Selected Topics in Numerical Linear Algebra and Control

Web page: `http://web.math.hr/~kressner/nlacontrol`

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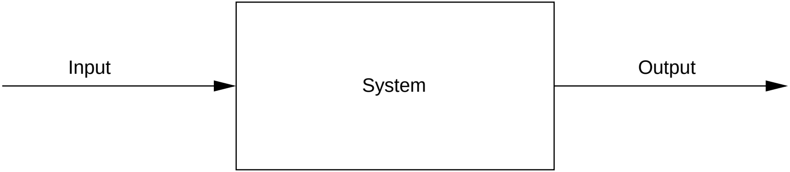
The goal of this course is to

- cover all important aspects of linear control systems;
- emphasize numerical algorithms, in particular from numerical linear algebra;
- lead to state-of-the-art techniques, e.g., in model reduction and optimal control.

Systems are separated from their outer environment.



Input variables:
External quantities acting on the system.

State variables:
Describe condition or state of the system.

Output variables:
System quantities that can be observed.

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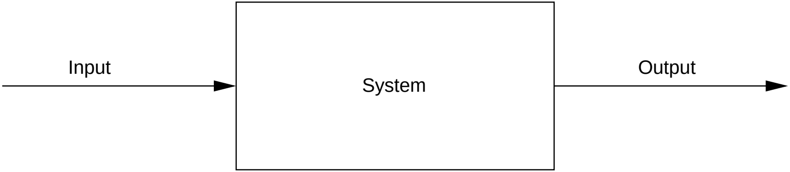

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Continuous time control systems

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), \qquad x(0) = x_0, \\ y(t) &= h(t, x(t), u(t)), \end{aligned}$$

where

$t \in [0, \infty)$ = time,
$x(t) \in \mathbb{R}^n$ = state vector at time t,
$u(t) \in \mathbb{R}^m$ = input vector at time t,
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and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$.

System = state equation (parameterized first-order ODE)
\+ output equation (function in time/state/input)

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Extensions

This is *not* the most general form of a continuous time control systems.

- higher order systems (state equation involves higher order derivatives of x);
- differential-algebraic systems (implicit state equation $F(t, x(t), \dot{x}(t), u(t)) = 0$);
- infinite-dimensional systems (state equation is PDE).

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Linear time-invariant (LTI) continuous time control systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where

$A \in \mathbb{R}^{n\times n}$ = state matrix,
$B \in \mathbb{R}^{n\times m}$ = input matrix,
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$D \in \mathbb{R}^{p\times m}$ = feedthrough matrix (often zero).

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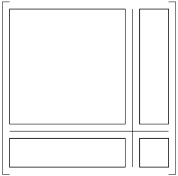
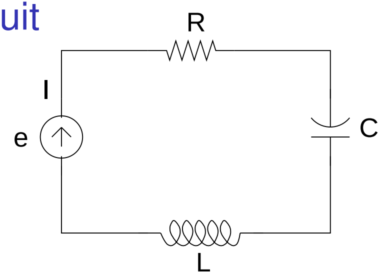


Linear RLC circuit



Kirchhoff's law states that the total voltage must be balanced:

$$e(t) - V_R(t) - V_C(t) - V_L(t) = 0.$$

Combined with

$$V_R(t) = I(t)R, \quad V_C(t) = Q(t)/C, \quad V_L(t) = L\dot{I}(t), \quad I(t) = \dot{Q}(t),$$

gives

$$e(t) = L\ddot{Q}(t) + R\dot{Q}(t) + Q(t)/C.$$

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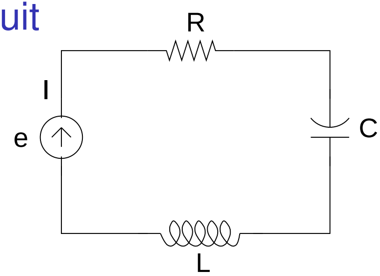

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Linear RLC circuit, ctd.

Second order → first order by introducing $x_1 = Q$, $x_2 = \dot{Q}$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e$$

Difficult to observe Q directly, instead current $I = \dot{Q} = x_2$:

$$y = x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Typical question of observability:

Is it possible to determine the complete state from this observation?

