Steiner 2-designs $S(2, 5, 41)$ with Automorphisms of Order 3

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Abstract

In this paper Steiner 2-designs $S(2,5,41)$ with automorphisms of order 3 are classified. Twelve such designs are found, nine of which are new ones. The total number of known $S(2,5,41)$ designs is now 14.

1 Introduction

A Steiner 2-design $S(2,k,v)$ is a set of v *points* together with a collection of k-element subsets (*lines*) such that every pair of points is contained in exactly one line. An *automorphism* is a line-preserving permutation of points. The following theorem about $S(2,5,41)$ designs was proved by R.Mathon and A.Rosa [3].

Theorem 1 *There are exactly four* $S(2,5,41)$ *designs admitting an automorphism of order* 5*.*

The four designs are denoted $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_6$ in table 2. Mathon and Rosa constructed one further example, $\mathcal{D}_4$, by applying a transformation to designs $\mathcal{D}_3$ and $\mathcal{D}_6$. An analogous transformation of $S(2,4,25)$ designs is described in [2]. Thus, five $S(2,5,41)$ designs had been known prior to this article. Performing an exhaustive search for $S(2,5,41)$'s with automorphisms of order 3 we find nine new designs, in addition to three already known ($\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$). Together with designs $\mathcal{D}_1$ and $\mathcal{D}_6$ this sums up to a total of 14 known $S(2,5,41)$ designs.

2 The Classification

First we need to determine possible sets of points and lines kept fixed by an automorphisms of order 3.

Theorem 2 *Let* α *be an automorphism of order* 3 *of a* $S(2,5,41)$ *design. The set of elements kept fixed by* α *is one of the following:*
(a) *five points and ten lines forming a complete pentagon (a* $(5_4,10_2)$ *configuration),*
(b) *five points on a single line,*
(c) *two points and the line joining them.*

Proof. Let f be the number of fixed points and g the number of fixed lines. Denote by p and q the number of point and line orbits of size 3, respectively. Clearly $f + 3p = 41$ and since the total number of lines is 82, $g + 3q = 82$. Furthermore, let b_i be the number of lines containing exactly i fixed points, for $i = 0, \ldots, 5$. If a line contains at least three fixed points, all of its points are fixed. Thus $b_3 = b_4 = 0$. The following two equations are also evident.

$$b_0 + b_1 + b_2 + b_5 = 82 \tag{1}$$

$$b_2 + b_5 = g = 82 - 3q \tag{2}$$

By counting pairs of fixed points and pairs of non-fixed points we get the next two equations.

$$b_2 + 10b_5 = \binom{f}{2} = \frac{(41-3p)(40-3p)}{2} \tag{3}$$

$$10b_0 + 6b_1 + 3b_2 = \binom{41-f}{2} = \frac{3p(3p-1)}{2} \tag{4}$$

The system of equations (1)–(4) has a unique solution in terms of the p's and q's.

$$\begin{aligned} b_0 &= \frac{3p(p-7)}{2} - 2q \\ b_1 &= \frac{3p(7-p)}{2} + 5q \\ b_2 &= \frac{p(27-p)}{2} - \frac{10q}{3} \\ b_5 &= 82 + \frac{p(p-27)}{2} + \frac{q}{3} \end{aligned} \tag{5}$$

On each line with exactly two fixed points there is a point orbit of size 3, hence $b_2 \leq p$. This condition is met by four pairs (p, q) yielding non-negative integer values for the b_i's. The four solutions are reported in table 1.

The solution in the first row is not possible, since there are two lines fixed pointwise ($b_5 = 2$) but only 8 fixed points. Solutions in rows two, three and four correspond to cases (a), (b) and (c), respectively. ∎

No.	p	q	b_0	b_1	b_2	b_5	f	g
1	11	24	18	54	8	2	8	10
2	12	24	42	30	10	0	5	10
3	12	27	36	45	0	1	5	1
4	13	27	63	18	1	0	2	1

Table 1: Solutions to equations (5).

Now we can state the main result.

Theorem 3 *There are exactly twelve* $S(2,5,41)$ *designs admitting an automorphism of order* 3*.*

In table 2 the twelve designs are denoted $\mathcal{D}_2,\ldots,\mathcal{D}_5$ and $\mathcal{D}_7,\ldots,\mathcal{D}_{14}$.

