-1}$ or $F = r^{-1}$, one easily checks that

$$F(x) = \sum_{s=0}^{\infty} F_s \frac{x^{s - \delta_{F,r-1}}}{(1-x)^{s-1}} \quad \text{for some} \quad F_s \in h^s \mathbb{C}[[h]], \quad (3.10)$$

so the lemma is verified by arguing as above. \square

³Note that the expression $\iota_{x, x_0} (1-x)^p F(x + x_0)$ is considered modulo U_0 because, otherwise, the aforementioned substitution would not be well-defined (although the same substitution is well-defined when applied to $(1-x)^p F(x + x_0)$ with $F(x + x_0)$ being regarded as a rational function with respect to the variables x and x_0).

We now recall a certain useful consequence of [31, Lemma 2.7], as given in [31, Rem. 2.8]: For any $A(z_1, z_2), B(z_1, z_2) \in \mathbb{C}((z_1, z_2))$, the equality

$$A(z_1, z_2) \Big|_{z_1=z_2e^{z_0}} = B(z_1, z_2) \Big|_{z_1=z_2e^{z_0}} \quad \text{implies} \quad A(z_1, z_2) = B(z_1, z_2). \quad (3.11)$$

Since the $\mathbb{C}[[h]]$ -module $\mathbb{C}((z_1, z_2))[[h]]$ is separable, implication (3.11) clearly extends to any $A(z_1, z_2), B(z_1, z_2) \in \mathbb{C}((z_1, z_2))[[h]]$.

Lemma 3.3. *In $\mathbb{C}((u))[[h]]$ we have*

$$r(e^{-u}) = \psi^2 g(e^u). \quad (3.12)$$

Moreover, for any integers $a_1, a_2, k > 0$ and $\alpha \in \mathbb{C}$ there exists an integer $p \geq 0$ such that the coefficients of all monomials (3.5) in

$$(z_1 - z_2)^p r(z_2 e^{-u_1+u_2-\alpha h} / z_1) \quad \text{and} \quad (z_1 - z_2)^p \psi^2 g(z_1 e^{u_1-u_2+\alpha h} / z_2) \quad (3.13)$$

coincide.

Proof. By [26, Prop. 2.1] we have $\psi^2 f(e^u) = f(e^{-u})^{-1}$. Therefore, using (3.1) we get

$$\begin{aligned} r(e^{-u}) &= -e^{-u+h}(1 - e^{-u+h})^{-1} f(e^{-u})^{-1} = -\psi^2 e^{-u+h}(1 - e^{-u+h})^{-1} f(e^u) \\ &= \psi^2 (1 - e^{u-h})^{-1} f(e^u) = \psi^2 g(e^u), \end{aligned}$$

as required, where the last equality follows from (1.15). Next, by Lemma 3.2 and (3.12), there exists $p \geq 0$ such that the coefficients of all monomials (3.5) in both expressions in (3.13) belong to $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ and such that the coefficients of all monomials (3.5) in

$$\begin{aligned} &((z_1 - z_2)^p r(z_2 e^{-u_1+u_2-\alpha h} / z_1)) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } u_1^{a_1}, u_2^{a_2}, h^k} \quad \text{and} \\ &((z_1 - z_2)^p \psi^2 g(z_1 e^{u_1-u_2+\alpha h} / z_2)) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } u_1^{a_1}, u_2^{a_2}, h^k} \end{aligned}$$

coincide. The second assertion of the lemma now follows by implication (3.11). \square

The next lemma, which relies on Lemma 3.3, will be used in the proof of $\widehat{\mathcal{S}}$ -locality of the ϕ -coordinated module map; see Lemma 3.8.

Lemma 3.4. (1) *Let $F = g^{\pm 1}$ or $F = r^{\pm 1}$. There exists $\widehat{F}(x, u, v)$ in $\mathbb{C}(x)[[u, v, h]]$ such that for all $\alpha \in \mathbb{C}$ the following equality in $\mathbb{C}((z))[[u, v, h]]$ holds:*

$$\widehat{F}(e^z, u, v - \alpha h) = F(e^{z+u-v+\alpha h}). \quad (3.14)$$

(2) *For any integers $n, m > 0$, the families of variables $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_m)$ and $c \in \mathbb{C}$ there exist functions $\widehat{G}(x, u, v), \widehat{H}(x, u, v) \in \mathbb{C}(x)[[u_1, \dots, u_n, v_1, \dots, v_m, h]]$ such that the following equalities hold in $\mathbb{C}((z))[[u_1, \dots, u_n, v_1, \dots, v_m, h]]$:*

$$\widehat{G}(e^z, u, v) = G(z, u, v) \quad \text{and} \quad \widehat{H}(e^z, u, v) = H(z, u, v), \quad \text{where}$$

$$G(z, u, v) = \prod_{i=1}^n \prod_{j=1}^m g(e^{z+u_i-v_j-h(N+c)})^{-1} g(e^{z+u_i-v_j+hc})^{-1} g(e^{z+u_i-v_j})^2, \quad (3.15)$$

$$H(z, u, v) = \prod_{i=1}^n \prod_{j=1}^m g(e^{-z+u_i-v_j-h(N+c)})^{-1} r(e^{z-u_i+v_j-hc})^{-1} g(e^{-z+u_i-v_j}) r(e^{z-u_i+v_j}). \quad (3.16)$$

(3) *Let $a_1, \dots, a_n, b_1, \dots, b_m, k > 0$ be arbitrary integers and $\iota = \iota_{z_1, z_2, u_1, \dots, u_n, v_1, \dots, v_m}$ the embedding. There exists an integer $p \geq 0$ such that the coefficients of all monomials*

$$u_1^{a'_1} \dots u_n^{a'_n} v_1^{b'_1} \dots v_m^{b'_m} h^{k'}, \quad \text{where} \quad 0 \leq a'_i < a_i, \quad 0 \leq b'_j < b_j \quad \text{and} \quad 0 \leq k' < k$$

in $(z_1 - z_2)^p \iota \widehat{G}(z_1/z_2, u, v)$ and $(z_1 - z_2)^p \iota \widehat{H}(z_2/z_1, u, v)$ coincide.

Proof. Due to (1.16), (3.3) and (3.10), we can regard $g(x)^{\pm 1}$ and $r(x)^{\pm 1}$ as elements of $\mathbb{C}(x)[[h]]$. Let $F = g^{\pm 1}$ or $F = r^{\pm 1}$ and write $F(x) = \sum_{s=0}^{\infty} F_s(x)h^s$ for some $F_s(x) \in \mathbb{C}(x)$. Applying formal Taylor Theorem (1.5) to $z \mapsto \iota_z F_s(e^z)$ we get for any $\alpha \in \mathbb{C}$

$$\iota_{z,u,v,h} F_s(e^{z+u-v+\alpha h}) = \sum_{l=0}^{\infty} \frac{(u-v+\alpha h)^l}{l!} \frac{\partial^l}{\partial z^l} \iota_z F_s(e^z) \quad \text{in } \mathbb{C}((z))[[u, v, h]].$$

The partial differential operator $\partial/\partial z$ commutes with the map ι_z and all $\frac{\partial^l}{\partial z^l} F_s(e^z)$ can be naturally regarded as elements of $\mathbb{C}(e^z)$. Hence we can introduce functions $\widehat{F}_{l,s}(x) \in \mathbb{C}(x)$ by the requirement $\widehat{F}_{l,s}(e^z) = \frac{\partial^l}{\partial z^l} F_s(e^z)$. The first statement of the lemma now clearly follows as the function $\widehat{F}(x, u, v) \in \mathbb{C}(x)[[u, v, h]]$ satisfying (3.14) can be defined by

$$\widehat{F}(x, u, v) = \sum_{s=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{(u-v)^l}{l!} \widehat{F}_{l,s}(x) \right) h^s.$$

The second statement is proved by applying the first statement on each factor of (3.15) and (3.16). Finally, by (3.12) we have $G(z, u, v) = H(-z, u, v)$, so the third statement follows by Lemma 3.3. \square

3.2. Establishing the ϕ -coordinated $\bar{\mathcal{V}}_c(\mathfrak{gl}_N)$ -module structure. Let W be a restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module of level $c \in \mathbb{C}$. In this subsection, we prove the first assertion of the Main Theorem, i.e. we show that (0.3) defines a unique structure of ϕ -coordinated $\bar{\mathcal{V}}_c(\mathfrak{gl}_N)$ -module on W , where $\phi(z_2, z_0) = z_2 e^{z_0}$. The proof is divided into four lemmas which verify all requirements imposed by Definition 2.7.

Lemma 3.5. *Formula (0.3), together with $Y_W(\mathbf{1}, z) = 1_W$, defines a unique $\mathbb{C}[[h]]$ -module map $\bar{\mathcal{V}}_c(\mathfrak{gl}_N) \otimes W \rightarrow W((z))[[h]]$.*

Proof. First, we note that the right hand side of (0.3) is well-defined, as was discussed in Remark 1.5. Next, we recall that the algebra $\bar{U}_h^+(\widehat{\mathfrak{gl}}_N)$ is spanned by all coefficients of all matrix entries of $T_{[n]}^+(u)$, $n \geq 1$, and $\mathbf{1}$; see [9, Sect. 3.4] or [26, Prop. 2.4]⁴. In order to prove the lemma, we have to show that $v \mapsto Y_W(v, z)$ preserves the ideal of relations (2.1). More specifically, it is sufficient to check that for any integers $n \geq 2$ and $i = 1, 2, \dots, n-1$ and the family of variables $u = (u_1, \dots, u_n)$ the expression

$$R_{ii+1}(e^{u_i - u_{i+1}}) T_{[n]}^+(u) \mathbf{1} - P_{ii+1} T_{[n]}^+(u_{i+1, i}) \mathbf{1} P_{ii+1} R_{ii+1}(e^{u_i - u_{i+1}}), \quad (3.17)$$

where $u_{i+1, i} = (u_1, \dots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \dots, u_n)$, belongs to the kernel of $v \mapsto Y_W(v, z)$.

