Selected Topics in Numerical Linear Algebra and Control

State and Output Feedback Control

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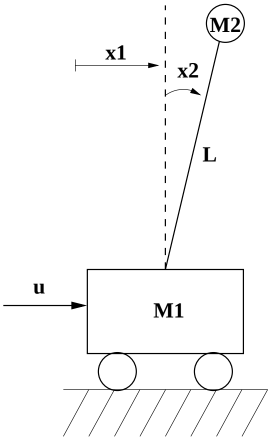
Inverted pendulum on a cart



$M_1 = 1kg,\ M_2 = 2kg$
$L = 1m,\ g = 9.8m/s^2$

Linearized model (after introducing $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$):

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

If only position of cart can be measured:

$$y = Cx,\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

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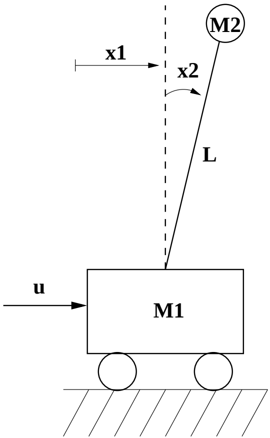

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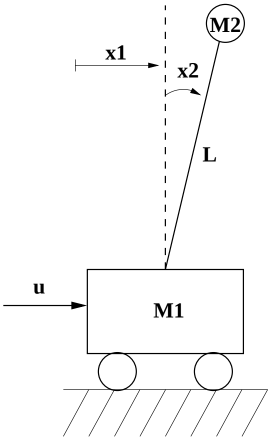

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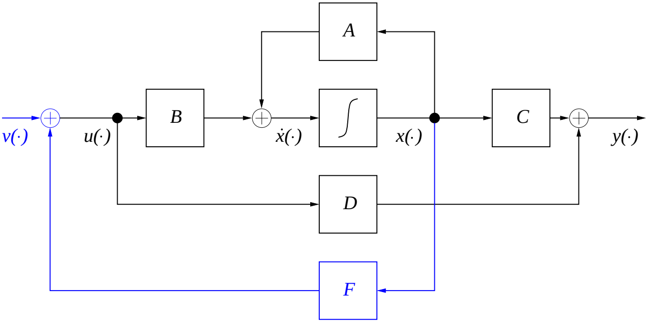
Inverted pendulum on a cart

Not so surprisingly, this is an unstable system

$$\text{eig}(A) = \{-5.42, 0, 0, 5.42\}$$

Effectively, the pendulum will drop down. To avoid this to happen state feedback can be applied:



Linear state feedback

With the linear state feedback law $u(t) = Fx(t) + v(t)$ we can turn

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

into

$$\begin{aligned} \dot{x}(t) &= (A+BF)x(t) + Bv(t) \\ y(t) &= (C+DF)x(t) + Dv(t). \end{aligned}$$

What properties of $A + BF$ can we ask for?

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Pole assignment

Even for the single-input case, we can assign any desired set of eigenvalues to $A + BF$, provided that the pair (A, B) is controllable.

Ackermann's formula (only for single input systems):
Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and $p(x) = (x - \sigma_1) \ldots (x - \sigma_n)$. Set

$$F = -e_n^T [B, AB, A^2B, \ldots, A^{n-1}B]^{-1} p(A).$$

Then the eigenvalues of $A + BF$ are Σ.

Theorem

Let $(\lambda_{k+1}, \ldots, \lambda_n)$ be the uncontrollable modes of (A, B). Then there exists a feedback matrix F with $\Lambda(A + BF) = \Sigma$ if and only if

$$\Sigma = \{\sigma_1, \ldots, \sigma_k, \lambda_{k+1}, \ldots, \lambda_n\}.$$

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Example

For the cart-pendulum example and with $\Sigma = \{-1, \dots, -4\}$, we obtain

$$F = -\text{acker}(A, B) = \left[\begin{array}{cccc} 2.4490 & 66.8490 & 5.1020 & 15.1020 \end{array} \right].$$

We have $\beta(A + BF) = 0.142$.

