Sveučilište u Dubrovniku 2005/2006
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# Logika, skupovi i diskretna matematika

**Exercise 1**

The assignments are due to 24.10.2005.

## Tutorial 1.1

1. Which of the following definitions properly defines a set?
   (a) $A = \{2, 4, 6, \ldots\}$.
   (b) $B = \{2, 18, 17, 5, \ldots\}$.
   (c) $C$ is the set of all books.
   (d) $D$ is the set of all good books.
2. Explain the difference between $A = \{a\}$ and $A = \{\{a\}\}$.
3. Explain the difference between $A = \emptyset$ and $A = \{\emptyset\}$.
4. Under which condition is $A \cup B = A \triangle B$ true?
5. Given three sets $A, B, C \subseteq X$, provide Venn diagrams and formal proofs of the following relations:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
   (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$;
   (c) $\overline{\overline{A}} = A$.
6. Show that if $X$ consists of $n$ elements then $2^X$ consists of $2^n$ elements.

## Tutorial 1.2

1. Which of the following definitions properly defines a function?
   (a) The function $f_1$ assigns to each country a famous author from this country.
   (b) The function $f_2$ assigns to each country its capital.
   (c) $f_3 : \mathbb{R} \to [-1, 1], \ f_3 : x \mapsto \sin(x)$.
   (d) $f_4 : \mathbb{R} \to (-1, 1), \ f_4 : x \mapsto \sin(x)$.
   (e) $f_5 : [-1, 1] \to \mathbb{R}, \ f_5 : x \mapsto 1/x$.
   (f) $f_6 : \{1, 2, 3\} \to \{a, b, c\}, \ f(1) = f(2) = f(3) = a$.
2. Are the following functions surjective, injective, or even bijective?
   (a) $f_1 : [0, \pi] \to [0, 1], \quad f_1 : x \mapsto \sin x$;
   (b) $f_2 : (-\infty, 0] \to [0, \infty), \quad f_2 : x \mapsto x^2$;
   (c) $f_3 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad f_3 : (a, b) \mapsto ab$.

3. Consider a matrix $A \in \mathbb{R}^{m \times n}$. Under which conditions is the function $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f : x \mapsto Ax$ surjective, injective, bijective?
4. Why do only bijective functions admit inverses?
5. Write down the inverse of the function $f : (-\infty, 0] \to [0, \infty), \quad f_2 : x \mapsto x^2$.
6. Show that if $A$ and $B$ are finite sets of the same cardinality then $f : A \to B$ is injective if and only if it is surjective.
7. Let $f : A \to B$ be a bijective function. Show that $f \circ f^{-1}$ and $f^{-1} \circ f$ are identities.

**Tutorial 1.3**

1. Is a finite set countable or uncountable? Are finite sets always equipotent? Are countable sets always equipotent? Is the set of all possible books that will ever get published finite, countable or uncountable?
2. Show that every infinite subset of a countable set is countable.
3. Show that the set of algebraic numbers is countable.

**Homework assignment 1.1** **8 points**

1. Given three sets $A, B, C \subseteq X$, provide Venn diagrams and formal proofs of the following relations:

    (a) $(A \cup B) \cup C = A \cup (B \cup C)$;

    (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

    (c) $\overline{A \cup B} = \overline{A} \cap \overline{B}$;

    (d) $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

2. Show that for any two sets $A$ and $B$, the following statements are equivalent:

    (a) $A \subseteq B$;

    (b) $A \cap B = A$;

    (c) $A \cup B = B$;

    (d) $A \setminus B = \emptyset$;

    (e) $\overline{B} \subseteq \overline{A}$.

3. Provide a counterexample which shows that

$$Y \cup Z = \{A \cup B : \; A \in Y, B \in Z\}$$

    is in general *not* true for $Y, Z \subseteq 2^X$.

4. If the sets $A_1, A_2, A_3$ have $n_1, n_2, n_3$ elements, respectively, how many elements does $A_1 \times A_2 \times A_3$ have?

**Homework assignment 1.2** **5 points**

1. Are the following functions surjective, injective, or even bijective?

    (a) $f_1 : [0, \pi] \to [-1, 1], \;\; f_1 : x \mapsto \sin x$;

    (b) $f_2 : \mathbb{R} \to [0, \infty), \quad f_2 : x \mapsto |x|$;

    (c) $f_3 : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}, \quad f_3(x) = 1/x$;

    (d) $f_4 : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, \quad f_4 : (a, b) \mapsto a/b$.

2. Consider a matrix $A \in \mathbb{R}^{n \times n}$. What is the inverse of $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f : x \mapsto Ax$?

**Homework assignment 1.3** **5 points**

1. Let $f : A \to B$ be surjective, where $A$ is a finite set. Show that $B$ is finite.

2. Show that the sets $(0, 1)$ and $(a, b)$ with $a < b$ are equipotent. Provide an argument why this implies that any two sets $(a, b)$ and $(c, d)$ with $a < b$ and $c < d$ are equipotent.

3. Is the set of all functions $f : \mathbb{N} \to \mathbb{N}$ countable?