Proof. We prove the theorem using the method of tactical decompositions (see, for example, [1] or [2]). Let α be an automorphism of order 3, $\mathcal{P}_1,\ldots,\mathcal{P}_m$ and $\mathcal{L}_1,\ldots,\mathcal{L}_n$ its point and line orbits. For any point $P\in\mathcal{P}_i$ denote by a_{ij} the number of lines of $\mathcal{L}_j$ incident with P. The orbits form a tactical decomposition, therefore a_{ij} does not depend on the choice of P. It can be shown that the following equations are met:

$$\sum_{j=1}^{n} a_{ij}=10,\quad \sum_{i=1}^{m}\frac{\nu_i}{\beta_j}a_{ij}=5,\quad \sum_{j=1}^{n}\frac{\nu_i}{\beta_j}a_{ij}a_{i'j}=\begin{cases}\nu_i, & \text{if } i\neq i'\\ \nu_i+9, & \text{if } i=i'\end{cases} \tag{6}$$

Here $\nu_i=|\mathcal{P}_i|$ and $\beta_j=|\mathcal{L}_j|$ are the orbit sizes.

A matrix $A=[a_{ij}]$ with non-negative integer entries satisfying relations (6) is called an *orbit matrix*. The first step of the classification is to find all orbit matrices. Matrices equivalent under rearrangements of rows and columns can be identified. For this task an algorithm producing orbit matrices in standard form was used. A matrix is said to be in *standard form* provided it is the greatest amongst all equivalent matrices, where ordering is lexicographical on vectors obtained by concatenating the rows.

The key observation is that a matrix in standard form with the last row removed remains in standard form. Orbit matrices were constructed by adding one row at each step and sifting out matrices that are not in standard form. This is essentially the same algorithm used by E.Spence to classify symmetric $(31, 10, 3)$ designs [6], Hadamard matrices of order 28 [7] and $S(2, 4, 25)$ designs [8].

The second step is usually called *indexing*. Entries in the orbit matrices are replaced by 3×3 circulant 0-1 matrices so as to obtain incidence matrices of $S(2, 5, 41)$ designs. A program based on a straightforward backtracking algorithm was fast enough to do the job.

Not every orbit matrix gives rise to designs. On the other hand, one matrix may lead to several non-isomorphic designs. As the final step incidence matrices of the designs need to be checked for isomorphism. For this B.D.McKay's `nauty` [4] was used.

Cases (a), (b) and (c) of theorem 2 need to be considered separately. In case (a) orbit matrices were constructed quite easily. There are 1904 non-equivalent matrices, none of which give rise to $S(2, 5, 41)$ designs. Thus, an automorphism fixing a complete pentagon is not possible.

In case (b) there are only 5 orbit matrices. However, the computation was more involved than in case (a) because of a large amount of partial matrices. Four of the orbit matrices (denoted $B_1, \ldots, B_4$ in table 2) give rise to a total of seven designs: $\mathcal{D}_5$, $\mathcal{D}_7$, $\mathcal{D}_8$, $\mathcal{D}_9$, $\mathcal{D}_{10}$, $\mathcal{D}_{12}$ and $\mathcal{D}_{13}$.

Case (c) was computationally the most difficult one. To make classification of orbit matrices feasible the standard form had to be considered into some detail. The required CPU time was several months, normalised to a Pentium at 133MHz (the actual computation was performed in parallel on a number of PCs and workstations). 47528 orbit matrices were constructed. The indexing phase proceeded at a much faster pace. Only 12 of the matrices (denoted $C_1, \ldots, C_{12}$) give rise to designs. Ten $S(2, 5, 41)$'s were found: $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_7$, $\mathcal{D}_8$, $\mathcal{D}_9$, $\mathcal{D}_{10}$, $\mathcal{D}_{11}$, $\mathcal{D}_{12}$ and $\mathcal{D}_{14}$. ∎

3 The Designs and Their Properties

Table 2 summarizes information about known $S(2, 5, 41)$ designs. In the second column orbit matrices corresponding to automorphisms of order 3 acting on the designs are listed. The third column contains a description of the full automorphism group. A dot denotes semidirect product and a cross '$\times$' direct product of groups. Factors are cyclic groups $\mathbb{Z}_n$, dihedral groups D_{2n}, alternating groups A_n (on n letters) and the quaternion group Q_8. Full automorphism groups were computed with `nauty` [4] and analyzed with GAP [5].

Orders of full automorphism groups are given in the fourth column.

