Selected Topics in Numerical Linear Algebra and Control

System Identification

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Consider LTI discrete-time control system

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k), \qquad x(0)=x_0, \\ y(k) &= Cx(k)+Du(k). \end{aligned}$$

Role of Laplace transform in continuous-time is played by z-transform:

$$\mathcal{Z}f(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k} = F(z).$$

Fundamental properties:

$$\begin{aligned} \mathcal{Z}[\alpha f+\beta g](z) &= \alpha \mathcal{Z}f(z)+\beta \mathcal{Z}g(z) \\ \mathcal{Z}f(\cdot+1)(z) &= z\cdot \mathcal{Z}f(z) \\ \mathcal{Z}(f*g)(z) &= \mathcal{Z}f(z)\cdot \mathcal{Z}g(z), \end{aligned}$$

with *convolution* $(f*g)(k)=\sum_{j=0}^{\infty} f(j)g(k-j)$.

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Applying z-transform to LTI system $\rightsquigarrow$

$$\begin{aligned} zX(z) &= AX(z) + BU(k), \\ Y(z) &= CX(z) + DU(z). \end{aligned}$$

$$\Rightarrow \quad Y(z) = [C(zI - A)^{-1}B + D]U(z) =: H(z)U(z).$$

Consider SISO case. Then $H(z)$ is a rational function

$$H(z) = \frac{P(z)}{Q(z)} = \frac{p_m z^m + p_{m-1} z^{m-1} + \cdots + p_1 z + p_0}{q_n z^n + q_{n-1} z^{n-1} + \cdots + q_1 z + q_0},$$

where $n \geq m$ (otherwise the system would not be causal).

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where $n \geq m$ (otherwise the system would not be causal).

The relation $Y(z) = H(z)U(z)$ can be reinterpreted as a convolution:

$$y(k) = \sum_{j=0}^{\infty} h(j)u(k-j), \tag{1}$$

with parameters $h(0), h(1), h(2), \ldots$.

These are exactly the parameters that are obtained from the impulse response: $y = h$ for $u = \{1, 0, \ldots, 0\}$.

In this case, $U(z) \equiv 1$ and $Y(z) = H(z)$, so we get from (1):

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \cdots = \frac{P(z)}{Q(z)}.$$

The task of system identification from the impulse response is to determine the (finitely many) parameters defining P and Q from the (infinitely many) parameters $h(0), h(1), \ldots$.

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Equating terms of equal powers of z in $H(z)Q(z) = P(z)$ yields

$$\left[\begin{array}{cccc} 0 & \cdots & 0 & h(0) \\ \vdots & \iddots & h(0) & h(1) \\ 0 & \iddots & \iddots & \vdots \\ h(0) & h(1) & & h(n) \\ \hline h(1) & & \iddots & h(n+1) \\ \vdots & \iddots & \iddots & \vdots \end{array}\right] \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{bmatrix} = \left[\begin{array}{c} p_n \\ p_{n-1} \\ \vdots \\ p_0 \\ \hline 0 \\ \vdots \end{array}\right]$$

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The Hankel matrix

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has rank n for $i \geq n$. Moreover, the leading $n \times n$ matrix is invertible.

But in the presence of noise it is better to work with rectangular matrices and solve a linear least-squares problem.

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Given an impulse response of length s, the following algorithm identifies the frequency-domain representation of the underlying system.

1. Estimate the rank of the $s \times i$ Hankel matrix H_i for increasing values of i until $n = \text{rank}(H_i) < i$.
2. Solve linear least-squares problem $\min \|H_n q + [h(n+1), h(n+2), \cdots, h(2n)]^T\|$.
3. Compute p by matrix-vector multiplication.

Computational issues:

- rank can be estimated recursively (URV updating algorithm);
- Hankel structure can be exploited to reduce flops from si^2 flops to si flops (generalized Schur algorithm);
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Suppose we have identified parameters $p_0, \ldots, p_n$ and $q_0, \ldots, q_n$ defining transfer function $H(z)$. How to obtain state space representation?

Simple by companion form (assume $q_n = 1$):

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \end{bmatrix}.$$

This system is always controllable. you proved it! Further, we have

$$C(zI - A)^{-1}B = \frac{p_{n-1}z^{n-1} + p_{n-1}z^{n-1} + \cdots + p_0}{z^n + q_{n-1}z^{n-1} + \cdots + q_0}$$

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Another possibility:

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$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad D = h(0).$$

This system is always observable.

There are many other systems. For invertible T, the following systems all have the same transfer function

$$\left[\begin{array}{cc} T^{-1}AT & T^{-1}B \\ CT & D \end{array}\right]$$

Are these all? Mostly yes, as we will soon see.