Let $x = (x_1, \dots, x_n)$ and $x_{i+1, i} = (x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)$. Using Yang–Baxter equation (1.3) and commutation relation (1.27) one can prove the identity

$$R_{ii+1}(x_i/x_{i+1}) \mathcal{L}_{[n]}(x) = P_{ii+1} \mathcal{L}_{[n]}(x_{i+1, i}) P_{ii+1} R_{ii+1}(x_i/x_{i+1}). \quad (3.18)$$

By Proposition 1.4, all matrix entries of $\mathcal{L}_{[n]}(x)$ belong to $\text{Hom}(W, W((x_1, \dots, x_n))[[h]])$, so all matrix entries in (3.18) belong to

$$\text{Hom}(W, W((x_{i+1}))((x_1, \dots, x_i, x_{i+2}, \dots, x_n))[[h]]).$$

Recall R -matrix decomposition (1.15). By (3.10) the function $g(x_i/x_{i+1})^{-1}$ belongs to $\mathbb{C}[x_{i+1}^{-1}][[h, x_i]]$, so we can multiply (3.18) by $g(x_i/x_{i+1})^{-1}$, thus getting

$$R_{ii+1}^+(x_i/x_{i+1}) \mathcal{L}_{[n]}(x) = P_{ii+1} \mathcal{L}_{[n]}(x_{i+1, i}) P_{ii+1} R_{ii+1}^+(x_i/x_{i+1}). \quad (3.19)$$

⁴ We should mention that the notation in this paper slightly differs from [26]. In particular, the algebra $U(R)$, as defined in [26, Sect. 2], coincides with the algebra $\bar{U}_h^+(\widehat{\mathfrak{gl}}_N)$ defined in Subsection 2.1.

Since the R -matrix $R_{ii+1}^+(x_i/x_{i+1})$ is a polynomial in x_i/x_{i+1} , all matrix entries of both sides in (3.19) belong to $\text{Hom}(W, W((x_1, \dots, x_n))[[h]])$. Therefore, we can apply the substitutions $x_i = ze^{u_i}$ with $i = 1, \dots, n$ to (3.19), thus getting the following equality in $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes \text{Hom}(W, W((z))[[h, u_1, \dots, u_n]])$:

$$R_{ii+1}^+(e^{u_i - u_{i+1}}) \cdot (\mathcal{L}_{[n]}(x)) \Big|_{x_i = ze^{u_i}} = P_{ii+1}(\mathcal{L}_{[n]}(x_{i+1, i})) \Big|_{x_i = ze^{u_i}} P_{ii+1} R_{ii+1}^+(e^{u_i - u_{i+1}}).$$

Multiplying the equality by $\psi g(e^{u_i - u_{i+1}}) \in \mathbb{C}((u_{i+1}))[[h, u_i]]$ and using (1.18) we find

$$R_{ii+1}(e^{u_i - u_{i+1}}) (\mathcal{L}_{[n]}(x)) \Big|_{x_i = ze^{u_i}} - P_{ii+1}(\mathcal{L}_{[n]}(x_{i+1, i})) \Big|_{x_i = ze^{u_i}} P_{ii+1} R_{ii+1}(e^{u_i - u_{i+1}}) = 0.$$

As the left hand side coincides with the image of (3.17), with respect to $Y_W(z)$, we conclude that (0.3) defines a $\mathbb{C}[[h]]$ -module map $\bar{\mathcal{V}}_c(\mathfrak{gl}_N) \otimes W \rightarrow W[[z^{\pm 1}]]$, as required. Moreover, by Remark 1.5 its image belongs to $W((z))[[h]]$. Finally, it is clear that the $\mathbb{C}[[h]]$ -module map $Y_W(z)$ is uniquely determined by (0.3). \square

The next lemma follows from $\widehat{\mathcal{S}}$ -locality property (2.31) which is verified in Lemma 3.8 below; recall Remark 2.8. Nonetheless, we provide the direct proof as the underlying calculations are required in the proof of Lemma 3.7.

Lemma 3.6. *The map $Y_W(z)$ satisfies (2.29), i.e. for any $u, v \in \bar{\mathcal{V}}_c(\mathfrak{gl}_N)$ and $k \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that*

$$(z_1 - z_2)^p Y_W(u, z_1) Y_W(v, z_2) \in \text{Hom}(W, W((z_1, z_2))) \pmod{h^k}. \quad (3.20)$$

Proof. For any integers $n, m \geq 1$ and families of variables $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ we have

$$Y_W(T_{[n]}^{+13}(u) \mathbf{1}, z_1) Y_W(T_{[m]}^{+23}(v) \mathbf{1}, z_2) = (\mathcal{L}_{[n]}^{13}(x)) \Big|_{x_i = z_1 e^{u_i}} (\mathcal{L}_{[m]}^{23}(y)) \Big|_{y_j = z_2 e^{v_j}}, \quad (3.21)$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$. The coefficients in (3.21) are operators on the multiple tensor product with superscripts 1, 2, 3 indicating the tensor factors:

$$\overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes m}}^2 \otimes \overbrace{\widehat{W}}^3.$$

Let us rewrite the right hand side in (3.21). The third assertion of Proposition 1.2 implies

$$\mathcal{L}_{[n]}^{13}(x) R_{nm}^{21}(ye^{-hc}/x) \mathcal{L}_{[m]}^{23}(y) = \mathcal{L}_{[n+m]}(x, y) R_{nm}^{21}(y/x). \quad (3.22)$$

By expressing the second crossing symmetry relation in (1.13) in the variable $x = y_j e^{-h(N+c)}/x_i$, then applying the transposition t_2 and finally conjugating the resulting equality by the permutation operator P we find

$$(D_1 R_{21}(y_j e^{-h(N+c)}/x_i)^{-1} D_1^{-1}) \Big|_{\text{RL}} \cdot R_{21}(y_j e^{-hc}/x_i) = 1.$$

Furthermore, due to Lemma 3.1, we can write this equality as

$$r(y_j e^{-h(N+c)}/x_i) (D_1 R_{12}^+(x_i e^{h(N+c)}/y_j) D_1^{-1}) \Big|_{\text{RL}} \cdot R_{21}(y_j e^{-hc}/x_i) = 1.$$

Hence we have

$$r(x, y) \left(D_{[n]}^1 R_{nm}^{+12}(x e^{h(N+c)}/y) (D_{[n]}^1)^{-1} \right) \Big|_{\text{RL}} \cdot R_{nm}^{21}(y e^{-hc}/x) = 1, \quad \text{where} \quad (3.23)$$

$$D_{[n]}^1 = D^{\otimes n} \otimes 1^{\otimes m} \quad \text{and} \quad r(x, y) = \prod_{i=1}^n \prod_{j=1}^m r(y_j e^{-h(N+c)}/x_i). \quad (3.24)$$

Using (3.23) we can move $R_{nm}^{21}(y e^{-hc}/x)$ in (3.22) to the right hand side, which gives us

$$\mathcal{L}_{[n]}^{13}(x) \mathcal{L}_{[m]}^{23}(y) = r(x, y) \left(D_{[n]}^1 R_{nm}^{+12}(x e^{h(N+c)}/y) (D_{[n]}^1)^{-1} \right) \Big|_{\text{RL}} \cdot (\mathcal{L}_{[n+m]}(x, y) R_{nm}^{21}(y/x))$$

$$= r(x, y)g(x, y) \left(D_{[n]}^1 R_{nm}^{+12}(xe^{h(N+c)}/y) (D_{[n]}^1)^{-1} \right)_{\text{RL}} \cdot (\mathcal{L}_{[n+m]}(x, y) R_{nm}^{+21}(y/x)), \quad (3.25)$$

where the second equality comes from (1.15) and the function $g(x, y)$ is given by

$$g(x, y) = \prod_{i=1}^n \prod_{j=1}^m g(y_j/x_i). \quad (3.26)$$

Let $a_1, \dots, a_n, b_1, \dots, b_m, k > 0$ be arbitrary integers. We now apply the substitutions

$$x_i = z_1 e^{u_i}, \quad y_j = z_2 e^{v_j} \quad \text{for} \quad i = 1, \dots, n, \quad j = 1, \dots, m \quad (3.27)$$

to (3.25), thus getting (3.21), and then consider the coefficients of all monomials

$$u_1^{a'_1} \dots u_n^{a'_n} v_1^{b'_1} \dots v_m^{b'_m} h^{k'}, \quad \text{where} \quad 0 \leq a'_i < a_i, \quad 0 \leq b'_j < b_j \quad \text{and} \quad k' < k. \quad (3.28)$$

First, as the R -matrix $R^+(w)$ is a polynomial with respect to the variable w , we conclude by Proposition 1.4 and Remark 1.5 that

$$\left(\left(D_{[n]}^1 R_{nm}^{+12}(xe^{h(N+c)}/y) (D_{[n]}^1)^{-1} \right)_{\text{RL}} \cdot (\mathcal{L}_{[n+m]}(x, y) R_{nm}^{+21}(y/x)) \right) \Big|_{x_i=z_1 e^{u_i}, y_j=z_2 e^{v_j}} \quad (3.29)$$

$$\in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \text{Hom}(W, W((z_1, z_2))[[u_1, \dots, u_n, v_1, \dots, v_n, h]]).$$

Next, by Lemma 3.2 there exists an integer $p \geq 0$, which depends on the choice of integers $a_1, \dots, a_n, b_1, \dots, b_m, k$, such that the coefficients of all monomials (3.28) in

$$(z_1 - z_2)^p (r(x, y)g(x, y)) \Big|_{x_i=z_1 e^{u_i}, y_j=z_2 e^{v_j}} \quad (3.30)$$

belong to $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$. Finally, we observe that the coefficients of all monomials (3.28) in the product of (3.29) and (3.30) coincide with the corresponding coefficients in

$$(z_1 - z_2)^p Y_W(T_{[n]}^{+13}(u) \mathbf{1}, z_1) Y_W(T_{[m]}^{+23}(v) \mathbf{1}, z_2).$$

Therefore, by the preceding discussion, these coefficients belong to

$$(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \text{Hom}(W, W((z_1, z_2))),$$

which implies the statement of the lemma. \square

Lemma 3.7. *The map $Y_W(z)$ satisfies weak associativity (2.30), i.e. for any $u, v \in \overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ and $k \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that (3.20) holds and such that*

$$\begin{aligned} & \left((z_1 - z_2)^p Y_W(u, z_1) Y_W(v, z_2) \right) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } h^k} \\ & - z_2^p (e^{z_0} - 1)^p Y_W(Y(u, z_0)v, z_2) \in h^k \text{Hom}(W, W[[z_0^{\pm 1}, z_2^{\pm 1}]]) \end{aligned} \quad (3.31)$$