Amongst linear codes associated with the designs only ternary codes are of interest. If p does not divide $r-1=9$, incidence matrices have full rank $v=41$ or rank $v-1=40$ over $GF(p)$ (the latter occurring only for $p=5$). Dimensions of the codes spanned over $GF(3)$ by incidence vectors of the lines are listed in the last column of table 2. However, minimal weights are not particularly large: three for design $\mathcal{D}_5$ and one or two for the rest.

The last two columns (order of automorphism group and 3-rank) discriminate all designs except $\mathcal{D}_7$ and $\mathcal{D}_9$. The two designs cannot be isomorphic because they arise from different sets of orbit matrices. Incidence matrices of the designs are available through the author's Web page `www.math.hr/~krcko`.

Design	Orbit matrices	Aut	$\|$Aut$\|$	3-rank
$\mathcal{D}_1$		$\mathbb{Z}_5 \cdot \mathbb{Z}_{41}$	205	41
$\mathcal{D}_2$	C_2	$\mathbb{Z}_2 \cdot A_5$	120	32
$\mathcal{D}_3$	C_3	$\mathbb{Z}_2 \cdot A_5$	120	36
$\mathcal{D}_4$	C_1	$\mathbb{Z}_2 \cdot A_4$	24	36
$\mathcal{D}_5$	B_1	$\mathbb{Z}_3 \cdot Q_8$	24	33
$\mathcal{D}_6$		$\mathbb{Z}_2 \cdot D_{10}$	20	36
$\mathcal{D}_7$	B_1, C_9, C_{10}	$\mathbb{Z}_3 \cdot D_6$	18	34
$\mathcal{D}_8$	B_1, C_8, C_{11}	$\mathbb{Z}_3 \cdot D_6$	18	32
$\mathcal{D}_9$	B_4, C_7, C_{10}	$\mathbb{Z}_3 \cdot D_6$	18	34
$\mathcal{D}_{10}$	B_3, C_7, C_{11}	$\mathbb{Z}_3 \cdot D_6$	18	33
$\mathcal{D}_{11}$	C_4	D_{12}	12	35
$\mathcal{D}_{12}$	B_2, C_5, C_6, C_{12}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	35
$\mathcal{D}_{13}$	B_1	$\mathbb{Z}_6$	6	36
$\mathcal{D}_{14}$	C_7	$\mathbb{Z}_6$	6	35

Table 2: Known $S(2,5,41)$ designs.

References

[1] Z.Janko and Tran van Trung, *Construction of a new symmetric block design for* $(78, 22, 6)$ *with the help of tactical decompositions*, Journal of Combinatorial Theory Series A **40** (1985), 451–455.

[2] E.S.Kramer, S.S.Magliveras and R.Mathon, *The Steiner systems* $S(2, 4, 25)$ *with nontrivial automorphism group*, Discrete Mathematics **77** (1989), 137–157.

[3] R. Mathon, private communication.

[4] B.D.McKay, `nauty` *User's guide (version 1.5)*, Technical Report TR-CS-90-02, Department of Computer Science, Australian National University, 1990.

[5] M.Schönert et al, *GAP – Groups, Algorithms and Programming*, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, 1995.

[6] E.Spence, *A complete classification of symmetric* $(31, 10, 3)$ *designs*, Designs, Codes and Cryptography **2** (1992), 127–136.

[7] E.Spence, *Classification of Hadamard matrices of orders* 24 *and* 28, Discrete Mathematics **140** (1995), 185–243.

[8] E.Spence, *The complete classification of Steiner Systems* $S(2, 4, 25)$, Journal of Combinatorial Designs **4** (1996), 295–300.

Appendix

In the appendix designs $\mathcal{D}_1, \ldots, \mathcal{D}_{14}$ are explicitly listed. Designs are described as sets of lines, where points are integers $1, 2, \ldots, 41$.