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Discrete time control systems

If time evolves in discrete steps $0, 1, 2, 3, \ldots$:

$$\begin{aligned} x(k+1) &= f(k, x(k), u(k)), \qquad x(0) = x_0, \\ y(k) &= h(k, x(k), u(k)). \end{aligned}$$

LTI case:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \qquad x(0) = x_0, \\ y(k) &= Cx(k) + Du(k). \end{aligned}$$

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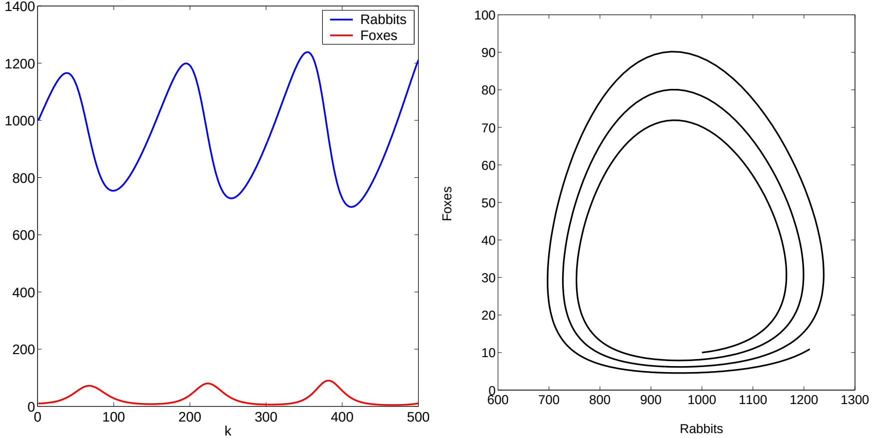
Predator-prey model

$x_1(k)$ = number of rabbits
$x_2(k)$ = number of foxes
a = growth rate of rabbits in absence of predation
b = death rate per encounter of rabbits due to predation
c = death rate of foxes in absence of food
e = efficiency of turning predated rabbits into foxes.

$$\begin{aligned} x_1(k+1) &= x_1(k) + a\,x_1(k) - b\,x_1(k)x_2(k) \\ x_2(k+1) &= x_2(k) + e\,b\,x_1(k)x_2(k) - c\,x_2(k) \end{aligned}$$

Predator-prey model, ctd.



Unstable situation! Typical question of stabilizibality:

Is it possible to stabilize the system by applying control?

System analysis

Introduce concepts of

- frequency domain approach
- time domain/state space approach
- stability
- controllability
- observability

In the following, we focus on:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Laplace transform

For real-valued function $f(t)$, Laplace transform is defined as

$$\hat{f}(s) = (\mathcal{L}f)(s) = \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t.$$

Some properties:

- $\mathcal{L}(\alpha f + \beta g) = \alpha(\mathcal{L}f) + \beta(\mathcal{L}g)$;
- $(\mathcal{L}\dot{f})(s) = s(\mathcal{L}f)(s) - f(0)$;
- $\left(\mathcal{L} \int f(\tau)\, \mathrm{d}\tau\right)(s) = \frac{1}{s}(\mathcal{L}f)(s)$.

Some special functions:

- $\hat{\delta}(s) \equiv 1$, where δ is the Dirac impulse at 0;
- $f(t) = e^{-\lambda t} \rightsquigarrow \hat{f}(s) = 1/(s+\lambda)$;
  $f(t) = \sin(bt) \rightsquigarrow \hat{f}(s) = b/(s^2+b^2)$;
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Transfer matrices

Applying Laplace transform to $\dot{x}(t) = Ax(t) + Bu(t)$ yields

$$s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s) \quad \Rightarrow \quad \hat{x}(s) = (sI - A)^{-1}B\hat{u}(s).$$

Inserting into $\hat{y}(s) = C\hat{y}(s) + D\hat{u}(s)$ gives

$$\hat{y}(s) = (C(sI - A)^{-1}B + D)\hat{u}(s).$$

The $p \times m$ matrix $G(s) = C(sI - A)^{-1}B + D$ is rational and called the transfer function/matrix.

Eigenvalues of A are poles. Zeros of $G(\cdot)$ are system zeros (SISO).

For the linear RCL circuit:

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Hardy spaces and H^p norms

Appropriate spaces for “Laplace transformed” functions.

Definition

$H^p(\mathbb{C}^m)$ is the space of all analytic $u : \mathbb{C}^+ \to \mathbb{C}^m$ with $\|u(\cdot)\|_{H^p} < \infty$ for

$$\|u(\cdot)\|_{H^p} = \begin{cases} \sup\limits_{\beta>0} \left(\int_{-\infty}^{\infty} \|u(\beta + \imath\omega)\|^p \, \mathrm{d}\omega \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{s\in\mathbb{C}^+} \|u(s)\| & \text{if } p = \infty. \end{cases}$$

Important property:

Define boundary function $u^0(\imath\omega) = \lim_{\beta\downarrow 0} u(\beta + \imath\omega)$. Then $u^0 \in L^p(\imath\mathbb{R}, \mathbb{C}^m)$ and $\|u\|_{H^p} = \|u^0\|_{L^p}$. In particular,

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H^p norms, ctd.

