Selected Topics in Numerical Linear Algebra and Control

Optimal Control

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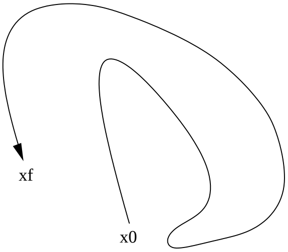
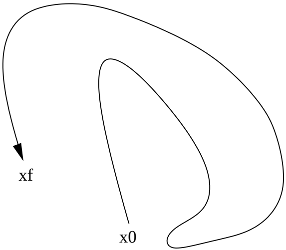


Goal: Find control u which drives $x(0) = x_0$ to $x(t_f) = x_f$.

By $x(t) \to x(t) - x_f$, we can w.l.o.g. assume that $x_f = 0$. Thus, at least asymptotically, our goal can be reached by any stabilizing control.



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Cart-pendulum example

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We want to bring cart and pendulum to rest at 0.

1. Moving poles to $\{-1, -2, -3, -4\} \rightsquigarrow \|u\|_2 = 25$.
2. Moving poles to $\{-10, -20, -30, -40\} \rightsquigarrow \|u\|_2 = 587$.
3. Lyapunov stabilization $\rightsquigarrow \|u\|_2 = 58900$.
4. Optimal control $\rightsquigarrow \|u\|_2 = 3$ (corresponding closed loop poles: $-5.4 \pm 0.0615i$, $-0.4129 \pm 0.4036i$).

Linear optimal control problem

Minimize

$$\mathcal{J}(u) = \frac{1}{2}\left(x(t_f)^T M x(t_f) + \int_0^{t_f} \left[x(t)^T Q x(t) + u(t)^T R u(t) \right] \, \mathrm{d}t \right)$$

under the dynamical constraint

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

Here, M, Q, R symmetric; u piecewise continuous.

1. $x(t_f)^T M x(t_f)$ measures the distance of $x(t_f)$ from 0;
2. $\int x(t)^T Q x(t)$ measures the energy of the state trajectory;
3. $\int u(t)^T R u(t)$ measures the energy of the input.

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LQ optimal control and boundary value problem

Application of Pontryagin's maximum principle.

Theorem

Let $u_\star$ be an optimal control with corresponding state trajectory $x_\star$. Then there is a co-state (Lagrange multiplier) $\mu(t) : [0, t_f] \to \mathbb{R}^n$ s.t. $x_\star(t), \mu(t), u_\star(t)$ solve

$$\begin{bmatrix} A & 0 & B \\ Q & A^T & 0 \\ 0 & B^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ \mu(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{u}(t) \end{bmatrix}$$

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$$x(0) = x_0, \qquad \mu(t_f) = Mx(t_f).$$

This gives necessary conditions; to also obtain sufficient conditions we need more assumptions.

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If Q, R, M are positive semidefinit, then any solution $x_\star(t), \mu(t), u_\star(t)$ of the BVP satisfies

$$\mathcal{J}(u) \geq \mathcal{J}(u_\star)$$

for all piecewise continuous u.

Usually, one also assumes positive definiteness of R ("free" controls are often not meaningful). Then we can set $u(t) = -R^{-1}B^T\mu(t)$ and the BVP becomes

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Turning the BVP into an IVP

Using *ansatz* $\mu(t) = X(t)x(t)$, we have

$$\dot{\mu}(t) = \dot{X}(t)x(t) + X\dot{x}(t), \quad \mu(t_f) = X(t_f)x(t_f).$$

The latter can be satisfied by requiring $X(t_f) = M$. Moreover,

$$\dot{X}(t) = -(Q + A^T X(t) + X(t)A - X(t)BR^{-1}B^T X(t)). \qquad \text{(DRE)}$$

This is the matrix version of the differential Riccati equation. Since $X^T(t)$ also solves (DRE), any solution must be symmetric.

Theorem

If $M, Q \geq 0$ and $R > 0$ then an optimal control $u_\star$ is given by

$$u_\star(t) = -R^{-1}B^T X_\star(t)x_\star(t),$$

where $X_\star$ solves (DRE).

$$\mathcal{J}(u_\star) = \frac{1}{2}x_0^T X_\star(0)x_0.$$

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$t_f \longrightarrow \infty$

We set $M = 0$ (closed loop system must be stable to attain goal).