$\mathcal{D}_1$	
2 3 4 5 6	7 8 9 10 11
1 2 12 27 37	1 3 13 28 38
1 4 14 29 39	1 5 15 30 40
1 6 16 31 41	1 7 17 22 32
1 8 18 23 33	1 9 19 24 34
1 10 20 25 35	1 11 21 26 36
2 7 13 16 40	3 8 12 14 41
4 9 13 15 37	5 10 14 16 38
6 11 12 15 39	2 10 15 19 33
3 11 16 20 34	4 7 12 21 35
5 8 13 17 36	6 9 14 18 32
2 9 26 28 41	3 10 22 29 37
4 11 23 30 38	5 7 24 31 39
6 8 25 27 40	2 8 22 30 39
3 9 23 31 40	4 10 24 27 41
5 11 25 28 37	6 7 26 29 38
2 11 31 32 35	3 7 27 33 36
4 8 28 32 34	5 9 29 33 35
6 10 30 34 36	2 14 20 23 36
3 15 21 24 32	4 16 17 25 33
5 12 18 26 34	6 13 19 22 35
2 17 21 29 34	3 17 18 30 35
4 18 19 31 36	5 19 20 27 32
6 20 21 28 33	2 18 24 25 38
3 19 25 26 39	4 20 22 26 40
5 21 22 23 41	6 17 23 24 37
7 14 19 28 30	8 15 20 29 31
9 16 21 27 30	10 12 17 28 31
11 13 18 27 29	7 15 23 25 34
8 16 24 26 35	9 12 22 25 36
10 13 23 26 32	11 14 22 24 33
7 18 20 37 41	8 19 21 37 38
9 17 20 38 39	10 18 21 39 40
11 17 19 40 41	12 16 19 23 29
12 13 20 24 30	13 14 21 25 31
14 15 17 26 27	15 16 18 22 28
12 32 33 38 40	13 33 34 39 41
14 34 35 37 40	15 35 36 38 41
16 32 36 37 39	22 27 31 34 38
23 27 28 35 39	24 28 29 36 40
25 29 30 32 41	26 30 31 33 37

$\mathcal{D}_2$	
2 3 4 5 6	7 8 9 10 11
1 2 7 12 17	1 3 8 13 18
1 4 9 14 19	1 5 10 15 20
1 6 11 16 21	1 22 27 32 37
1 23 28 33 38	1 24 29 34 39
1 25 30 35 40	1 26 31 36 41
2 11 22 25 31	3 7 23 26 27
4 8 22 24 28	5 9 23 25 29
6 10 24 26 30	2 9 24 37 41
3 10 25 37 38	4 11 26 38 39
5 7 22 39 40	6 8 23 40 41
2 10 27 28 35	3 11 28 29 36
4 7 29 30 32	5 8 30 31 33
6 9 27 31 34	2 8 32 34 38
3 9 33 35 39	4 10 34 36 40
5 11 32 35 41	6 7 33 36 37
2 14 15 26 33	3 15 16 22 34
4 12 16 23 35	5 12 13 24 36
6 13 14 25 32	2 13 16 30 39
3 12 14 31 40	4 13 15 27 41
5 14 16 28 37	6 12 15 29 38
2 19 20 23 36	3 20 21 24 32
4 17 21 25 33	5 17 18 26 34
6 18 19 22 35	2 18 21 29 40
3 17 19 30 41	4 18 20 31 37
5 19 21 27 38	6 17 20 28 39
7 15 19 24 25	8 16 20 25 26
9 12 21 22 26	10 13 17 22 23
11 14 18 23 24	7 13 21 28 31
8 14 17 27 29	9 15 18 28 30
10 16 19 29 31	11 12 20 27 30
7 14 20 34 35	8 15 21 35 36
9 16 17 32 36	10 12 18 32 33
11 13 19 33 34	7 16 18 38 41
8 12 19 37 39	9 13 20 38 40
10 14 21 39 41	11 15 17 37 40
12 25 28 34 41	13 26 29 35 37
14 22 30 36 38	15 23 31 32 39
16 24 27 33 40	17 24 31 35 38
18 25 27 36 39	19 26 28 32 40
20 22 29 33 41	21 23 30 34 37