Proof. Let $n, m, a_1, \dots, a_n, b_1, \dots, b_m, k > 0$ be arbitrary integers, $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ the families of variables. Consider the coefficients of all monomials (3.28) in the expression

$$\left((z_1 - z_2)^p Y_W(T_{[n]}^{+13}(u) \mathbf{1}, z_1) Y_W(T_{[m]}^{+23}(v) \mathbf{1}, z_2) \right) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } u_1^{a_1}, \dots, u_n^{a_n}, v_1^{b_1}, \dots, v_m^{b_m}, h^k}, \quad (3.32)$$

which corresponds to the first summand in (3.31). As demonstrated in the proof of Lemma 3.6, they coincide with the coefficients of all monomials (3.28) in the product

$$\left(\left(\left(D_{[n]}^1 R_{nm}^{+12}(xe^{h(N+c)}/y) (D_{[n]}^1)^{-1} \right)_{\text{RL}} \cdot (\mathcal{L}_{[n+m]}(x, y) R_{nm}^{+21}(y/x)) \right) \Big|_{x_i=z_1 e^{u_i}, y_j=z_2 e^{v_j}} \right) \Big|_{z_1=z_2 e^{z_0}} \quad (3.33)$$

$$\times \left((z_1 - z_2)^p \cdot (r(x, y)g(x, y)) \Big|_{x_i=z_1 e^{u_i}, y_j=z_2 e^{v_j}} \right) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } u_1^{a_1}, \dots, u_n^{a_n}, v_1^{b_1}, \dots, v_m^{b_m}, h^k} \quad (3.34)$$

for a suitably chosen integer $p \geq 0$ (which depends on $a_1, \dots, a_n, b_1, \dots, b_m, k$). Recall that the functions r and g are given by (3.24) and (3.26). First, we observe that the coefficients of all monomials (3.28) in factor (3.33) coincide with the corresponding coefficients in

$$\left(D_{[n]}^1 R_{nm}^{+12} (e^{z_0+u-v+h(N+c)}) (D_{[n]}^1)^{-1} \right)_{\text{RL}} \cdot \left(\left(\mathcal{L}_{[n+m]}(x, y) \Big|_{\substack{x_i=z_1 e^{u_i} \\ y_j=z_2 e^{v_j}}} \right) \Big|_{z_1=z_2 e^{z_0}} R_{nm}^{+21} (e^{-z_0-u+v}) \right). \quad (3.35)$$

Next, we turn to factor (3.34). Due to Lemma 3.2, we can assume that the integer p was chosen so that the coefficients of all monomials (3.28) in factor (3.34) coincide with the coefficients of all monomials (3.28) in

$$z_2^p (e^{z_0} - 1)^p \prod_{i=1}^n \prod_{j=1}^m r(e^{-z_0-u_i+v_j-h(N+c)}) g(e^{-z_0-u_i+v_j}).$$

Moreover, by (3.12), this is equal to

$$z_2^p (e^{z_0} - 1)^p \psi^{2nm} \prod_{i=1}^n \prod_{j=1}^m g(e^{z_0+u_i-v_j+h(N+c)}) g(e^{-z_0-u_i+v_j}). \quad (3.36)$$

Finally, we conclude that the coefficients of all monomials (3.28) in (3.32) coincide with the coefficients of the corresponding monomials in the product of (3.35) and (3.36).

Consider the expression

$$z_2^p (e^{z_0} - 1)^p Y_W(Y(T_{[n]}^{+13}(u) \mathbf{1}, z_0) T_{[m]}^{+23}(v) \mathbf{1}, z_2), \quad (3.37)$$

which corresponds to the second summand in (3.31). By (2.15) it is equal to

$$z_2^p (e^{z_0} - 1)^p Y_W(T_{[n]}^{+13}(u|z_0) T_{[n]}^{*13}(u|z_0 + hc/2)^{-1} T_{[m]}^{+23}(v) \mathbf{1}, z_2). \quad (3.38)$$

Since $T^*(u) \mathbf{1} = \mathbf{1}$, by combining relation (2.11) and the first crossing symmetry relation in (2.17) we obtain

$$\begin{aligned} & T_{[n]}^{*13}(u|z_0 + hc/2)^{-1} T_{[m]}^{+23}(v) \mathbf{1} \\ &= (D_{[n]}^1 R_{nm}^{12} (e^{z_0+u-v+h(N+c)}) (D_{[n]}^1)^{-1})_{\text{RL}} \cdot \left(T_{[m]}^{+23}(v) \mathbf{1} R_{nm}^{12} (e^{z_0+u-v})^{-1} \right). \end{aligned} \quad (3.39)$$

Introduce the functions

$$g_1(u, v, z_0) = \psi^{nm} \prod_{i=1}^n \prod_{j=1}^m g(e^{z_0+u_i-v_j+h(N+c)}), \quad g_2(u, v, z_0) = \psi^{nm} \prod_{i=1}^n \prod_{j=1}^m g(e^{-z_0-u_i+v_j}).$$

By (1.18) we have

$$R_{nm}^{12} (e^{z_0+u-v+h(N+c)}) = g_1(u, v, z_0) R_{nm}^{+12} (e^{z_0+u-v+h(N+c)}). \quad (3.40)$$

Furthermore, by combining (1.18) and unitarity property (1.19) we find

$$R_{nm}^{12} (e^{z_0+u-v})^{-1} = R_{nm}^{21} (e^{-z_0-u+v}) = g_2(u, v, z_0) R_{nm}^{+21} (e^{-z_0-u+v}). \quad (3.41)$$

Using (3.40) and (3.41) we rewrite the right hand side of (3.39) as

$$g_1(u, v, z_0) g_2(u, v, z_0) (D_{[n]}^1 R_{nm}^{+12} (e^{z_0+u-v+h(N+c)}) (D_{[n]}^1)^{-1})_{\text{RL}} \cdot \left(T_{[m]}^{+23}(v) \mathbf{1} R_{nm}^{+21} (e^{-z_0-u+v}) \right). \quad (3.42)$$

Next, we employ (3.42) and then (0.3) to express (3.38) as

$$\begin{aligned} & (D_{[n]}^1 R_{nm}^{+12} (e^{z_0+u-v+h(N+c)}) (D_{[n]}^1)^{-1})_{\text{RL}} \cdot \left(Y_W(T_{[n]}^{+13}(u|z_0) T_{[m]}^{+23}(v) \mathbf{1}, z_2) R_{nm}^{+21} (e^{-z_0-u+v}) \right) \\ & \times z_2^p (e^{z_0} - 1)^p g_1(u, v, z_0) g_2(u, v, z_0) \end{aligned}$$

$$= (D_{[n]}^1 R_{nm}^{+12}(e^{z_0+u-v+h(N+c)})(D_{[n]}^1)^{-1}) \cdot_{\text{RL}} \left(\mathcal{L}_{[n+m]}(x, y) \Big|_{\substack{x_i=z_2 e^{z_0+u_i} \\ y_j=z_2 e^{v_j}}} R_{nm}^{+21}(e^{-z_0-u+v}) \right) \\ \times z_2^p (e^{z_0} - 1)^p g_1(u, v, z_0) g_2(u, v, z_0).$$

Note that $z_2^p (e^{z_0} - 1)^p g_1(u, v, z_0) g_2(u, v, z_0)$ is equal to (3.36) and that

$$\mathcal{L}_{[n+m]}(x, y) \Big|_{\substack{x_i=z_2 e^{z_0+u_i} \\ y_j=z_2 e^{v_j}}} = \left(\mathcal{L}_{[n+m]}(x, y) \Big|_{\substack{x_i=z_1 e^{u_i} \\ y_j=z_2 e^{v_j}}} \right) \Big|_{z_1=z_2 e^{z_0}}.$$

Therefore, the product of (3.35) and (3.36) is equal to (3.37), so we conclude that the coefficients of all monomials (3.28) in (3.32) and in (3.37) coincide, as required. \square

Lemma 3.8. *The map $Y_W(z)$ satisfies $\widehat{\mathcal{S}}$ -locality (2.31), i.e. for any $u, v \in \overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ and $k \in \mathbb{Z}_{\geq 0}$ there exists $p \in \mathbb{Z}_{\geq 0}$ such that for all $w \in W$*

$$(z_1 - z_2)^p Y_W(z_1) (1 \otimes Y_W(z_2)) \iota_{z_1, z_2} (\widehat{\mathcal{S}}(z_1/z_2)(u \otimes v) \otimes w) \\ - (z_1 - z_2)^p Y_W(z_2) (1 \otimes Y_W(z_1)) (v \otimes u \otimes w) \in h^k W[[z_1^{\pm 1}, z_2^{\pm 1}]]. \quad (3.43)$$

Proof. Let $n, m, a_1, \dots, a_n, b_1, \dots, b_m, k > 0$ be arbitrary integers, $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ the families of variables. We will apply $Y_W(z_1) (1 \otimes Y_W(z_2)) \iota_{z_1, z_2} \widehat{\mathcal{S}}(z_1/z_2)$, which corresponds to the first summand in (3.43), to

$$T_{[n]}^{+13}(u) T_{[m]}^{+24}(v) (\mathbf{1} \otimes \mathbf{1}) \quad (3.44)$$

and then consider the coefficients of all monomials (3.28) in the resulting expression. Note that the superscripts 1, 2, 3, 4 in (3.44) indicate the tensor factors as follows:

$$\overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes m}}^2 \otimes \overbrace{\overline{\mathcal{V}}_c(\mathfrak{gl}_N)}^3 \otimes \overbrace{\overline{\mathcal{V}}_c(\mathfrak{gl}_N)}^4.$$