At the moment, we have

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Closing the circle

If we consider the formal Neumann series

$$(zI - A)^{-1} = z^{-1}(I - z^{-1}A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + z^{-4}A^3 + \cdots,$$

we have

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B + D \\ &= D + CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + \cdots \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \cdots \end{aligned}$$

Hence,

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Mimimal realization

Definition

A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of $G(z)$ is minimial if A has smallest possible dimension (the so called McMillan degree).

Theorem

A state space realization (A, B, C, D) is minimal if and only if (A, B) is controllable and (A, C) is observable.

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If (A, B, C, D) and (A', B', C', D') are two minimal realizations of the same transfer function then $\exists$ invertible T with

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This theorem is proven by setting

$$T = [(\mathcal{O})^T\mathcal{O}]^{-1}\mathcal{O}^T\mathcal{O}' = \mathcal{C}\mathcal{C}'^T[\mathcal{C}'\mathcal{C}'^T]^{-1}.$$

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As another consequence, if we have two full rank factorizations $H_j = \mathcal{O}\mathcal{C} = \mathcal{O}'\mathcal{C}'$ then

$$\mathcal{O}' = \mathcal{O}T, \quad \mathcal{C}' = T^{-1}\mathcal{C}.$$

$\rightsquigarrow$ *all* full rank factorization of H_j for $j > n$ lead to state space representations, and vice versa!

If $H_j = \mathcal{O}_j\mathcal{C}_j$, where $\mathcal{O}_j, \mathcal{C}_j$ have structure of controllability/observability matrices, i.e., we can partition

$$\mathcal{O}_j = \begin{bmatrix} \mathcal{O}_- \\ CA^{j-1} \end{bmatrix} = \begin{bmatrix} C \\ \mathcal{O}_+ \end{bmatrix}, \quad \mathcal{C}_j = \begin{bmatrix} \mathcal{C}_- & A^{j-1}B \end{bmatrix} = \begin{bmatrix} B & \mathcal{C}_+ \end{bmatrix}$$

$\rightsquigarrow$ A can be recovered from the relations

$$\mathcal{O}_- A = \mathcal{O}_+, \quad A\mathcal{C}_- = \mathcal{C}_+.$$

Properties of this realization

If we use compact SVD $H_j = U\Sigma V^T$ and set $\mathcal{O}_j = U\Sigma^{1/2}$, $\mathcal{C}_j = \Sigma^{1/2} V^T$ then

$$\mathcal{O}_j^T \mathcal{O}_j = \mathcal{C}_j \mathcal{C}_j^T = \Sigma.$$

The (finite horizon) Gramians

$$\mathcal{O}_j^T \mathcal{O}_j = \sum_{k=0}^{j-1} A^{kT} C^T C A^k, \quad \mathcal{C}_j \mathcal{C}_j^T = \sum_{k=0}^{j-1} A^k B B^T A^{kT}$$

are thus balanced.

Instead of $\{1, 0, 0, \dots\}, \{h(0), h(1), h(2), \dots\}$, we now consider general SISO pair

$$\{u(0), u(1), u(2), \dots, \}, \quad \{y(0), y(1), y(2), \dots, \}$$

generated by a system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of order n.

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From $Y(z)Q(z) = U(z)P(z)$ we have $y \star q = u \star p$, which can be rewritten as

$$\begin{bmatrix} 0 & \cdots & 0 & y(0) \\ \vdots & \iddots & \iddots & y(1) \\ 0 & \iddots & \iddots & y(2) \\ y(0) & \iddots & \iddots & \vdots \\ y(1) & \iddots & \iddots & \\ y(2) & \iddots & & \\ \vdots & & & \end{bmatrix} \begin{bmatrix} q_0 \\ \vdots \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & u(0) \\ \vdots & \iddots & \iddots & u(1) \\ 0 & \iddots & \iddots & u(2) \\ u(0) & \iddots & \iddots & \vdots \\ u(1) & \iddots & \iddots & \\ u(2) & \iddots & & \\ \vdots & & & \end{bmatrix} \begin{bmatrix} p_0 \\ \vdots \\ \vdots \\ p_n \end{bmatrix}.$$

Defining this matrices to be $\mathcal{U}_{:,n+1}$ and $\mathcal{Y}_{:,n+1}$ we can find $p.$ and $q.$ from the kernel of $[\mathcal{U}_{:,n+1}, \mathcal{Y}_{:,n+1}]$.

(The columns of $[\mathcal{U}_{:,n+1}, \mathcal{Y}_{:,n+1}]$ can be reordered to yield a block Hankel matrix with 1×2 blocks.)