$\mathcal{D}_3$	
2 3 4 5 6	7 8 9 10 11
1 2 7 12 17	1 3 8 13 18
1 4 9 14 19	1 5 10 15 20
1 6 11 16 21	1 22 27 32 37
1 23 28 33 38	1 24 29 34 39
1 25 30 35 40	1 26 31 36 41
2 11 22 25 29	3 7 23 26 30
4 8 22 24 31	5 9 23 25 27
6 10 24 26 28	2 9 24 37 38
3 10 25 38 39	4 11 26 39 40
5 7 22 40 41	6 8 23 37 41
2 10 30 31 34	3 11 27 31 35
4 7 27 28 36	5 8 28 29 32
6 9 29 30 33	2 8 33 36 39
3 9 32 34 40	4 10 33 35 41
5 11 34 36 37	6 7 32 35 38
2 14 15 26 32	3 15 16 22 33
4 12 16 23 34	5 12 13 24 35
6 13 14 25 36	2 13 16 28 40
3 12 14 29 41	4 13 15 30 37
5 14 16 31 38	6 12 15 27 39
2 19 20 23 35	3 20 21 24 36
4 17 21 25 32	5 17 18 26 33
6 18 19 22 34	2 18 21 27 41
3 17 19 28 37	4 18 20 29 38
5 19 21 30 39	6 17 20 31 40
7 15 18 24 25	8 16 19 25 26
9 12 20 22 26	10 13 21 22 23
11 14 17 23 24	7 13 19 29 31
8 14 20 27 30	9 15 21 28 31
10 16 17 27 29	11 12 18 28 30
7 14 21 33 34	8 15 17 34 35
9 16 18 35 36	10 12 19 32 36
11 13 20 32 33	7 16 20 37 39
8 12 21 38 40	9 13 17 39 41
10 14 18 37 40	11 15 19 38 41
12 25 31 33 37	13 26 27 34 38
14 22 28 35 39	15 23 29 36 40
16 24 30 32 41	17 22 30 36 38
18 23 31 32 39	19 24 27 33 40
20 25 28 34 41	21 26 29 35 37

$\mathcal{D}_4$	
1 2 3 4 5	1 6 28 34 40
1 7 12 19 41	1 8 15 23 37
1 9 11 30 36	1 10 13 21 27
1 14 17 25 32	1 16 22 31 33
1 18 24 35 39	1 20 26 29 38
2 6 30 31 37	2 7 14 18 40
2 8 12 17 27	2 9 20 24 41
2 10 16 35 36	2 11 25 28 29
2 13 19 33 38	2 15 22 26 32
2 21 23 34 39	3 6 29 35 41
3 7 25 36 39	3 8 13 24 40
3 9 15 28 33	3 10 19 26 37
3 11 18 22 27	3 12 16 21 32
3 14 20 30 34	3 17 23 31 38
4 6 32 36 38	4 7 15 20 27
4 8 14 29 33	4 9 18 21 37
4 10 24 28 31	4 11 16 23 40
4 12 26 30 39	4 13 22 25 41
4 17 19 34 35	5 6 27 33 39
5 7 21 29 31	5 8 22 34 36
5 9 17 26 40	5 10 14 23 41
5 11 19 24 32	5 12 18 28 38
5 13 15 30 35	5 16 20 25 37
27 32 37 40 41	6 7 8 9 10
6 11 12 13 14	6 15 16 17 18
6 19 20 21 22	6 23 24 25 26
15 19 29 36 40	19 23 27 28 30
7 23 32 33 35	7 11 34 37 38
11 15 31 39 41	12 20 31 35 40
16 24 27 29 34	8 20 28 32 39
12 24 33 36 37	8 16 30 38 41
21 25 30 33 40	9 25 27 35 38
9 13 31 32 34	13 17 29 37 39
17 21 28 36 41	10 22 38 39 40
14 26 27 31 36	10 18 29 30 32
14 22 28 35 37	18 26 33 34 41
7 13 16 26 28	10 11 17 20 33
14 15 21 24 38	8 18 19 25 31
9 12 22 23 29	10 12 15 25 34
9 14 16 19 39	13 18 20 23 36
7 17 22 24 30	8 11 21 26 35

$\mathcal{D}_5$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 22	2 7 9 19 23
2 8 10 20 21	2 12 16 30 35
2 13 17 31 33	2 14 15 32 34
2 24 29 36 40	2 25 27 37 41
2 26 28 38 39	3 6 10 32 33
3 7 11 30 34	3 8 9 31 35
3 12 17 24 28	3 13 15 25 29
3 14 16 26 27	3 18 23 37 39
3 19 21 38 40	3 20 22 36 41
4 6 24 30 38	4 7 25 31 36
4 8 26 32 37	4 9 14 20 39
4 10 12 18 40	4 11 13 19 41
4 15 23 28 33	4 16 21 29 34
4 17 22 27 35	5 6 23 27 31
5 7 21 28 32	5 8 22 29 30
5 9 13 24 37	5 10 14 25 38
5 11 12 26 36	5 15 20 35 40
5 16 18 33 41	5 17 19 34 39
6 8 28 34 41	6 7 29 35 39
7 8 27 33 40	6 13 14 21 36
7 12 14 22 37	8 12 13 23 38
6 16 19 20 37	7 17 18 20 38
8 15 18 19 36	6 17 25 26 40
7 15 24 26 41	8 16 24 25 39
9 11 16 28 40	9 10 17 29 41
10 11 15 27 39	9 18 25 30 32
10 19 26 30 31	11 20 24 31 32
9 21 22 26 33	10 22 23 24 34
11 21 23 25 35	9 27 34 36 38
10 28 35 36 37	11 29 33 37 38
12 20 25 33 34	13 18 26 34 35
14 19 24 33 35	12 19 27 29 32
13 20 27 28 30	14 18 28 29 31
12 21 31 39 41	13 22 32 39 40
14 23 30 40 41	15 17 21 30 37
15 16 22 31 38	16 17 23 32 36