Moving poles further to the left, does generally not increase $\beta(A + BF)$:

$$\Sigma = \{-10, \dots, -40\} \quad \Rightarrow \quad \beta(A + BF) = 0.047.$$

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Pole placement via Schur decomposition

Ackermann's formula is numerically unreliable, for several reasons (e.g., exponentially growing condition number of $[B, AB, \ldots, A^{n-1}B]$).

Better alternatives are based on Schur decomposition of A:

$$Q_0^T A Q_0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,1} \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}, \quad Q_0^T B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Setting $F_1 = (\sigma_1 - a_{nn})/b_n \cdot e_n^T Q_0^T$ leads to

$$Q_0^T (A + BF) Q_0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & \star \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & \sigma_1 \end{bmatrix}$$

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Pole placement via Schur decomposition

Compute orthogonal matrix Q_1 s.t.

$$(Q_0 Q_1)^T (A + BF)(Q_0 Q_1) = \left[\begin{array}{c|ccc} \sigma_1 & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \hline 0 & \tilde{a}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{a}_{n-1,1} \\ 0 & \cdots & 0 & \tilde{a}_{nn} \end{array}\right].$$

Repeat procedure with $(Q_0 Q_1)^T (A + BF)(Q_0 Q_1)$ and $(Q_0 Q_1)^T B$.

After n applications $\rightsquigarrow$

$$(Q_0 Q_1 \cdots Q_n)^T (A + BF)(Q_0 Q_1 \cdots Q_n) = \begin{bmatrix} \sigma_1 & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{a}_{n-1,1} \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix}$$

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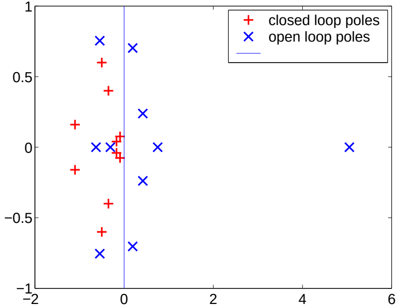
Example

Random matrix pair (A, B) with $n = 10$.



Maximal eigenvalue error using Ackermann's formula: 2×10^{-5}

Maximal eigenvalue error using Schur decomposition: 1×10^{-12}

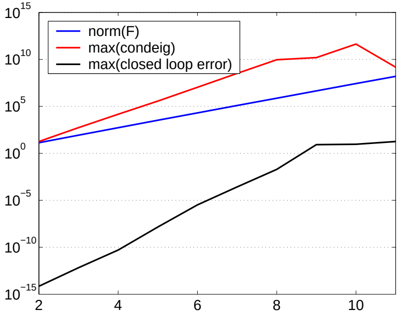
Condition numbers of closed loop eigenvalues: $9 \times 10^2 \ldots 2 \times 10^4$

Pole placement can be extremely ill-conditioned!

Three sources of error:

1. small distance to uncontrollability;
2. large norm of F;
3. ill-conditioned closed loop eigenvalues.

Example: $A = \text{diag}(1, \ldots, n)$, $B = [1, \ldots, 1]^T$, $\Sigma = \{-1, \ldots, -n\}$.



For this example, an eigenvector matrix of $A - BF$ is given by

$$H = \left[\frac{1}{i+j}\right]_{i,j=1}^{n},$$

which is nearly the Hilbert matrix, being famous for its exponentially growing condition number.

Robust pole placement

For multi-input systems, *nm* parameters for placing *n* eigenvalues. Additional freedom can be used to minimize the eigenvalue condition numbers of $A + BF$.

W.l.o.g., we assume $\operatorname{rank}(B) = m$. The goal is to find an $m \times n$ matrix F and a nonsingular matrix X s.t.

$$(A + BF)X = X\Sigma, \quad \Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n). \tag{1}$$

Lemma

Let $B = [U_0, U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix}$ *with unitary* $[U_0, U_1]$ *and square, invertible* Z*. There exists* F *solving (1)* $\Leftrightarrow$

$$U_1^*(AX - X\Sigma) = 0.$$

Moreover, $F = Z^{-1}U_0^*(A - X\Sigma X^{-1})$*.*

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Moreover, $F = Z^{-1} U_0^* (A - X\Sigma X^{-1})$.