Applying the map $\widehat{\mathcal{S}}(z_1/z_2)$, given by (2.22), to (3.44) and then using (1.15) we get

$$\widehat{G}(z_1/z_2, u, v) (D_{[m]}^2 R_{nm}^{+12}(z_1 e^{u-v-h(N+c)}/z_2)^{-1} (D_{[m]}^2)^{-1}) \\ \cdot_{\text{LR}} \left(R_{nm}^{+12}(z_1 e^{u-v}/z_2) T_{[n]}^{+13}(u) R_{nm}^{+12}(z_1 e^{u-v+hc}/z_2)^{-1} T_{[m]}^{+24}(v) R_{nm}^{+12}(z_1 e^{u-v}/z_2) (\mathbf{1} \otimes \mathbf{1}) \right), \quad (3.45)$$

where the function $\widehat{G}(x, u, v)$ is given by Lemma 3.4. As we only consider the coefficients of monomials (3.28), it is sufficient to carry out the calculations modulo U , where

$$U = \sum_{i=1}^n u_i^{a_i} V + \sum_{j=1}^m v_j^{b_j} V + h^k V \quad \text{and}$$

$$V = (\text{End } \mathbb{C}^N)^{\otimes(n+m)} \otimes \overline{\mathcal{V}}_c(\mathfrak{gl}_N)^{\otimes 2}((z_1))((z_2))[[u_1, \dots, u_n, v_1, \dots, v_m, h]].$$

By Lemma 3.4, there exists an integer $p \geq 0$ such that the image of the product of $(z_1 - z_2)^p$ and (3.45) with respect to the map $\iota = \iota_{z_1, z_2, u_1, \dots, u_n, v_1, \dots, v_m}$ coincides with

$$(z_1 - z_2)^p \iota \widehat{H}(z_2/z_1, u, v) (D_{[m]}^2 R_{nm}^{+12}(z_1 e^{u-v-h(N+c)}/z_2)^{-1} (D_{[m]}^2)^{-1}) \\ \cdot_{\text{LR}} \left(R_{nm}^{+12}(z_1 e^{u-v}/z_2) T_{[n]}^{+13}(u) R_{nm}^{+12}(z_1 e^{u-v+hc}/z_2)^{-1} T_{[m]}^{+24}(v) R_{nm}^{+12}(z_1 e^{u-v}/z_2) (\mathbf{1} \otimes \mathbf{1}) \right) \quad (3.46)$$

modulo U , where the function $\widehat{H}(x, u, v)$ is given by Lemma 3.4. Note that there are only finitely many monomials (3.28). Therefore, due to Lemma 3.2, we can assume that $p = 4p_0$ for some integer $p_0 \geq 0$ such that all coefficients of monomials (3.28) in

$$\iota \left((z_1 - z_2)^{p_0} R_{nm}^{12}(z_1 e^{u-v-h(N+c)}/z_2)^{-1} \right), \quad \iota \left((z_1 - z_2)^{p_0} R_{nm}^{12}(z_1 e^{u-v}/z_2) \right), \\ \iota \left((z_1 - z_2)^{p_0} R_{nm}^{21}(z_2 e^{-u+v-hc}/z_1) \right), \quad \iota \left((z_1 - z_2)^{p_0} R_{nm}^{21}(z_2 e^{-u+v}/z_1)^{-1} \right)$$

belong to $(\text{End } \mathbb{C}^N)^{\otimes(n+m)}((z_1, z_2))$. Due to the definition of the function $\widehat{H}(x, u, v)$, see in particular (3.16), we conclude by (1.15) and (3.2) that the expression in (3.46) equals

$$(z_1 - z_2)^p \iota \left(D_{[m]}^2 R_{nm}^{12} (z_1 e^{u-v-h(N+c)}/z_2)^{-1} (D_{[m]}^2)^{-1} \right) \cdot_{\text{LR}} \left(R_{nm}^{12} (z_1 e^{u-v}/z_2) \right. \\ \left. \times T_{[n]}^{+13}(u) R_{nm}^{21} (z_2 e^{-u+v-hc}/z_1) T_{[m]}^{+24}(v) R_{nm}^{21} (z_2 e^{-u+v}/z_1)^{-1} (\mathbf{1} \otimes \mathbf{1}) \right) \pmod{U}. \quad (3.47)$$

Next, we apply $Y_W(z_1)(1 \otimes Y_W(z_2))$ to (3.47), thus getting

$$(z_1 - z_2)^p \left(D_{[m]}^2 \iota \left(R_{nm}^{12} (z_1 e^{u-v-h(N+c)}/z_2)^{-1} \right) (D_{[m]}^2)^{-1} \right) \cdot_{\text{LR}} \left(\iota \left(R_{nm}^{12} (z_1 e^{u-v}/z_2) \right) \right. \\ \left. \times \mathcal{L}_{[n]}^{13}(x) \Big|_{x_i=z_1 e^{u_i}} \iota \left(R_{nm}^{21} (z_2 e^{-u+v-hc}/z_1) \right) \mathcal{L}_{[m]}^{23}(y) \Big|_{y_j=z_2 e^{v_j}} \iota \left(R_{nm}^{21} (z_2 e^{-u+v}/z_1)^{-1} \right) \right) \quad (3.48)$$

modulo U_0 , where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are the families of variables and

$$U_0 = \sum_{i=1}^n u_i^{a_i} V_0 + \sum_{j=1}^m v_j^{b_j} V_0 + h^k V_0 \quad \text{for}$$

$$V_0 = (\text{End } \mathbb{C}^N)^{\otimes(n+m)} \otimes \text{Hom}(W, W[[z_1^{\pm 1}, z_2^{\pm 2}]])[[u_1, \dots, u_n, v_1, \dots, v_m]].$$

By employing quantum current commutation relation (1.34) we rewrite (3.48) as

$$\left(D_{[m]}^2 \iota \left((z_1 - z_2)^{p_0} R_{nm}^{12} (z_1 e^{u-v-h(N+c)}/z_2)^{-1} \right) (D_{[m]}^2)^{-1} \right) \\ \cdot_{\text{LR}} \left(\iota \left((z_1 - z_2)^{p_0} R_{nm}^{12} (z_1 e^{u-v}/z_2) \right) \iota \left((z_1 - z_2)^{p_0} R_{nm}^{12} (z_1 e^{u-v}/z_2)^{-1} \right) \right) \\ \times \mathcal{L}_{[m]}^{23}(y) \Big|_{y_j=z_2 e^{v_j}} \iota \left((z_1 - z_2)^{p_0} R_{nm}^{12} (z_1 e^{u-v-hc}/z_2) \right) \mathcal{L}_{[n]}^{13}(x) \Big|_{x_i=z_1 e^{u_i}} \pmod{U_0}. \quad (3.49)$$

Observe that all products in (3.49) are well-defined modulo U_0 due to our choice of the integer $p = 4p_0$ and Remark 1.5. Canceling the R -matrices $R_{nm}^{12} (z_1 e^{u-v}/z_2)^{\pm 1}$ and then using the following consequence of the second crossing symmetry relation in (1.13) which is verified by arguing as in Remark 2.4,

$$\left(D_{[m]}^2 R_{nm}^{12} (z_1 e^{u-v-h(N+c)}/z_2)^{-1} (D_{[m]}^2)^{-1} \right) \cdot_{\text{LR}} R_{nm}^{12} (z_1 e^{u-v-hc}/z_2),$$

the expression in (3.49) simplifies to

$$(z_1 - z_2)^p \mathcal{L}_{[m]}^{23}(y) \Big|_{y_j=z_2 e^{v_j}} \mathcal{L}_{[n]}^{13}(x) \Big|_{x_i=z_1 e^{u_i}} \pmod{U_0}. \quad (3.50)$$

Finally, consider the expression which corresponds to the second summand in (3.43), i.e. which is obtained by applying $(z_1 - z_2)^p Y_W(z_2)(1 \otimes Y_W(z_1))$ to $T_{[n]}^{+14}(u) T_{[m]}^{+23}(v)(\mathbf{1} \otimes \mathbf{1})$. Clearly, all its coefficients with respect to monomials (3.28) coincide with the corresponding coefficients in (3.50), so the $\widehat{\mathcal{S}}$ -locality follows. \square

3.3. Establishing the restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module structure. Let (W, Y_W) be a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -module for some $c \in \mathbb{C}$, where $\phi(z_2, z_0) = z_2 e^{z_0}$. In this subsection, which consists of two lemmas, we finish the proof of the Main Theorem in the \mathfrak{gl}_N case.

Lemma 3.9. *Formula (0.4) defines a unique structure of restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module of level c on W .*

Proof. The uniqueness is clear as (0.4) determines the action of all generators of $U_h(\widehat{\mathfrak{gl}}_N)$ on W . We now use the Jacobi-type identity given in Proposition 2.10 to check that (0.4) satisfies defining relation (1.27) for the algebra $U_h(\widehat{\mathfrak{gl}}_N)$ at the level c . Let $n \geq 0$ be an arbitrary integer. Choose $p \geq 0$ such that the expressions

$$\iota_{z_2, z_1} (z_1 - z_2)^p R_{12}(z_2/z_1)^{-1} T_{23}^+(0) R_{12}(z_2 e^{-hc}/z_1) T_{14}^+(0) (\mathbf{1} \otimes \mathbf{1}) \quad \text{and} \quad (3.51)$$

$$\iota_{z_1, z_2} (z_1 - z_2)^p R_{12}(z_2/z_1)^{-1} T_{23}^+(0) R_{12}(z_2 e^{-hc}/z_1) T_{14}^+(0) (\mathbf{1} \otimes \mathbf{1}), \quad (3.52)$$

whose coefficients belong to $(\text{End } \mathbb{C}^N)^{\otimes 2} \otimes \overline{\mathcal{V}}_c(\mathfrak{gl}_N)^{\otimes 2}$, coincide modulo h^n . Note that the embedding map ι_{z_1, z_2} in (3.52) can be omitted as both R -matrices are Taylor series in z_2/z_1 , i.e. they consist of nonnegative powers of z_2/z_1 . Furthermore, we can assume that the integer p is chosen so that expression (3.51) modulo h^n is a polynomial in the variables $z_1^{\pm 1}, z_2^{\pm 1}$. Hence the embedding map ι_{z_2, z_1} can be also omitted when regarding (3.51) modulo h^n . Applying first term (2.32) of the Jacobi identity on (3.52) we get

$$\begin{aligned} & (z_2 z)^{-1} \delta \left(\frac{z_1 - z_2}{z_2 z} \right) Y_W(z_1) (1 \otimes Y_W(z_2)) \\ & \times (z_1 - z_2)^p R_{12}(z_2/z_1)^{-1} T_{23}^+(0) R_{12}(z_2 e^{-hc}/z_1) T_{14}^+(0) (\mathbf{1} \otimes \mathbf{1}). \end{aligned}$$