$\mathcal{D}_6$	
2 3 4 5 6	7 8 9 10 11
1 2 7 12 17	1 3 8 13 18
1 4 9 14 19	1 5 10 15 20
1 6 11 16 21	1 22 27 32 37
1 23 28 33 38	1 24 29 34 39
1 25 30 35 40	1 26 31 36 41
2 11 22 25 28	3 7 23 26 29
4 8 22 24 30	5 9 23 25 31
6 10 24 26 27	2 9 24 40 41
3 10 25 37 41	4 11 26 37 38
5 7 22 38 39	6 8 23 39 40
2 10 29 30 36	3 11 30 31 32
4 7 27 31 33	5 8 27 28 34
6 9 28 29 35	2 8 33 35 37
3 9 34 36 38	4 10 32 35 39
5 11 33 36 40	6 7 32 34 41
2 14 15 26 34	3 15 16 22 35
4 12 16 23 36	5 12 13 24 32
6 13 14 25 33	2 13 16 27 38
3 12 14 28 39	4 13 15 29 40
5 14 16 30 41	6 12 15 31 37
2 19 20 23 32	3 20 21 24 33
4 17 21 25 34	5 17 18 26 35
6 18 19 22 36	2 18 21 31 39
3 17 19 27 40	4 18 20 28 41
5 19 21 29 37	6 17 20 30 38
7 16 19 24 25	8 12 20 25 26
9 13 21 22 26	10 14 17 22 23
11 15 18 23 24	7 15 21 28 30
8 16 17 29 31	9 12 18 27 30
10 13 19 28 31	11 14 20 27 29
7 13 20 35 36	8 14 21 32 36
9 15 17 32 33	10 16 18 33 34
11 12 19 34 35	7 14 18 37 40
8 15 19 38 41	9 16 20 37 39
10 12 21 38 40	11 13 17 39 41
12 22 29 33 41	13 23 30 34 37
14 24 31 35 38	15 25 27 36 39
16 26 28 32 40	17 24 28 36 37
18 25 29 32 38	19 26 30 33 39
20 22 31 34 40	21 23 27 35 41

$\mathcal{D}_7$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 22	2 7 9 19 23
2 8 10 20 21	2 12 17 30 34
2 13 15 31 35	2 14 16 32 33
2 24 29 38 39	2 25 27 36 40
2 26 28 37 41	3 6 10 32 34
3 7 11 30 35	3 8 9 31 33
3 12 16 25 29	3 13 17 26 27
3 14 15 24 28	3 18 23 36 41
3 19 21 37 39	3 20 22 38 40
4 6 26 30 38	4 7 24 31 36
4 8 25 32 37	4 9 14 18 40
4 10 12 19 41	4 11 13 20 39
4 15 23 27 34	4 16 21 28 35
4 17 22 29 33	5 6 21 29 31
5 7 22 27 32	5 8 23 28 30
5 9 13 25 38	5 10 14 26 36
5 11 12 24 37	5 15 20 33 41
5 16 18 34 39	5 17 19 35 40
6 8 27 35 39	6 7 28 33 40
7 8 29 34 41	6 13 14 23 37
7 12 14 21 38	8 12 13 22 36
6 16 19 20 36	7 17 18 20 37
8 15 18 19 38	6 17 24 25 41
7 15 25 26 39	8 16 24 26 40
9 11 16 27 41	9 10 17 28 39
10 11 15 29 40	9 20 24 30 32
10 18 25 30 31	11 19 26 31 32
9 21 22 26 34	10 22 23 24 35
11 21 23 25 33	9 29 35 36 37
10 27 33 37 38	11 28 34 36 38
12 18 26 33 35	13 19 24 33 34
14 20 25 34 35	12 20 27 28 31
13 18 28 29 32	14 19 27 29 30
12 23 32 39 40	13 21 30 40 41
14 22 31 39 41	15 17 21 32 36
15 16 22 30 37	16 17 23 31 38