It remains to determine $X = [x_1, \ldots, x_n]$.

Lemma

The eigenvector x_j of $A + BF$ belonging to σ_j has to satisfy

$$x_j \in \mathcal{S}_j = kern(U_1^*(A - \sigma_j I)).$$

The dimension of $\mathcal{S}_j$ is $m + k_j$, where

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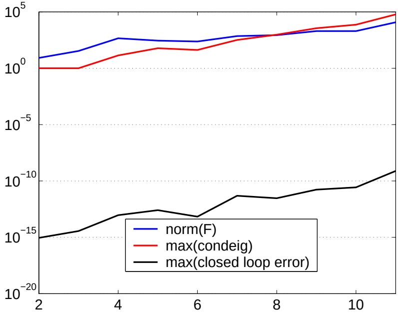
If we let S_j be an orthonormal basis of $\mathcal{S}_j$ for $j = 1, \ldots, n$, choose vectors w_j of length $(m + k_j)$ s.t.

$$X = [S_1, \ldots, S_n] \operatorname{diag}(w_1, \ldots, w_n) := SD.$$

Minimizing the spectral condition $\|SD\| \cdot \|(SD)^{-1}\|$ is difficult for $m > 1$. Instead, we minimize

$$\|X^{-1}\|_F \qquad \text{under the condition } \|D\|_F = n.$$

Example: $A = \mathrm{diag}(1, \ldots, n)$, $B = \mathrm{rand}(n, 3)$, $\Sigma = \{-1, \ldots, -n\}$.



Stabilization with Lyapunov equations

Theorem

Let (A, B) be stabilizable and $\beta \in \mathbb{R}$ with $\beta > |\Lambda(A)|$. If Z solves

$$(A + \beta I)Z + Z(A + \beta I)^T = 2BB^T, \tag{2}$$

then $F = -B^T Z^+$ is a stabilizing feedback.

Leads to the following method.

1. Let $\beta = 2\|A\|$.
2. Solve (2).
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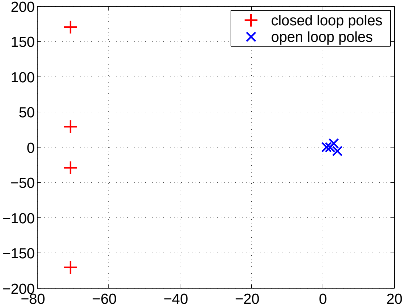
For the cart-pendulum example, we get

$$F \approx \begin{bmatrix} 2 \times 10^7 & 2 \times 10^7 & 6 \times 10^5 & 6 \times 10^5 \end{bmatrix}.$$



Distance to instability: $\beta(A + BF) = 0.0007$.

(These unpleasant effects are caused by the rapidly decaying singular values of Z: 3×10^{-2}, 1×10^{-6}, 2×10^{-9}, 9×10^{-14}).

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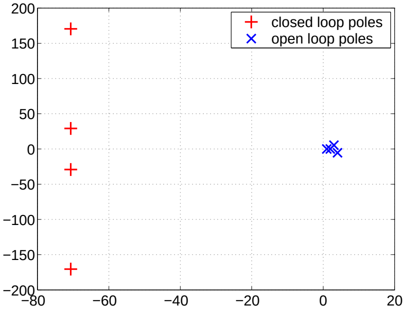

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Extension to discrete-time systems

Pole placement is the same (moving all poles to zeros is called deadbeat control).

Stabilization with Stein equations:

1. Choose $0 < \alpha < \min\{1, \min|\Lambda(A)|\}$.
2. Solve $AZA^T - \alpha^2 Z = 2BB^T$.
3. Set $F = -B^T(Z + BB^T)^+ A$.

Design of state observers

In many cases, full information about states is not available.

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t). \tag{3}$$

Observability is concerned with the question whether we can observe $x(t)$ from $y(t)$.

Definition

The system (3) is *observable* if the following statement holds. Let $x(\cdot)$ and $x'(\cdot)$ be solutions of (3) for the same control $u(\cdot)$. Then

$$Cx(t) = Cx'(t), \quad \text{for all } t \leq t_f$$

implies $x(t) = x'(t)$ for all $t \leq t_f$.