Using (0.4) we rewrite this as

$$(z_2 z)^{-1} \delta \left(\frac{z_1 - z_2}{z_2 z} \right) (z_1 - z_2)^p R_{12}(z_2/z_1)^{-1} \mathcal{L}_2(z_1) R_{12}(z_2 e^{-hc}/z_1) \mathcal{L}_1(z_2).$$

Due to the well-known δ -function identity,

$$x \delta(x) = \delta(x), \quad (3.53)$$

by multiplying by $(z_2 z)^{-p}$ and then taking the residue $\text{Res}_{z_2 z}$ we obtain

$$R_{12}(z_2/z_1)^{-1} \mathcal{L}_2(z_1) R_{12}(z_2 e^{-hc}/z_1) \mathcal{L}_1(z_2). \quad (3.54)$$

We now turn to second term (2.33) of the Jacobi identity. Choose $r \geq 0$ such that

$$\begin{aligned} A(z_1, z_2) &:= \iota_{z_2, z_1}(z_1 - z_2)^r R_{12}(z_2 e^{hc}/z_1)^{-1}, & B(z_1, z_2) &:= \iota_{z_2, z_1}(z_1 - z_2)^r \psi^2 R_{21}(z_1 e^{-hc}/z_2), \\ C(z_1, z_2) &:= \iota_{z_2, z_1}(z_1 - z_2)^r R_{12}(z_2/z_1), & D(z_1, z_2) &:= \iota_{z_2, z_1}(z_1 - z_2)^r \psi^{-2} R_{21}(z_1/z_2)^{-1} \end{aligned}$$

belong to $(\text{End } \mathbb{C}^N)^{\otimes 2}((z_1, z_2))$ modulo h^n . As with (3.52), observe that the embedding map ι_{z_2, z_1} can be omitted in the definitions of $B(z_1, z_2)$ and $D(z_1, z_2)$ above. By combining (1.18) and unitarity property (1.19) we find

$$A(z_1, z_2) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } h^n} = B(z_1, z_2) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } h^n} \quad \text{and} \quad C(z_1, z_2) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } h^n} = D(z_1, z_2) \Big|_{z_1=z_2 e^{z_0}}^{\text{mod } h^n}.$$

Therefore, by the implication in (3.11) we conclude that

$$A(z_1, z_2) = B(z_1, z_2) \pmod{h^n} \quad \text{and} \quad C(z_1, z_2) = D(z_1, z_2) \pmod{h^n}. \quad (3.55)$$

Consider (3.51) modulo h^n . Applying second term (2.33) of the Jacobi identity we get⁵

$$\begin{aligned} & - (z_2 z)^{-1} \delta \left(\frac{z_2 - z_1}{-z_2 z} \right) Y_W(z_2) (1 \otimes Y_W(z_1)) \iota_{z_2, z_1}(\widehat{\mathcal{S}}(z_2/z_1)) \\ & \times (z_1 - z_2)^p R_{12}(z_2/z_1)^{-1} T_{24}^+(0) R_{12}(z_2 e^{-hc}/z_1) T_{13}^+(0) (\mathbf{1} \otimes \mathbf{1}) \pmod{h^n}. \end{aligned} \quad (3.56)$$

By using explicit formula (2.23) for the map $\widehat{\mathcal{S}}(x)$ we rewrite (3.56) as

$$\begin{aligned} & - (z_2 z)^{-1} \delta \left(\frac{z_2 - z_1}{-z_2 z} \right) Y_W(z_2) (1 \otimes Y_W(z_1)) \\ & \times (z_1 - z_2)^p T_{13}^+(0) \iota_{z_2, z_1} (R_{12}(z_2 e^{hc}/z_1)^{-1}) T_{24}^+(0) \iota_{z_2, z_1} (R_{12}(z_2/z_1)) (\mathbf{1} \otimes \mathbf{1}) \pmod{h^n}. \end{aligned}$$

Next, the application of (0.4) gives us

$$\begin{aligned} & - (z_2 z)^{-1} \delta \left(\frac{z_2 - z_1}{-z_2 z} \right) (z_1 - z_2)^p \mathcal{L}_1(z_2) \\ & \times \iota_{z_2, z_1} (R_{12}(z_2 e^{hc}/z_1)^{-1}) \mathcal{L}_2(z_1) \iota_{z_2, z_1} (R_{12}(z_2/z_1)) \pmod{h^n}. \end{aligned} \quad (3.57)$$

⁵Note that, in contrast with (2.32) and (2.34), the vectors u and v in (2.33) are swapped, so that in (3.56) we have $T_{24}^+(0)$ and $T_{13}^+(0)$ instead of $T_{23}^+(0)$ and $T_{14}^+(0)$.

Note that (3.53) implies

$$\delta\left(\frac{z_2 - z_1}{-z_2 z}\right) = \frac{(z_2 - z_1)^r}{(-z_2 z)^r} \delta\left(\frac{z_2 - z_1}{-z_2 z}\right),$$

so that we can use both equalities in (3.55) to rewrite (3.57) as

$$-(z_2 z)^{-1} \delta\left(\frac{z_2 - z_1}{-z_2 z}\right) (z_1 - z_2)^p \mathcal{L}_1(z_2) R_{21}(z_1 e^{-hc}/z_2) \mathcal{L}_2(z_1) R_{21}(z_1/z_2)^{-1} \pmod{h^n},$$

where the embedding maps ι_{z_2, z_1} are omitted as both R -matrices consist of nonnegative powers of z_1/z_2 . Finally, multiplying by $(z_2 z)^{-p}$ and taking the residue $\text{Res}_{z_2 z}$ we get

$$-\mathcal{L}_1(z_2) R_{21}(z_1 e^{-hc}/z_2) \mathcal{L}_2(z_1) R_{21}(z_1/z_2)^{-1} \pmod{h^n}. \quad (3.58)$$

By applying third term (2.34) of the Jacobi identity to (3.51) we get

$$\begin{aligned} & z_1^{-1} \delta\left(\frac{z_2(1+z)}{z_1}\right) Y_W(z_2) (Y(\log(1+z)) \otimes 1) \\ & \times \iota_{z_2, z_1}(z_1 - z_2)^p R_{12}(z_2/z_1)^{-1} T_{23}^+(0) R_{12}(z_2 e^{-hc}/z_1) T_{14}^+(0) (\mathbf{1} \otimes \mathbf{1}) \pmod{h^n}. \end{aligned} \quad (3.59)$$

As before, by (1.18) and unitarity property (1.19) there exists $s \geq 0$ such that

$$\iota_{z_2, z_1}(z_1 - z_2)^s R_{12}(z_2 e^{-hc}/z_1) = \iota_{z_2, z_1}(z_1 - z_2)^s \psi^{-2} R_{21}(z_1 e^{hc}/z_2)^{-1} \pmod{h^n}. \quad (3.60)$$

Using the δ -function identities

$$\left(\frac{z_1}{z_2}\right)^l \delta\left(\frac{z_2(1+z)}{z_1}\right) = (1+z)^l \delta\left(\frac{z_2(1+z)}{z_1}\right), \quad (3.61)$$

which follow directly from (3.53), one can easily derive

$$\delta\left(\frac{z_2(1+z)}{z_1}\right) = \frac{(z_1 - z_2)^k}{(z_2 z)^k} \delta\left(\frac{z_2(1+z)}{z_1}\right),$$

in particular for $k = p, s$. Therefore, we can employ (3.60) to rewrite (3.59) as

$$\begin{aligned} & \psi^{-2} z_1^{-1} \delta\left(\frac{z_2(1+z)}{z_1}\right) (z_2 z)^p Y_W(z_2) (Y(\log(1+z)) \otimes 1) \\ & \times \iota_{z_2, z_1}(R_{12}(z_2/z_1)^{-1}) T_{23}^+(0) \iota_{z_2, z_1}(R_{21}(z_1 e^{hc}/z_2)^{-1}) T_{14}^+(0) (\mathbf{1} \otimes \mathbf{1}) \pmod{h^n}. \end{aligned}$$

Next, using definition (2.15) of the vertex operator map and (3.61) we get

$$\begin{aligned} & \psi^{-2} z_1^{-1} \delta\left(\frac{z_2(1+z)}{z_1}\right) (z_2 z)^p Y_W(z_2) \iota_{z_2, z_1}(R_{12}(z_2/z_1)^{-1}) \\ & \times T_{23}^+(\log(1+z)) T_{23}^*(\log(1+z) + hc/2)^{-1} R_{21}((1+z)e^{hc})^{-1} T_{13}^+(0) \mathbf{1} \pmod{h^n}. \end{aligned}$$

Finally, we use relation (2.11) to swap the operators T_{23}^* and T_{13}^+ , and then we employ the identity $T^*(x) \mathbf{1} = \mathbf{1}$, thus getting

$$\begin{aligned} & \psi^{-2} z_1^{-1} \delta\left(\frac{z_2(1+z)}{z_1}\right) (z_2 z)^p Y_W(z_2) \iota_{z_2, z_1}(R_{12}(z_2/z_1)^{-1}) \\ & \times T_{23}^+(\log(1+z)) T_{13}^+(0) \iota_{z_2, z_1}(R_{21}(z_1/z_2)^{-1}) \mathbf{1} \pmod{h^n}. \end{aligned} \quad (3.62)$$

It is clear that the application of the module map $Y_W(z_2)$ in (3.62) will not produce any negative powers of the variable z . Therefore, multiplying (3.62) by $(z_2 z)^{-p}$ and then taking the residue $\text{Res}_{z_2 z}$ we obtain $0 \pmod{h^n}$. Hence combining the Jacobi-type identity from Proposition 2.10 with (3.54) and (3.58) we obtain the equality

$$\begin{aligned} & \mathcal{L}_1(z_2) R_{21}(z_1 e^{-hc}/z_2) \mathcal{L}_2(z_1) R_{21}(z_1/z_2)^{-1} \\ & - R_{12}(z_2/z_1)^{-1} \mathcal{L}_2(z_1) R_{12}(z_2 e^{-hc}/z_1) \mathcal{L}_1(z_2) = 0 \pmod{h^n} \end{aligned}$$

for operators on W . As the integer n was arbitrary, we conclude that the given equality holds for all n . Hence we proved that (0.4) satisfies quantum current commutation relation (1.27), so that it defines the structure of $U_h(\widehat{\mathfrak{gl}}_N)$ -module of level c on W , as required. In the end, in order to finish the proof, it remains to observe that W is a topologically free $\mathbb{C}[[h]]$ -module and, furthermore, restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module by Definition 2.7. \square

The next lemma completes the proof of the **Main Theorem** for $\mathfrak{gl}_N = \mathfrak{gl}_N$.