Alternatively, by normalizing $d_n = 1$, we can write

$$[\mathcal{U}_{:,n+1}, \mathcal{Y}_{:,n}] \begin{bmatrix} p_0 \\ \vdots \\ p_n \\ -q_0 \\ \vdots \\ -q_{n-1} \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix}.$$

How many rows are sufficient? $2n+1$

Some computational issues:

- Order estimation by incremental SVDs/URVs of $[\mathcal{U}_{:,n+1}, \mathcal{Y}_{:,n+1}]$.
- If more data is available, this should be used (typically, resulting in large tall matrices); system is solved in least-squares sense.
- Real time data implemented by updated QR decompositions (with optional forgetting factor).

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Let $u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^p$. One possibility is to identify transfer matrix componentwise ⇝ *order overestimation*.

If we assume that states $x(\cdot)$ are available, then

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} x(2) & x(3) & \cdots & x(N) \\ y(1) & y(2) & \cdots & y(N-1) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(1) & x(2) & \cdots & x(N-1) \\ u(1) & u(2) & \cdots & u(N-1) \end{bmatrix}$$

⇝ problem is solved if states are known.

Let $u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^p$. One possibility is to identify transfer matrix componentwise $\rightsquigarrow$ *order overestimation*.

If we assume that states $x(\cdot)$ are available, then

$$\left[\begin{array}{c} x(k+1) \\ y(k) \end{array}\right] \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \left[\begin{array}{c} x(k) \\ u(k) \end{array}\right]$$

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$\rightsquigarrow$ problem is solved if states are known.

We have

$$\begin{bmatrix} y(k) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x(k) + \begin{bmatrix} D & 0 \\ CD & D \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix}$$

More general,

$$\begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+i-1) \end{bmatrix} = \mathbf{C}_i x(k) + \mathbf{D}_i \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+i-1) \end{bmatrix},$$

where

$$\mathbf{C}_i = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}, \quad \mathbf{D}_i = \begin{bmatrix} D & & & \\ CB & D & & \\ \vdots & & \ddots & \\ CA^{i-2}B & \cdots & CB\cdots & D \end{bmatrix}.$$

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Define

$$X_k = \left[\begin{array}{cccc} x(k) & x(k+1) & \cdots & x(k+j-1) \end{array}\right],$$

$$Y_{k,i} = \left[\begin{array}{ccc} y(k) & \cdots & y(k+j-1) \\ \vdots & & \vdots \\ y(k+i-1) & \cdots & y(k++i+j-2) \end{array}\right],$$

and similarly $U_{k,i}$:

$$Y_{k,i} = \mathbf{C}_i X_k + \mathbf{D}_i U_{k,i}.$$

Assuming (A, B, C, D) to be minimal ⇝ for $i \geq n$ is X_k in the row space of $Y_{k,i}$ and $U_{k,i}$:

$$\operatorname{Im}(X_k^T) \subseteq \operatorname{Im}\left[\begin{array}{cc} Y_{k,i}^T & U_{k,i}^T \end{array}\right].$$

Of course, also

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Provided that $i \geq n$, $j \geq (m+p)i$, and $u(\cdot)$, $y(\cdot)$ is persistently exciting:

$$\mathrm{Im}(X_{k+i}^T) = \mathrm{Im}\begin{bmatrix} Y_{k,i}^T & U_{k,i}^T \end{bmatrix} \cap \mathrm{Im}\begin{bmatrix} Y_{k+i,i}^T & U_{k+i,i}^T \end{bmatrix}$$

(Theorem 2.8 in [Van Dooren])

By QR decompositions find orthogonal Q and V s.t.

$$\left[\begin{array}{c} Y_{k,i} \\ U_{k,i} \\ \hline Y_{k+i,i} \\ U_{k+i,i} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & Q \end{array}\right] \left[\begin{array}{cc|c} H_{11} & H_{12} & 0 \\ \hline H_{21} & 0 & 0 \\ H_{31} & H_{32} & H_{33} \end{array}\right] V^T,$$

where $[H_{11}, H_{12}]$ and H_{21} have full column rank, and H_{33} has full row rank. $\rightsquigarrow$ first n rows of V^T are a representation of $X_{k+i,j}$.

- Block Hankel structure can be exploited.
- Real-time data processing possible.

There are plenty of extensions! See [Van Dooren] and [Datta].

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Software

- MATLAB system identification toolbox;
- SLICOT system identification toolbox.

Database for benchmark examples:
`http://homes.esat.kuleuven.be/~tokka/daisydata.html`