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Selected Topics in Numerical Linear Algebra and Control

Exercise 2

Assignment 1.1

In the following, $\beta(A)$ denotes the distance of a real matrix A to instability.

1. Prove that $\beta(A) = |\max\{\mathrm{Re}(\lambda) : \lambda \text{ is an eigenvalue of } A\}|$ if A is a symmetric and stable matrix.

2. Prove that $\beta(A) \leq \|A + A^T\|$ for a general complex matrix A. (Hint: one possibility is to construct a perturbation which makes all eigenvalues purely imaginary).

3. The Lyapunov equation $AX + XA^T$ is uniquely solvable for every right hand side if and only if A has no eigenvalues on the imaginary axis. Use the Kronecker representation of the Lypanov equation to prove
$$\frac{1}{2}\sigma_{\min}(I \otimes A + A \otimes I) \leq \beta(A).$$
(Hint: Consider two different classes of perturbations making the matrix $I \otimes A + A \otimes I$ singular.)

Assignment 1.2

Consider the matrix pair

$$(A, B) = \left(\begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 \\ 0 & \beta & 1 & 2 \\ \alpha & 0 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

depending on the two real parameters α and β.

1. Inspect the contour plot of
$$\sigma_{\min}([A - (x + \mathrm{i}y)I, B])$$
in the range $x, y \in [-5, 5]$ for $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (0, 0)$. Try to guess the distance of this pair to uncontrollability for both parameter settings.

2. Compute the controllability form of (A, B) for $(\alpha, \beta) = (1, 1)$ and use the heuristics explained in the lecture to obtain an upper bound on the distance to uncontrollability.

Assignment 1.3

Download the file `impulse.dat` from the web page of this course; it represents the impulse response of a discrete-time LTI system, which is known to be corrupted by an error of roughly 10^{-4}.

1. Estimate the order t of the system by inspecting the singular values of the Hankel matrix for sufficiently large n.

2. Identify the frequency-space representation of the system in two different ways:

(a) using a $t \times t$ Hankel matrix and solving the corresponding linear system for identifying the coefficients of Q;

(b) using a $100 \times t$ Hankel (composed of all available data) and solving the corresponding linear least-squares system for identifying the coefficients of Q.

(Note that there was a sign error in the slides presented in the lecture; this is corrected now. Also be careful with indices, the first entry of the impulse response corresponds to $h(0)$.)

3. Compute the norm of the error between the original and identified impulse responses for both approaches (use `impulse`).

Assignment 1.4

For continuous-time LTI systems the concept of minimal realizations is defined in the same way as for discrete-time LTI systems, i.e., a state space representation is minimal if there is no system of smaller order having the same transfer function.

1. Find a minimal realization of the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 4 & 3 \\ -9/2 & -7/2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 3 & 2 \end{bmatrix} x(t).\end{aligned}$$

2. Prove that a realization of order n is minimal if and only if the transfer function can be written as $G(s) = P(s)/Q(s)$, where P and Q are coprime polynomials and Q has degree n (this holds for both continuous-time and discrete-time systems).

Assignment 1.5

Get comfortable with the SLICOT system identification toolbox and write small MATLAB programs, which identify the discrete-time LTI models of a glass furnace and a CD-player arm from the input-output data available from DaISy (use `findABCD` without Kalman predictor). Plot the poles of the identified systems. Are the identified systems stable (all poles must be within the unit disc)?