Prof. dr. sc. Zlatko Drmač
Dr. sc. Daniel Kressner

Selected Topics in Numerical Linear Algebra and Control

Exercise 3

Assignment 3.1

1. In the proof of the single-input pole placement problem we have reduced Ackermann's formula to the form
$$\tilde{A} = A - e_n e_1^T p(A),$$
where $p(A) = (A - \sigma_1 I) \cdots (A - \sigma_n I)$ and
$$A = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}.$$
Here, $x^n + \alpha_{n-1}x^{n-1} + \alpha_1 x + \alpha_0$ is the characteristic polynomial of A. Prove that the eigenvalues of $\tilde{A}$ are given by $\sigma_1, \dots, \sigma_n$. (Hint: One way to prove this is to show first that $e_1^T(A - \sigma_1 I) \cdots (A - \sigma_j I)$ contains the coefficients of the polynomial $(x - \sigma_1) \cdots (x - \sigma_j)$ for $j < n$.)

2. Consider the closed loop matrix $A + BF$ obtained after placing the eigenvalues of $A = \mathrm{diag}(1, \dots, n)$ to $\Sigma = \{-1, \dots, -n\}$ with $B = [1, \dots, 1]^T$. Show that
$$H^{-1}\tilde{A}H = \mathrm{diag}(-1, \dots, -n),$$
where
$$H = \left[\frac{1}{i+j}\right]_{i,j=1}^{n}.$$
(Hint: Use Ackermann's formula and prove $\tilde{A}H = H\mathrm{diag}(-1, \dots, -n)$.)

3. Implement the Schur decomposition method for pole placement discussed in the lecture assuming that A is a real matrix having only real eigenvalues. Use `schord` to reorder the eigenvalues of a triangular matrix. Test your implementation on the data
$$A = \begin{bmatrix} -4 & -4 & -4 \\ -9 & -9 & -8 \\ -9 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \Sigma = \{-1, -2, -3\},$$
and compare the computed closed loop matrix with the one obtained by applying the Matlab function `acker` or `place`.

4. Consider Example 1.8 from the CTDSX benchmark collection (generated by `[E,A,B,C,D] = ctdsx([1 8]);` E is the identity matrix and may thus be ignored). Compute the stability radius of A. Try to improve the stability radius by placing the largest eigenvalue of A to -0.1,
   (a) using `acker` and only the first column of B as input matrix;
   (b) using `place` and all three columns of B as input matrix.

   Compute the stability radii of the obtained closed loop matrices.

Assignment 3.2

This assignment is concerned with solving the algebraic Riccati equation

$$\mathcal{R}(X) = Q + A^T X + XA - XBR^{-1}B^T X = 0. \tag{1}$$

1. Implement the Newton method for solving (1) presented in the lecture. (Use `lyap` to solve the arising Lyapunov equations.) Test your implementation by applying it to the CTDSX benchmark example 1.6 with

   $$Q = I_n, \quad R = I_3, \quad X_0 = 0$$

   and plot the residual norms $\|\mathcal{R}(X_k)\|_F$ for $k = 0, 1, \ldots$.

2. In the following, we enhance the Newton method with exact line search. For this purpose, we replace the Newton update $X_{k+1} = X_k + N_k$ by $X_{k+1} = X_k + t_k N_k$, where t_k satisfies

   $$\|\mathcal{R}(X_k + t_k N_k)\|_F = \inf_{t\in[0,2]} \|\mathcal{R}(X_k + tN_k)\|_F \tag{2}$$

   (a) Show that $\|\mathcal{R}(X_k + tN_k)\|_F$ is a fourth-order polynomial in t and implement a MATLAB function solving (2) (Hint: use the trace characterization of $\|\cdot\|_F$.)

   (b) Implement the Newton method with exact line search and repeat the numerical experiments from 1.