Plancherel-like equality (important for transferring results between time and frequency domain):

> The normalized Laplace transform $\sqrt{2\pi}\mathcal{L}$ is a linear *isometry* from $L^2((0,\infty);\mathbb{C}^m)$ onto $H^2(\mathbb{C}^m)$.

H^∞ is often the space easier to work with. Transfer function $G(s) \in \mathbb{C}^{p\times m}$:

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(Computational methods closely related to stability radius.)

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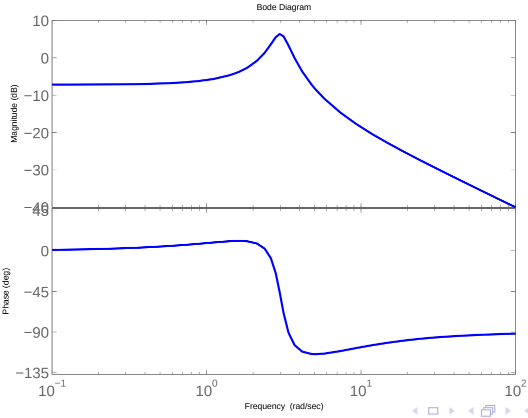
Bode plot

SISO case: For input $u(t) = u_0 \sin(\omega t)$, $\approx$ output for large t

$$y^{ss}(t) = u_0 \, |G(\imath\omega)| \, \sin(\omega t + \arg(G(\imath\omega))).$$

Example: $g(s) = \frac{(s-1)(s+4)}{(s-1)(s+0.4-3\imath)(s+0.4+3\imath)}$.



Don’t use transfer functions for computations!