$\mathcal{D}_8$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 23	2 7 9 19 21
2 8 10 20 22	2 12 17 30 34
2 13 15 31 35	2 14 16 32 33
2 24 28 38 40	2 25 29 36 41
2 26 27 37 39	3 6 10 32 34
3 7 11 30 35	3 8 9 31 33
3 12 16 25 27	3 13 17 26 28
3 14 15 24 29	3 18 22 37 41
3 19 23 38 39	3 20 21 36 40
4 6 26 31 38	4 7 24 32 36
4 8 25 30 37	4 9 13 20 39
4 10 14 18 40	4 11 12 19 41
4 15 23 27 33	4 16 21 28 34
4 17 22 29 35	5 6 21 29 30
5 7 22 27 31	5 8 23 28 32
5 9 14 25 38	5 10 12 26 36
5 11 13 24 37	5 15 20 34 41
5 16 18 35 39	5 17 19 33 40
6 8 27 35 40	6 7 28 33 41
7 8 29 34 39	6 13 14 22 36
7 12 14 23 37	8 12 13 21 38
6 16 19 20 37	7 17 18 20 38
8 15 18 19 36	6 17 24 25 39
7 15 25 26 40	8 16 24 26 41
9 11 16 29 40	9 10 17 27 41
10 11 15 28 39	9 18 26 30 32
10 19 24 30 31	11 20 25 31 32
9 22 23 24 34	10 21 23 25 35
11 21 22 26 33	9 28 35 36 37
10 29 33 37 38	11 27 34 36 38
12 20 24 33 35	13 18 25 33 34
14 19 26 34 35	12 18 28 29 31
13 19 27 29 32	14 20 27 28 30
12 22 32 39 40	13 23 30 40 41
14 21 31 39 41	15 17 21 32 37
15 16 22 30 38	16 17 23 31 36

$\mathcal{D}_9$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 23	2 7 9 19 21
2 8 10 20 22	2 12 17 30 34
2 13 15 31 35	2 14 16 32 33
2 24 29 37 39	2 25 27 38 40
2 26 28 36 41	3 6 10 24 28
3 7 11 25 29	3 8 9 26 27
3 12 16 37 41	3 13 17 38 39
3 14 15 36 40	3 18 22 32 34
3 19 23 30 35	3 20 21 31 33
4 6 29 32 36	4 7 27 30 37
4 8 28 31 38	4 9 14 18 39
4 10 12 19 40	4 11 13 20 41
4 15 22 24 33	4 16 23 25 34
4 17 21 26 35	5 6 16 21 38
5 7 17 22 36	5 8 15 23 37
5 9 25 33 41	5 10 26 34 39
5 11 24 35 40	5 12 20 27 32
5 13 18 28 30	5 14 19 29 31
6 8 25 35 39	6 7 26 33 40
7 8 24 34 41	6 13 14 27 34
7 12 14 28 35	8 12 13 29 33
6 17 19 20 37	7 15 18 20 38
8 16 18 19 36	6 22 30 31 41
7 23 31 32 39	8 21 30 32 40
9 11 28 32 37	9 10 29 30 38
10 11 27 31 36	9 13 22 23 40
10 14 21 23 41	11 12 21 22 39
9 16 17 24 31	10 15 17 25 32
11 15 16 26 30	9 20 34 35 36
10 18 33 35 37	11 19 33 34 38
12 18 25 26 31	13 19 24 26 32
14 20 24 25 30	12 23 24 36 38
13 21 25 36 37	14 22 26 37 38
15 19 27 39 41	16 20 28 39 40
17 18 29 40 41	15 21 28 29 34
16 22 27 29 35	17 23 27 28 33