The initial state can be reconstructed from equation

$$\underbrace{\int_0^{t_f} e^{A^T\tau} C^T C e^{A\tau}\,\mathrm{d}\tau}_{=:W^\star(0,t_f)}\, x_0 = \int_0^{t_f} e^{A^T\tau} C^T g(\tau)\,\mathrm{d}\tau$$

where $g(t) = y(t) - \int_0^{t_f} Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$.

If system is *not* observable then there exists $x_0 \neq 0$ with $W^\star(0,t_f)x_0 = 0$. Equivalently, $Ce^{At}x_0 = 0$.

Connection to observability form:

$$\left[\begin{array}{c|c} Q^TAQ & Q^TB \\ \hline CQ & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \\ \hline 0 & C_2 & D \end{array}\right],$$

where A_{11} is $k \times k$ containing the k unobservable modes. The space spanned by the first k columns of Q contains all x_0 with $Ce^{At}x_0 = 0$.

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If unobservable modes are stable then the states can still be asymptotically correctly observed.

Definition

The system (3) is *detectable* if every element of the unobservable subspace is contained in the stable subspace. In other words, for every x_0 with $Ce^{At}x_0 = 0$ we have $Ce^{At}x_0 \to 0$ as $t \to \infty$.

The dual system $\dot{x}(t) = A^T x(t) + C^T$ is stabilizable $\Leftrightarrow$ (3) is detectable. Hence:

Lemma

If (3) is detectable, we can find an $n \times p$ matrix K such that

$$A + KC$$

is stable.

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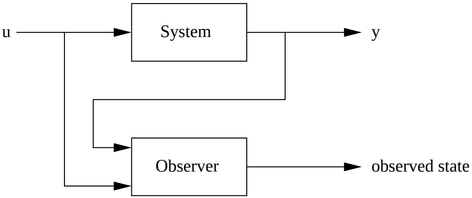
Dynamical observers



The observed state satisfies

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - K(y(t) - C\hat{x}(t))$$

with some $n \times k$ matrix K. When is $\hat{x}$ a good approximation to x? Define

$$e(t) = x(t) - \hat{x}(t).$$

Then

$$\dot{e}(t) = (A + KC)e(t).$$

We can force $e \to 0$ by moving the unstable modes of $A + KC$ to the left half plane. (Always possible if system is detectable.)

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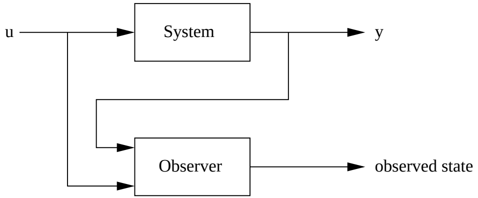

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Stabilization by output feedback

Instead of the state feedback $u(t) = Fx(t)$ we make us of the approximate observed state: $u(t) = F\hat{x}(t)$.

How can we choose F and K to obtain a stable system? The systems

$$\dot{x} = Ax + BF\hat{x}, \quad \dot{\hat{x}} = A\hat{x} + BF\hat{x} - K(Cx - C\hat{x}).$$

can be embedded into

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BF \\ -KC & A + BF + KC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}.$$

By an equivalence transformation

$$\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} A & BF \\ -KC & A + BF + KC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC \end{bmatrix}.$$

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Theorem

The system $\dot{x} = Ax + Bu$, $y = Cx$, can be stabilized by output feedback if and only if it is stabilizable and detectable.

Some interactions between Control and NLA

- The Arnoldi method applied to A with starting vector B produces the controllable subspace of (A, B). In other words, Arnoldi breaks down if and only if (A, B) is not controllable.
- The two-sided Lanczos process applied to A with starting vectors B and C^T produces the minimal realization of (A, B, C) [Parlett'92].
- Pole placement can be used to accelerate the convergence of GMRES [Calvetti/Reichel'03].
- Deadbeat control could be used to speed up general iterative methods.
- The distance to uncontrollability is implicitly used in new deflation techniques for the QR algorithm [Braman/Byers/Mathias'02].