MATLAB script:

```
n = 50;
A = diag(1:n); B = ones(n,1); C = ones(1,n); D = 0;
sys = ss(A,B,C,D); tra = tf(sys);
p = eig(tra);
plot(p);
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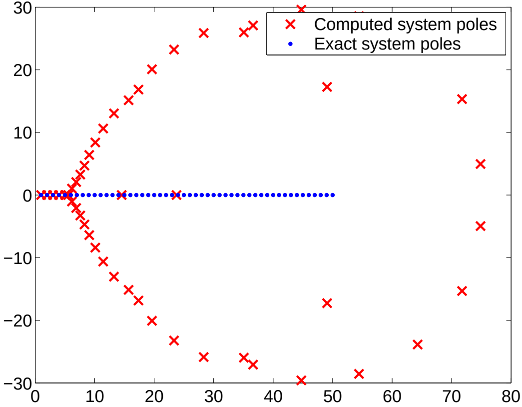

The exponential matrix function

Important tool for developing state space representations:

$$e^{A} := I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Some useful properties:

- $e^{A(t+s)} = e^{At} e^{As}, \quad e^{-At} = \left[e^{At}\right]^{-1}, \quad \frac{\partial}{\partial t} e^{At} = A\, e^{At};$
- $e^{P^{-1}AP} = P^{-1} e^{A} P;$
- if λ is an eigenvalue of A then e^{λ} is an eigenvalue of e^{A};
- But: $e^{A+B} \neq e^{A} e^{B}$ (unless A and B commute).

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Solution of state equation

The state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

is a linear ODE and thus *always* uniquely solvable. Solution:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau.$$

Inserting into output equation $y(t) = Cx(t) + Du(t)$ yields

$$y(t) = C\,e^{At}x_0 + \int_0^t C\,e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t).$$

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Stability

1. The system $\dot{x}(t) = Ax(t)$ is called asymptotically stable if for each x_0 the solution $x(t)$ satisfies $\lim_{t\to\infty} x(t) = 0$.
2. If for each x_0 there exists a constant $C > 0$ so that $\|x(t)\| \le C$ for all $t > 0$ then the system is called stable.
3. In any other case, the system is called unstable.

For $\dot{x}(t) = Ax(t) + Bu(t)$, let $x(t)$ be solution corresponding to initial state x_0 and let $\hat{x}(t)$ be solution corresponding to perturbed $\hat{x}_0$. Then the error between both trajectories $e(t) = \hat{x}(t) - x(t)$ satisfies $\dot{e}(t) = Ae(t)$.

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Eigenvalue criterion for stability

The system $\dot{x}(t) = Ax(t)$ is

1. asymptotically stable $\Leftrightarrow$ $\lambda(A) \subset \mathbb{C}^-$;
2. stable $\Leftrightarrow$ $\lambda(A) \subset \mathbb{C}^- \cup \imath\mathbb{R}$ and each purely imaginary eigenvalue is semi-simple.

Examples:

$$A = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \quad \Rightarrow \quad e^{At} = \begin{bmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

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Lyapunov's direct method

Consider time-invariant system with equilibrium point $\bar{x}$:

$$\dot{x}(t) = f(x(t)), \quad f(\bar{x}) = 0.$$

Let D be an open neighborhood of $\hat{x}$. Differentiable $V : D \to \mathbb{R}$ is called Lyapunov function if

$$V(\bar{x}) = 0, \quad V(x) > 0,\ x \in D \backslash \{\bar{x}\}, \quad \dot{V}(x) \leq 0,\ x \in D,$$

where $\dot{V}(x) = \langle \operatorname{grad} V(x), f(x) \rangle$.
Existence of Lyapunov function $\Rightarrow$ $\bar{x}$ is stable.

V is a strict Lyapunov function if additionally $\dot{V}(x) < 0,\ x \in D \backslash \{\bar{x}\}$.
Existence of strict Lyapunov function $\Rightarrow$ $\bar{x}$ is asymptotically stable.

Lyapunov's direct method

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Lyapunov function for LTI

For $\dot{x}(t) = Ax(t)$ we take $V(x) = x^T Px$ for some symmetric $P \in \mathbb{R}^{n \times n}$.

$$\dot{V}(x) = x^T A^T Px + x^T PAx = x^T (A^T P + PA)x.$$

For a strict Lyapunov function P and $Q = -(A^T P + PA)$ must be positive definite.

Theorem

$\dot{x}(t) = Ax(t)$ is asymptotically stable $\Leftrightarrow$ the Lyapunov matrix equation

$$A^T P + PA = -Q$$

has a s.p.d. solution P for any s.p.d. Q.

Idea of proof: $P = \int_0^\infty e^{A^T t} Q e^{At} \, \mathrm{d}t$.

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Definition of controllability

The system $\dot{x}(t) = Ax(t) + Bu(t)$ is called controllable if there is a (continuous) control law $u(t)$ that transfers any initial state x_0 to any x_f at some time $t_f > 0$.

Remarks:

- State depends linearly on $x_0 \rightsquigarrow$ w.l.o.g. $x_f = 0$.
- Time-invariance of system $\rightsquigarrow$ x_0 can be transferred in any time $t_f > 0$.

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Kalman test

(A, B) is controllable if and only if

$$\mathrm{rank}[B, AB, A^2B, \cdots, A^{n-1}B] = n.$$

Note: Proof facilitates equivalent condition

$$P(t_f) = \int_0^{t_f} e^{-A\tau} BB^T e^{-A^T\tau}\, \mathrm{d}\tau$$

is p.d. for all/some t_f. $t_f \to \infty$: Controllability Gramian

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Direct application of Kalman test is numerically infeasible!

$$A = \begin{bmatrix} 1 & & & & \\ 20 & 2 & & & \\ & \ddots & \ddots & & \\ & & 20 & 19 & \\ & & & 20 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 20 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$\mathcal{C} = [B, AB, \cdots, A^{19}B]$ (MATLAB command `ctrb`) has singular values

$$2.6 \times 10^{28}, 2.2 \times 10^{28}, \ldots, 78.2, 20.$$

$\mathcal{C}$ is numerically singular!

Hautus test

Is p an eigenvector of A^T, then $p^T B \neq 0$.

Variations:

1. $\operatorname{rank}([A - \lambda I, B]) = n$ for all $\lambda \in \Lambda(A)$;
2. $\operatorname{rank}([A - \lambda I, B]) = n$ for all $\lambda \in \mathbb{C}$.

For example from above, $[A - 20 \cdot I, B]$ has singular values

