Prof. dr. sc. Zlatko Drmač
Dr. sc. Daniel Kressner

Selected Topics in Numerical Linear Algebra and Control

Exercise 1

Assignment 1.1

Read Chapters 2 and 3 of Datta's book.

Assignment 1.2

Consider the following LTI system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 4 & 3 \\ -9/2 & -7/2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 3 & 2 \end{bmatrix} x(t). \end{aligned}$$

1. Write down the transfer function of this system and determine its zeros and poles.
2. Is this system controllable and/or observable?

Assignment 1.3

1. Prove that the pair

$$A = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

   is controllable.

2. Prove that if the pair (A, B) is controllable, then so is the pair $(A - BK, B)$.
3. Rosenbrock test: Show that the pair (A, B) is controllable if and only if the $n^2 \times n(n+m-1)$ matrix

$$\begin{bmatrix} I & & & & & & B \\ -A & I & & & & B & \\ & -A & \ddots & & \cdots & & \\ & & \ddots & I & B & & \\ & & & -A & B & & \end{bmatrix}$$

   has rank n^2.

4. Prove that the following statements are equivalent
   (a) (A, B) is stabilizable;
   (b) if p is an eigenvector of A^T belonging to an eigenvalue λ with $\mathrm{Re}(\lambda) \geq 0$ then $p^T B \neq 0$;
   (c) $\mathrm{rank}([A - \lambda I,\ B]) = n$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.

Assignment 1.4

1. Read Section 7.6 of Datta's book and implement Algorithm 7.6.1 in MATLAB.
2. Test your implementation for computing the distance to instability of a continuous time system with the state matrix

$$A = \begin{bmatrix} -0.5 & 1 & 1 & 1 & 1 & 1 \\ 0 & -0.5 & 1 & 1 & 1 & 1 \\ 0 & 0 & -0.5 & 1 & 1 & 1 \\ 0 & 0 & 0 & -0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & -0.5 \end{bmatrix}$$

   What can you conclude from the obtained result?

Assignment 1.5

In the following, we consider the Sylvester matrix equation

$$AX + XB = C \tag{1}$$

with coefficient matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times n}$.

1. Write down the corresponding Kronecker product formulation (see page 31 of Datta's book).
2. Assume that A and B be in block upper triangular form, i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

   Show how (1) can be solved by solving four lower-dimensional Sylvester equations and a number of matrix-matrix multiplications.

3. Implement a MATLAB program for solving (1) based on the following idea:
   (a) First A and B are reduced to real Schur form, and C is transformed correspondingly (use MATLAB's `schur`).
   (b) The (block) triangular Sylvester equation is solved recursively using the ideas from 2. (Hints: good performance can be obtained by a balanced partition of A and B; avoid cutting 2×2 blocks of the real Schur forms; use the Kronecker product approach – `sylv` – for sufficiently small Sylvester equations.)
   (c) The obtained solution is transformed back using the transformation matrices from (a).

   Compare the execution time of your implementation with the time of the MATLAB function `axxbc` for 500×500 random matrices A, B and C (Note that `axxbc` and `sylv` are only available if the μ toolbox is installed).

4. Apply your implementation to solve the Lyapunov equation $AX + XA^T = -BB^T$ for the benchmark examples 1.6 and 1.9 from the CTDSX collection (see `http://web.math.hr/~kressner/nlacontrol`). Plot the eigenvalues and singular values of X. What theoretical properties would you expect from the solution X? Are they numerically satisfied?