Lemma 3.10. *A topologically free $\mathbb{C}[[h]]$ -submodule W_1 of W is a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -submodule of W if and only if it is an $U_h(\widehat{\mathfrak{gl}}_N)$ -submodule of W .*

Proof. Suppose that W_1 is a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -submodule of W . Then

$$\mathcal{L}(z)w = Y_W(T^+(0) \mathbf{1}, z)w \in \text{End } \mathbb{C}^N \otimes W_1((z))[[h]] \quad \text{for any } w \in W_1,$$

so W_1 is clearly an $U_h(\widehat{\mathfrak{gl}}_N)$ -submodule of W .

Conversely, suppose that W_1 is a topologically free $U_h(\widehat{\mathfrak{gl}}_N)$ -submodule of W . Clearly, W_1 is a restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module of level c , so by Proposition 1.4 we have

$$\mathcal{L}_{[n]}(x_1, \dots, x_n)w \in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes W_1((x_1, \dots, x_n))[[h]] \quad \text{for all } n \geq 1 \text{ and } w \in W_1.$$

Applying the substitutions $x_i = ze^{u_i}$ with $i = 1, \dots, n$ we get

$$\mathcal{L}_{[n]}(x) \Big|_{x_i = ze^{u_i}} w = Y_W(T_{[n]}^+(u) \mathbf{1}, z)w \in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes W_1((z))[[u_1, \dots, u_n, h]].$$

By [9, Sect. 3.4], see also [26, Prop. 2.4], the coefficients of all matrix entries of $T_{[n]}^+(u)$, $n \geq 1$, and $\mathbf{1}$ span an h -adically dense $\mathbb{C}[[h]]$ -submodule of $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$, so we conclude that W_1 is a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ -submodule of W , as required. \square

3.4. Proof of the Main Theorem in the \mathfrak{sl}_N case. For any integer $n = 1, \dots, N$ set

$$u_{[n]} = (u, u - h, \dots, u - (n - 1)h) \quad \text{and} \quad x_{[n]} = (x, xe^{-h}, \dots, xe^{-(n-1)h}).$$

Let $P^{(n)}: x_1 \otimes \dots \otimes x_n \mapsto x_n \otimes \dots \otimes x_1$ be the permutation operator on $(\mathbb{C}^N)^{\otimes n}$. Write

$$\begin{aligned} \mathcal{L}_{[n]}(x_{[n]}) &= \mathcal{L}_{[n]}(x_1, \dots, x_n) \Big|_{x_1=x, \dots, x_n=xe^{-(n-1)h}}, \\ \tilde{\mathcal{L}}_{[n]}(x_{[n]}) &= P^{(n)} \mathcal{L}_{[n]}(x_n, \dots, x_1) \Big|_{x_1=x, \dots, x_n=xe^{-(n-1)h}} P^{(n)}. \end{aligned}$$

We first list some useful properties of the anti-symmetrizer $A^{(n)}$ defined by (1.35).

Lemma 3.11. *For any $n = 1, \dots, N$ we have*

$$A^{(n)} \mathcal{L}_{[n]}(x_{[n]}) = \tilde{\mathcal{L}}_{[n]}(x_{[n]}) A^{(n)}, \quad (3.63)$$

$$A^{(n)} D_1 \dots D_n = D_1 \dots D_n A^{(n)}, \quad (3.64)$$

$$A^{(N)} \overrightarrow{R}_{1N}^{12}(y/x_{[N]}) = \overleftarrow{R}_{1N}^{12}(y/x_{[N]}) A^{(N)} = A^{(N)} e^{-(N-1)h/2} \frac{x - e^{(N-1)h}y}{x - y}, \quad (3.65)$$

where the arrow in $\overrightarrow{R}_{1N}^{12}(y/x_{[N]})$ indicates the reversed order of factors. The coefficients in (3.65) belong to $\text{End } \mathbb{C}^N \otimes (\text{End } \mathbb{C}^N)^{\otimes N}[[h]]$ and the anti-symmetrizer $A^{(N)}$ is applied on the tensor factors $2, \dots, N + 1$.

Proof. Equality (3.63) is verified by using Yang–Baxter equation (1.3), generalized quantum current commutation relation (1.34) and the following case of the fusion procedure for the two-parameter R -matrix $\overline{R}(x, y) = (xe^{-h/2} - ye^{h/2})\overline{R}(x/y)$ going back to [4],

$$\prod_{i=1, \dots, n-1}^{\rightarrow} \prod_{j=i+1, \dots, n}^{\rightarrow} \overline{R}_{ij}(xe^{-(i-1)h}, xe^{-(j-1)h}) = n! x^{\frac{n(n-1)}{2}} \prod_{0 \leq i < j \leq n-1} (e^{-ih} - e^{-jh}) A^{(n)}. \quad (3.66)$$

Equality (3.64) follows from the identities

$$D_1 D_2 = D_2 D_1 \quad \text{and} \quad \overline{R}(x, y) D_1 D_2 = D_2 D_1 \overline{R}(x, y)$$

while (3.65) is established in the proof of [13, Lemma 4.3]. \square

Let W be a restricted $U_h(\widehat{\mathfrak{sl}}_N)$ -module of level $c \in \mathbb{C}$.

Lemma 3.12. *Formula (0.3), together with $Y_W(\mathbf{1}, z) = 1_W$, defines a unique structure of ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{sl}_N)$ -module on W , where $\phi(z_2, z_0) = z_2 e^{z_0}$.*

Proof. In order to prove the lemma, it is sufficient to verify that (0.3), together with $Y_W(\mathbf{1}, z) = 1_W$, defines a $\mathbb{C}[[h]]$ -module map $\overline{\mathcal{V}}_c(\mathfrak{sl}_N) \otimes W \rightarrow W((z))[[h]]$. Indeed, all other properties of the aforementioned map are recovered by arguing as in the $\mathfrak{gl}_N = \mathfrak{gl}_N$ case; see Subsection 3.2. Therefore, we have to show that the map $v \mapsto Y_W(v, z)$ preserves the ideal of relations (2.1) and (2.4). However, it is sufficient to consider (2.4) as relations (2.1) are already taken care of in the proof of Lemma 3.5.

Let n and m be nonnegative integers. Introduce the families of variables $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_m)$. Consider the image of the expression

$$T_{[n]}^{+13}(v) \text{qdet} T^+(u) T_{[m]}^{+23}(w) \mathbf{1} \in (\text{End } \mathbb{C}^N)^{\otimes(n+m)} \otimes \overline{\mathcal{V}}_c(\mathfrak{sl}_N)[[v_1, \dots, v_n, u, w_1, \dots, w_m]] \quad (3.67)$$

with respect to $Y_W(z)$. Introduce the tensor product

$$\overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes N}}^2 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes m}}^3 \otimes \overbrace{\overline{\mathcal{V}}_c(\mathfrak{sl}_N)}^4 \quad (3.68)$$

and write

$$A_2^{(N)} = 1^{\otimes n} \otimes A^{(N)} \otimes 1^{\otimes m} \quad \text{and} \quad D_{[N]}^2 = 1^{\otimes n} \otimes D_{[N]} \otimes 1^{\otimes m} = D_{n+1} \dots D_{n+N}.$$

By the definition of quantum determinant given by (2.2), using the labels in (3.68) to indicate the corresponding tensor factors, (3.67) can be expressed as

$$\text{tr}_{n+1, \dots, n+N} A_2^{(N)} T_{[n]}^{+14}(v) T_{[N]}^{+24}(u_{[N]}) T_{[m]}^{+34}(w) D_{[N]}^2 \mathbf{1}. \quad (3.69)$$

By (0.3), the image of (3.69) with respect to $Y_W(z)$ equals

$$\text{tr}_{n+1, \dots, n+N} A_2^{(N)} \mathcal{L}_{[n+N+m]}^{(N)}(\overline{x}, x_{[N]}, \overline{y}) \Big|_{\substack{x_1=ze^{v_1}, \dots, x_n=ze^{v_n}, x=ze^u \\ y_1=ze^{w_1}, \dots, y_n=ze^{w_m}}} D_{[N]}^2, \quad (3.70)$$

where $\overline{x} = (x_1, \dots, x_n)$ and $\overline{y} = (y_1, \dots, y_m)$. Using generalized quantum current commutation relation (1.34) we transform $A_2^{(N)} \mathcal{L}_{[n+N+m]}^{(N)}(\overline{x}, x_{[N]}, \overline{y})$ and bring it to the form

$$\begin{aligned} & A_2^{(N)} \mathcal{L}_{[n]}^{14}(\overline{x}) R_{nN}^{21}(x_{[N]} e^{-hc}/\overline{x}) R_{nm}^{31}(\overline{y} e^{-hc}/\overline{x}) \mathcal{L}_{[N]}^{24}(x_{[N]}) \\ & \times R_{Nm}^{32}(\overline{y} e^{-hc}/x_{[N]}) \mathcal{L}_{[m]}^{34}(\overline{y}) R_{Nm}^{32}(\overline{y}/x_{[N]})^{-1} R_{nm}^{31}(\overline{y}/\overline{x})^{-1} R_{nN}^{21}(x_{[N]}/\overline{x})^{-1}. \end{aligned} \quad (3.71)$$

By employing (1.11) and (3.65) one can verify the following identities:

$$\begin{aligned} A_2^{(N)} R_{nN}^{21}(x_{[N]} e^{-hc}/\overline{x}) &= e^{-n(N-1)h/2} A_2^{(N)}, & A_2^{(N)} R_{nN}^{21}(x_{[N]}/\overline{x})^{-1} &= e^{n(N-1)h/2} A_2^{(N)}, \\ A_2^{(N)} R_{Nm}^{32}(\overline{y} e^{-hc}/x_{[N]}) &= e^{-m(N-1)h/2} A_2^{(N)}, & A_2^{(N)} R_{Nm}^{32}(\overline{y}/x_{[N]})^{-1} &= e^{m(N-1)h/2} A_2^{(N)}. \end{aligned}$$