New *ansatz*: $\mu(t) = Xx(t)$ with constant matrix X. Then (DRE) reduces to algebraic Riccati equation

$$Q + A^T X + XA - XBR^{-1}B^T X = 0. \qquad (ARE)$$

In contrast to (DRE), (ARE) has infinitely many solutions for $n > 1$.

However, together with the stabilization property

$$\Lambda(A - BR^{-1}B^T X) \subseteq \mathbb{C}^-,$$

(ARE) is uniquely solvable under fairly mild conditions.

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ARE and Hamiltonian matrices

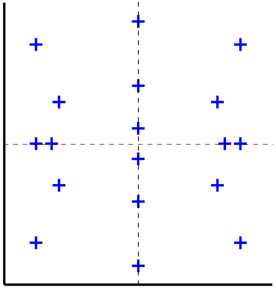
Let

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}, \quad T = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix},$$

where X solves (ARE). Then

$$T^{-1}HT = \begin{bmatrix} A - BR^{-1}B^T X & -BR^{-1}B^T \\ 0 & -(A - BR^{-1}B^T X)^T \end{bmatrix}.$$

Remember that eigenvalues of H come in pairs $(\lambda, -\lambda)$.



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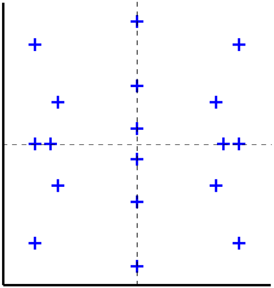

We want $A - BR^{-1}B^T X$ to contain only stable eigenvalues. This is possible iff H has no eigenvalues on the imaginary axis.

Theorem

If (A, B) is stabilizable and (A, Q) is detectable with $Q \geq 0$. Then the corresponding Hamiltonian matrix H has no eigenvalues on the imaginary axis.

Compute block Schur decomposition of H such that

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^T H \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}$$

such that $\Lambda(H_{11}) \subset \mathbb{C}^-$. Then $X_\star = U_{21}U_{11}^{-1}$ is a stabilizing solution of (ARE).

One can show that solution is unique and s.p.sd.

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One can show that solution is unique and s.p.sd.

Theorem

Let $Q \geq 0$ and $R > 0$, (A, B) stabilizable, (A, Q) detectable. Then an optimal, stabilizing control $u_\star$ for $t_f = \infty$ is given by

$$u_\star(t) = -R^{-1}B^T X_\star x_\star(t),$$

where $X_\star$ is the stabilizing solution of (ARE).

$$\mathcal{J}(u_\star) = \frac{1}{2}x_0^T X_\star x_0.$$

Newton for ARE

A different approach for solving (ARE) is obtained by applying Newton to

$$\mathcal{R}(X) = Q + A^T X + XA - XBR^{-1}B^T X = 0$$

The Fréchet derivative of $\mathcal{R}$ in a direction Z can be computed as

$$\begin{aligned} \mathcal{R}'_X(Z) &= \lim_{t\to\infty} \frac{1}{t}(\mathcal{R}(X+tZ) - \mathcal{R}(X)) \\ &= (A - BR^{-1}B^T X)^T Z + Z(A - BR^{-1}B^T X). \end{aligned}$$

Newton step given by $X_{k+1} = X_k + N_k = X_K - \mathcal{R}'_{X_k}(Z)^{-1}\mathcal{R}(X_k)$, i.e.,

$$(A - BR^{-1}B^T X_k)^T N_k + N_k(A - BR^{-1}B^T X_k) = -\mathcal{R}(X_k).$$

Let X_0 be symmetric s.t. $\Lambda(A - BR^{-1}B^TX_0) \subset \mathbb{C}^-$.

for $k = 0, 1, 2 \ldots$

1. Solve Lyapunov eq.
   $(A - BR^{-1}B^TX_k)^TN_k + N_k(A - BR^{-1}B^TX_k) = -\mathcal{R}(X_k)$.
2. Set $X_{k+1} = X_k + N_k$.

Then it can be shown (assuming (A, Q) detectable).

- $\Lambda(A - BR^{-1}B^TX_k) \subset \mathbb{C}^-$ for all k;
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