$$36.6, \ldots, 5.9.$$

Far from being singular!

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System transformations

State space transformation $x \rightarrow T^{-1}x$ for nonsingular T corresponds to system transformation

$$\begin{array}{c|c} A & B \\ \hline C & D \end{array} \qquad \Rightarrow \qquad \begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}.$$

Does not change the input-output map. To attain numerical stability, restrict to orthogonal coordinate transformations.

Controllability form for SI systems

Consider single input case ($m = 1$): Let $m = 1$ and Q be an orthogonal matrix which maps B to multiple of e_1:

$$[A|B] \rightsquigarrow [Q^T AQ|Q^T B] = \left[\begin{array}{cccc|c} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & 0 \end{array}\right].$$

Reduce A further to Hessenberg form.

$$\rightsquigarrow \left[\begin{array}{cccc|c} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & \times & \times & \times & 0 \end{array}\right].$$

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Controllability matrix $\mathcal{C}$ of reduced system is upper triangular with diagonal elements

$$b_1, a_{21} b_1, a_{32} a_{21} b_1, a_{43} a_{32} a_{21} b_1, \ldots$$

System is controllable $\Leftrightarrow b_1 \neq 0, a_{21} \neq 0, a_{32} \neq 0, \ldots$

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Controllability form for SI systems

If subdiagonal element is zero, transformed system can be decomposed into

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right]$$

Two subsystems:

$\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ is controllable ($\lambda(A_{11})$ are controllable modes);

$\left[\begin{array}{c|c} A_{22} & 0 \\ \hline C_2 & D \end{array}\right]$ is uncontrollable ($\lambda(A_{22})$ are uncontrollable modes).

Staircase form for MI systems

There exists an orthogonal matrix Q s.t.

$$[Q^T AQ|Q^T B] = \left[\begin{array}{cccc|c||c} A_{1,1} & A_{1,2} & \cdots & A_{1,k} & A_{1,k+1} & X_1 \\ X_2 & A_{2,2} & & \vdots & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ & \ddots & X_k & A_{k,k} & A_{k,k+1} & 0 \\ \hline & & 0 & {\color{blue}0} & A_{k+1,k+1} & 0 \end{array}\right],$$

where

A_{ii} are $\rho_i \times \rho_i$ matrices;
X_i are $\rho_i \times \rho_{i-1}$ matrices of full row rank.

Eigenvalues of $A_{k+1,k+1}$ are uncontrollable modes.

Definition of stabilizability

System $\dot{x}(t) = Ax(t) + Bu(t)$ is called stabilizable if for any x_0 there is a continuous control $u(t)$ such that $\lim_{t\to\infty} x(t) = 0$.

Later, we will see that the poles of a controllable system can be placed *anywhere* by $u(t) = -BFx(t)$.

$\rightsquigarrow$ for systems in staircase controllability form

$$\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}$$

By the controllabilty of (A_{11}, B_1), the poles of A_{11} can always be placed in the open left half plane by some control. But nothing can be done about the poles of A_{22}.

Stabilizability $\Leftrightarrow \lambda(A_{22}) \in \mathbb{C}^-$.

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System $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t)$ is called observable if there is a t_f such that for arbitrary $u(t)$ we can determine $x(0)$ from knowledge of $u(t)$ and $y(t)$ in $[0, t_f]$.

Remark:

- For LTI systems, it is sufficient to show statement for one u.

Let $g(t) = y(t) - \int_0^{t_f} Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau - Du(t)$, then

$$\int_0^{t_f} e^{A^T\tau}C^TCe^{A\tau}\,\mathrm{d}\tau\, x_0 = \int_0^{t_f} e^{A^T\tau}C^Tg(\tau)\,\mathrm{d}\tau$$

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Duality to controllability

Compare both Gramians:

$$Q(t_f) = \int_0^{t_f} e^{A^T\tau} C^T C e^{A\tau}\, \mathrm{d}\tau, \quad P(t_f) = \int_0^{t_f} e^{-A\tau} B B^T e^{-A^T\tau}\, \mathrm{d}\tau$$

Replacing A by $-A^T$ and B by C^T in P yields Q.

Gives equivalent conditions for observability:

1. Observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank.
2. rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$ for all $\lambda \in \lambda(C)$.
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Observability form for SO systems

By a variation of algorithm for controllability form:

$$\left[\frac{Q^T A Q}{CQ}\right] = \left[\begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ \hline 0 & 0 & 0 & \times \end{array}\right]$$

If subdiagonal element is zero, decomposition

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Subsystem $\begin{array}{c|c} A_{11} & B_1 \\ \hline 0 & D \end{array}$ is not observable

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Combining controllability and observability forms

There is an orthogonal matrix Q s.t.

$$\left[\begin{array}{c|c} Q^T A Q & Q^T B \\ \hline CQ & D \end{array}\right] = \left[\begin{array}{ccc|c} A_{c\bar{o}} & \times & \times & B_{c\bar{o}} \\ 0 & A_{co} & \times & B_{co} \\ 0 & 0 & A_{\bar{c}} & 0 \\ \hline 0 & C_{co} & C_{\bar{o}} & D \end{array}\right]$$

where

$A_{c\bar{o}}$ contains unobservable modes;
$A_{\bar{c}}$ contains uncontrollable modes.

Subsystem $\begin{array}{c|c} A_{co} & B_{co} \\ \hline C_{co} & D \end{array}$ is controllable and observable.