As the anti-symmetrizer $A_2^{(N)}$ commutes with the terms $R_{nm}^{31}(\overline{y} e^{-hc}/\overline{x})$, $R_{nm}^{31}(\overline{y}/\overline{x})^{-1}$, $\mathcal{L}_{[n]}^{14}(\overline{x})$ and $\mathcal{L}_{[m]}^{34}(\overline{y})$, by combining the above identities and (3.63), we rewrite (3.71) as

$$\mathcal{L}_{[n]}^{14}(\overline{x}) R_{nm}^{31}(\overline{y} e^{-hc}/\overline{x}) A_2^{(N)} \mathcal{L}_{[N]}^{24}(x_{[N]}) \mathcal{L}_{[m]}^{34}(\overline{y}) R_{nm}^{31}(\overline{y}/\overline{x})^{-1}. \quad (3.72)$$

Note that the expression in (3.70) is obtained from (3.72) by applying the substitutions

$$x_1 = ze^{v_1}, \dots, x_n = ze^{v_n}, x = ze^u, y_1 = ze^{w_1}, \dots, y_n = ze^{w_m},$$

then multiplying by $D_{[N]}^2$ from the right and, finally, taking the trace $\text{tr}_{n+1, \dots, n+N}$. However, as $D_{[N]}^2$ commutes with the terms $\mathcal{L}_{[m]}^{34}(\bar{y})$ and $R_{nm}^{31}(\bar{y}/\bar{x})^{-1}$, it is clear that applying the aforementioned transformations to (3.72) and using definition of quantum determinant (1.36) results in

$$\mathcal{L}_{[n]}^{13}(\bar{x}) R_{nm}^{21}(\bar{y}e^{-hc}/\bar{x}) \text{qdet } \mathcal{L}(x) \mathcal{L}_{[m]}^{23}(\bar{y}) R_{nm}^{21}(\bar{y}/\bar{x})^{-1} \Big|_{\substack{x_1=ze^{v_1}, \dots, x_n=ze^{v_n}, x=ze^u, \\ y_1=ze^{w_1}, \dots, y_n=ze^{w_m}}} \quad (3.73)$$

where, due to application of the trace, the tensor factors in (3.73) are now labeled in accordance with (3.67). As $\text{qdet } \mathcal{L}(x) = 1$ in $U_h(\widehat{\mathfrak{sl}}_N)$ we conclude by quantum current commutation relation (1.34) that (3.73) is equal to

$$\mathcal{L}_{[n+m]}(\bar{x}, \bar{y}) \Big|_{\substack{x_1=ze^{v_1}, \dots, x_n=ze^{v_n} \\ y_1=ze^{w_1}, \dots, y_n=ze^{w_m}}} = Y_W(T_{[n]}^{+13}(v)T_{[m]}^{+23}(w) \mathbf{1}, z).$$

Therefore, the images of (3.67) and $T_{[n]}^{+13}(v)T_{[m]}^{+23}(w) \mathbf{1}$ with respect to $Y_W(z)$ coincide, so we conclude that the $\mathbb{C}[[h]]$ -module map $\bar{\mathcal{V}}_c(\mathfrak{sl}_N) \otimes W \rightarrow W((z))[[h]]$ is well-defined by (0.3), as required. \square

Let (W, Y_W) be a ϕ -coordinated $\bar{\mathcal{V}}_c(\mathfrak{sl}_N)$ -module for some $c \in \mathbb{C}$, where $\phi(z_2, z_0) = z_2e^{z_0}$. In order to prove that (0.4) defines a unique structure of restricted $U_h(\widehat{\mathfrak{sl}}_N)$ -module of level c on W , we need the following identity.

Lemma 3.13. *For any positive integer n the identity*

$$\begin{aligned} Y_W((T_1^+((n-1)h)T_2^+((n-2)h) \dots T_n^+(0) \mathbf{1}, ze^{-(n-1)h}) \\ = \mathcal{L}_{[n]}(x_1, x_2, \dots, x_n) \Big|_{x_1=z, x_2=ze^{-h}, \dots, x_n=ze^{-(n-1)h}} \end{aligned} \quad (3.74)$$

holds for operators on W , where the action of $\mathcal{L}_{[n]}(x_1, \dots, x_n)$ on W is given by formula (1.33) with $\mathcal{L}(x) = Y_W(T^+(0) \mathbf{1}, x)$.

Proof. We derive (3.74) using the weak associativity property. Let k be a positive integer. By (2.29) and (2.30) there exists an integer $p \geq 0$ such that

$$(z_1 - z_2)^p Y_W(T_1^+(0) \mathbf{1}, z_1) Y_W(T_2^+(0) \mathbf{1}, z_2) = (z_1 - z_2)^p \mathcal{L}_1(z_1) \mathcal{L}_2(z_2)$$

belongs to $(\text{End } \mathbb{C}^N)^{\otimes 2} \otimes \text{Hom}(W, W((z_1, z_2)))$ modulo h^k and such that

$$((z_1 - z_2)^p \mathcal{L}_1(z_1) \mathcal{L}_2(z_2)) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } h^k} \quad \text{and} \quad z_2^p (e^{z_0} - 1)^p Y_W(Y(T_1^+(0) \mathbf{1}, z_0)T_2^+(0) \mathbf{1}, z_2) \quad (3.75)$$

coincide modulo h^k . Using relation (2.11) and then the first crossing symmetry property in (2.17) we express the second term in (3.75) as

$$z_2^p (e^{z_0} - 1)^p (D_1 R(e^{z_0+h(c+N)}) D_1^{-1})_{\text{RL}} \cdot (Y_W(T_1^+(z_0)T_2^+(0) \mathbf{1}, z_2) R(e^{z_0})^{-1}). \quad (3.76)$$

The first crossing symmetry property in (2.17) and unitarity (1.19) imply the identities

$$R_{21}(e^{-z_0-hc})_{\text{RL}} \cdot (D_1 R(e^{z_0+h(c+N)}) D_1^{-1}) = 1 \quad \text{and} \quad R(e^{z_0})^{-1} R_{21}(e^{-z_0})^{-1} = 1,$$

which enable us to move the R -matrices appearing in (3.76) from the second term in (3.75) to the first term in (3.75). Hence we find that

$$\left(R_{21}(e^{-z_0-hc})_{\text{RL}} \cdot ((z_1 - z_2)^p \mathcal{L}_1(z_1) \mathcal{L}_2(z_2)) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } h^k} \right) R_{21}(e^{-z_0})^{-1} \quad (3.77)$$

$$\text{and} \quad z_2^p (e^{z_0} - 1)^p Y_W(T_1^+(z_0)T_2^+(0) \mathbf{1}, z_2) \quad (3.78)$$

coincide modulo h^k . Without loss of generality we can assume that the integer p is sufficiently large, so that we conclude by Lemma 3.2 that (3.77) is equal to

$$((z_1 - z_2)^p \mathcal{L}_1(z_1) R_{21}(z_2e^{-hc}/z_1) \mathcal{L}_2(z_2) R_{21}(z_2/z_1)^{-1}) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } h^k}. \quad (3.79)$$

By employing (1.33) for $n = 2$ and the relation

$$\mathcal{L}_{[2]}(z_1, z_2) \in (\text{End } \mathbb{C}^N)^{\otimes 2} \otimes \text{Hom}(W, W((z_1, z_2))[[h]]),$$

which is verified by arguing as in the proof of Proposition 1.4, we rewrite (3.79) as

$$((z_1 - z_2)^p \mathcal{L}_{[2]}(z_1, z_2)) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } h^k} = z_2^p (e^{z_0} - 1)^p (\mathcal{L}_{[2]}(z_1, z_2)) \Big|_{z_1=z_2e^{z_0}}^{\text{mod } h^k}. \quad (3.80)$$

Thus we proved that (3.78) and (3.80) coincide modulo h^k . Hence, multiplying (3.78) and (3.80) by $z_2^{-p}(e^{z_0} - 1)^{-p}$ we find that

$$\mathcal{L}_{[2]}(z_1, z_2) \Big|_{z_1=z_2e^{z_0}} \quad \text{and} \quad Y_W(T_1^+(z_0)T_2^+(0) \mathbf{1}, z_2)$$

coincide modulo h^k . Moreover, by setting $z_0 = h$ and $z_2 = ze^{-h}$ we conclude that

$$\mathcal{L}_{[2]}(z_1, z_2) \Big|_{z_1=z, z_2=ze^{-h}} \quad \text{and} \quad Y_W(T_1^+(h)T_2^+(0) \mathbf{1}, ze^{-h})$$

coincide modulo h^k . As the integer $k > 0$ was arbitrary, this implies equality (3.74) for $n = 2$. The general case is proved by induction on n . \square

The next two lemmas complete the proof of the [Main Theorem](#) for $\mathfrak{g}_N = \mathfrak{sl}_N$. The second lemma follows by the same arguments as for Lemma 3.10, so we omit its proof.

Lemma 3.14. *Formula (0.4) defines a unique structure of restricted $U_h(\widehat{\mathfrak{sl}}_N)$ -module of level c on W .*

Proof. Due to the proof of Lemma 3.9, it is sufficient to verify the equality $\text{qdet } \mathcal{L}(z) = 1$ on W , where the action of $\mathcal{L}(z)$ on W is given by (0.4). By (1.36) and (3.74), the action of quantum determinant of $\mathcal{L}(z)$ on W is given by

$$\begin{aligned} & \text{tr}_{1, \dots, N} A^{(N)} Y_W((T_1^+((N-1)h)T_2^+((N-2)h) \dots T_N^+(0) \mathbf{1}, ze^{-(N-1)h}) D_1 \dots D_N \\ & = Y_W(\text{tr}_{1, \dots, N} A^{(N)} (T_1^+((N-1)h)T_2^+((N-2)h) \dots T_N^+(0) \mathbf{1} D_1 \dots D_N, ze^{-(N-1)h})). \end{aligned}$$

By applying (2.2) with $u = (N-1)h$ and (2.4) the given expression takes the form

$$Y_W(\text{qdet } T^+((N-1)h) \mathbf{1}, ze^{-(N-1)h}) = Y_W(\mathbf{1}, ze^{-(N-1)h}).$$

Finally, Definition 2.7 implies $Y_W(\mathbf{1}, ze^{-(N-1)h}) = 1$, which completes the proof. \square

Lemma 3.15. *A topologically free $\mathbb{C}[[h]]$ -submodule W_1 of W is a ϕ -coordinated $\overline{\mathcal{V}}_c(\mathfrak{sl}_N)$ -submodule of W if and only if W_1 is an $U_h(\widehat{\mathfrak{sl}}_N)$ -submodule of W .*

4. IMAGE OF THE CENTER OF THE QUANTUM AFFINE VERTEX ALGEBRA

In this section, we briefly discuss a connection between families of central elements for the quantum affine vertex algebra and the quantum affine algebra established by the ϕ -coordinated module map from the [Main Theorem](#).

4.1. Noncritical level. Following [22], we define the *center* of the quantum vertex algebra $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$ as the $\mathbb{C}[[h]]$ -submodule

$$\mathfrak{z}(\overline{\mathcal{V}}_c(\mathfrak{gl}_N)) = \{v \in \overline{\mathcal{V}}_c(\mathfrak{gl}_N) : Y(w, z)v \in \overline{\mathcal{V}}_c(\mathfrak{gl}_N)[[z]] \text{ for all } w \in \overline{\mathcal{V}}_c(\mathfrak{gl}_N)\}.$$

For more details on the notion of center of quantum vertex algebra see [5, Thm. 1.4] and [22, Sect. 3.2]. Observe that (0.3) implies the identity

$$Y_W(\text{qdet } T^+(0) \mathbf{1}, z) = \text{qdet } \mathcal{L}(z) \quad (4.1)$$

on any restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module W of level $c \in \mathbb{C}$. By [26, Prop. 3.10] the coefficients of the quantum determinant $\text{qdet } T^+(u)$, as given by (2.3), belong to the center of the quantum vertex algebra $\overline{\mathcal{V}}_c(\mathfrak{gl}_N)$ for any $c \in \mathbb{C}$. The next proposition, which is well-known, provides a quantum affine algebra counterpart of this fact; cf. [13]. We formulate

the proposition and outline its proof in terms of Ding's quantum current realization for completeness.

Proposition 4.1. *For any $c \in \mathbb{C}$ all coefficients d_r of the quantum determinant $\text{qdet } \mathcal{L}(z)$, as given by (1.37), belong to the center of the quantum affine algebra $U_h(\widehat{\mathfrak{gl}}_N)_c$.*

Proof. It is sufficient to prove the equality

$$\mathcal{L}(y) \text{qdet } \mathcal{L}(x) = \text{qdet } \mathcal{L}(x) \mathcal{L}(y) \quad (4.2)$$

in $\text{End } \mathbb{C}^N \otimes U_h(\widehat{\mathfrak{gl}}_N)_c$. By (1.36) the left hand side in (4.2) equals

$$\text{tr}_{1, \dots, N} \mathcal{L}_0(y) A^{(N)} \mathcal{L}_{[N]}(x_{[N]}) D_{[N]}, \quad \text{where } D_{[N]} = D_1 \dots D_N \quad (4.3)$$

and the coefficients of the expression under the trace belong to the tensor product $\text{End } \mathbb{C}^N \otimes (\text{End } \mathbb{C}^N)^{\otimes N} \otimes U_h(\widehat{\mathfrak{gl}}_N)_c$. The copies of $\text{End } \mathbb{C}^N$ in (4.3) are labeled by $0, \dots, N$. The matrix $\mathcal{L}(y)$ is applied on the tensor factor 0 while the remaining terms, $A^{(N)}$, $\mathcal{L}_{[N]}(x_{[N]})$ and $D_{[N]}$ are applied on the tensor factors $1, \dots, N$. By $\mathcal{L}_0(y) A^{(N)} = A^{(N)} \mathcal{L}_0(y)$ and generalized quantum current commutation relation (1.34) we rewrite (4.3) as

$$\text{tr}_{1, \dots, N} A^{(N)} \left(A \cdot_{\text{RL}} \left((B \mathcal{L}_{[N]}(x_{[N]}) C \mathcal{L}_0(y)) E \right) \right) D_{[N]}, \quad (4.4)$$

where

$$\begin{aligned} A &= D_{[N]}^{-1} R_{1N}^{21}(x_{[N]} e^{-(N+c)h}/y)^{-1} D_{[N]}, & B &= R_{1N}^{12}(y/x_{[N]})^{-1}, \\ C &= R_{1N}^{12}(y e^{-hc}/x_{[N]}), & E &= R_{1N}^{21}(x_{[N]}/y). \end{aligned}$$

Note that the element A is found via the second crossing symmetry property in (1.13); see also Remark 2.4. Next, by using (1.11) and (3.65) one can verify the following equalities:

$$A^{(N)} Z = \lambda_Z A^{(N)} \text{ for } Z = A, B, C, E \text{ and } \lambda_A = \lambda_B = \lambda_C^{-1} = \lambda_E^{-1} = e^{(N-1)h/2}. \quad (4.5)$$

Using (3.63) and (4.5) we move the anti-symmetrizer in (4.4) to the right, thus getting

$$\text{tr}_{1, \dots, N} \tilde{\mathcal{L}}_{[N]}(x_{[N]}) \mathcal{L}_0(y) A^{(N)} D_{[N]} = \text{tr}_{1, \dots, N} \tilde{\mathcal{L}}_{[N]}(x_{[N]}) A^{(N)} D_{[N]} \mathcal{L}_0(y).$$

Finally, we use (3.63) to move the anti-symmetrizer $A^{(N)}$ to the left, thus getting the right hand side in (4.2), as required. \square

Following [14, Sect. 3.3], we define the *submodule of invariants* of the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ as the $\mathbb{C}[[h]]$ -submodule

$$\mathfrak{z}(\mathcal{V}_c(\mathfrak{gl}_N)) = \{v \in \mathcal{V}_c(\mathfrak{gl}_N) : \mathcal{L}(z)v \in \mathcal{V}_c(\mathfrak{gl}_N)[[z]]\}.$$

Recall Corollary 0.1. By setting $W = \mathcal{V}_c(\mathfrak{gl}_N)$ in (4.1) and then applying the resulting equality on $1 \in \mathcal{V}_c(\mathfrak{gl}_N)$ one recovers the invariants of the vacuum module; cf. [13].

Corollary 4.2. *For any $c \in \mathbb{C}$ all coefficients of the series*

$$\bar{\ell}_N(z) := Y_{\mathcal{V}_c(\mathfrak{gl}_N)}(\text{qdet } T^+(0), z) 1 = \text{qdet } \mathcal{L}(z) 1 \in \mathcal{V}_c(\mathfrak{gl}_N)[[z]].$$

belong to the submodule of invariants $\mathfrak{z}(\mathcal{V}_c(\mathfrak{gl}_N))$.

Proof. The Corollary follows by applying identity (4.2) on $1 \in \mathcal{V}_c(\mathfrak{gl}_N)$. \square

4.2. Critical level. Consider the quantum affine vertex algebra at the critical level $\bar{\mathcal{V}}_{\text{cri}}(\mathfrak{gl}_N) = \bar{\mathcal{V}}_{-N}(\mathfrak{gl}_N)$. The following family of central elements for the quantum vertex algebra $\bar{\mathcal{V}}_{\text{cri}}(\mathfrak{gl}_N)$ was given by Molev and the author [26, Prop. 3.5].

Proposition 4.3. *All coefficients of the series*

$$\phi_n(u) := \text{tr}_{1, \dots, n} A^{(n)} T_{[n]}^+(u, u-h, \dots, u-(n-1)h) D_1 \dots D_n \mathbf{1} \in \bar{\mathcal{V}}_{\text{cri}}(\mathfrak{gl}_N)[[u]]$$

with $n = 1, \dots, N$ belong to the center of the quantum vertex algebra $\bar{\mathcal{V}}_{\text{cri}}(\mathfrak{gl}_N)$.

Now consider the quantum affine algebra at the critical level $U_h(\widehat{\mathfrak{gl}}_N)_{\text{cri}} = U_h(\widehat{\mathfrak{gl}}_N)_{-N}$. The next theorem goes back to Frappat, Jing, Molev and Ragoucy [13, Thm. 3.2]. Although it is originally given in terms of the *RLL* realization of the quantum affine algebra, we formulate the theorem using Ding’s quantum current realization. The direct proof in terms of Ding’s realization is carried out by arguing as in the proof of [27, Thm. 2.14] and using Lemma 3.11.

Theorem 4.4. *All coefficients of the series*

$$\ell_n(z) := \text{tr}_{1, \dots, n} A^{(n)} \mathcal{L}_{[n]}(z_1, \dots, z_n) \Big|_{z_1=z, \dots, z_n=ze^{-(n-1)h}} D_1 \cdots D_n \in U_h(\widehat{\mathfrak{gl}}_N)_{\text{cri}}[[z^{\pm 1}]]$$

with $n = 1, \dots, N$ belong to the center of the algebra $U_h(\widehat{\mathfrak{gl}}_N)_{\text{cri}}$.

Finally, let W be any restricted $U_h(\widehat{\mathfrak{gl}}_N)$ -module of level $-N$. Then the identities

$$Y_W(\phi_n(0), z) = \ell_n(z) \quad \text{for } n = 1, \dots, N \quad (4.6)$$

hold for operators on W , where the map $Y_W(z)$ is given by (0.3). Recall Corollary 0.1. By setting $W = \mathcal{V}_{\text{cri}}(\mathfrak{gl}_N)$ in (4.6) and then applying the resulting equality on $1 \in \mathcal{V}_{\text{cri}}(\mathfrak{gl}_N)$ one recovers the invariants of the vacuum module; see [13, Corollary 3.3].

Corollary 4.5. *All coefficients of the series*

$$\bar{\ell}_n(z) := Y_{\mathcal{V}_{\text{cri}}(\mathfrak{gl}_N)}(\phi_n(0), z)1 = \ell_n(z)1 \in \mathcal{V}_{\text{cri}}(\mathfrak{gl}_N)[[z]]$$

with $n = 1, \dots, N$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{V}_{\text{cri}}(\mathfrak{gl}_N))$.

ACKNOWLEDGEMENT

The author would like to thank Naihuan Jing and Mirko Primc for stimulating discussions. The research reported in this paper was finalized during the author’s visit to Max Planck Institute for Mathematics in Bonn. The author is grateful to the Institute for its hospitality and financial support. This work has been supported in part by Croatian Science Foundation under the project 8488.

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