$\mathcal{D}_{10}$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 23	2 7 9 19 21
2 8 10 20 22	2 12 17 30 34
2 13 15 31 35	2 14 16 32 33
2 24 29 37 39	2 25 27 38 40
2 26 28 36 41	3 6 10 37 41
3 7 11 38 39	3 8 9 36 40
3 12 16 18 22	3 13 17 19 23
3 14 15 20 21	3 24 28 32 34
3 25 29 30 35	3 26 27 31 33
4 6 21 29 36	4 7 22 27 37
4 8 23 28 38	4 9 14 35 39
4 10 12 33 40	4 11 13 34 41
4 15 19 24 30	4 16 20 25 31
4 17 18 26 32	5 6 16 30 38
5 7 17 31 36	5 8 15 32 37
5 9 18 25 41	5 10 19 26 39
5 11 20 24 40	5 12 23 27 35
5 13 21 28 33	5 14 22 29 34
6 8 19 27 34	6 7 20 28 35
7 8 18 29 33	6 13 14 26 40
7 12 14 24 41	8 12 13 25 39
6 17 24 25 33	7 15 25 26 34
8 16 24 26 35	6 22 31 32 39
7 23 30 32 40	8 21 30 31 41
9 11 22 26 30	9 10 23 24 31
10 11 21 25 32	9 13 27 29 32
10 14 27 28 30	11 12 28 29 31
9 16 17 28 37	10 15 17 29 38
11 15 16 27 36	9 20 33 34 38
10 18 34 35 36	11 19 33 35 37
12 19 20 32 36	13 18 20 30 37
14 18 19 31 38	12 21 26 37 38
13 22 24 36 38	14 23 25 36 37
15 18 28 39 40	16 19 29 40 41
17 20 27 39 41	15 22 23 33 41
16 21 23 34 39	17 21 22 35 40

$\mathcal{D}_{11}$	
1 2 39 40 41	1 3 6 9 12
1 4 7 10 13	1 5 8 11 14
1 15 18 21 24	1 16 19 22 25
1 17 20 23 26	1 27 30 33 36
1 28 31 34 37	1 29 32 35 38
2 3 8 15 19	2 4 6 16 20
2 5 7 17 18	2 9 13 27 32
2 10 14 28 30	2 11 12 29 31
2 21 26 34 36	2 22 24 35 37
2 23 25 33 38	3 5 25 30 39
3 4 26 31 40	4 5 24 32 41
3 7 35 36 41	4 8 33 37 39
5 6 34 38 40	3 10 11 23 32
4 9 11 21 30	5 9 10 22 31
3 14 20 33 34	4 12 18 34 35
5 13 19 33 35	3 13 21 22 38
4 14 22 23 36	5 12 21 23 37
3 16 17 24 28	4 15 17 25 29
5 15 16 26 27	3 18 27 29 37
4 19 27 28 38	5 20 28 29 36
6 8 21 27 41	6 7 22 28 39
7 8 23 29 40	6 10 15 36 37
7 11 16 37 38	8 9 17 36 38
6 11 25 26 35	7 9 24 26 33
8 10 24 25 34	6 13 14 24 29
7 12 14 25 27	8 12 13 26 28
6 17 31 32 33	7 15 30 32 34
8 16 30 31 35	6 18 19 23 30
7 19 20 21 31	8 18 20 22 32
9 14 16 18 40	10 12 17 19 41
11 13 15 20 39	9 15 23 28 35
10 16 21 29 33	11 17 22 27 34
9 20 25 37 41	10 18 26 38 39
11 19 24 36 40	9 19 29 34 39
10 20 27 35 40	11 18 28 33 41
12 15 22 33 40	13 16 23 34 41
14 17 21 35 39	12 16 32 36 39
13 17 30 37 40	14 15 31 38 41
12 20 24 30 38	13 18 25 31 36
14 19 26 32 37	21 25 28 32 40
22 26 29 30 41	23 24 27 31 39

$\mathcal{D}_{12}$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 22	2 7 9 19 23
2 8 10 20 21	2 12 17 30 34
2 13 15 31 35	2 14 16 32 33
2 24 29 37 39	2 25 27 38 40
2 26 28 36 41	3 6 10 32 34
3 7 11 30 35	3 8 9 31 33
3 12 16 26 27	3 13 17 24 28
3 14 15 25 29	3 18 23 36 40
3 19 21 37 41	3 20 22 38 39
4 6 24 30 41	4 7 25 31 39
4 8 26 32 40	4 9 14 18 37
4 10 12 19 38	4 11 13 20 36
4 15 23 28 34	4 16 21 29 35
4 17 22 27 33	5 6 29 31 38
5 7 27 32 36	5 8 28 30 37
5 9 13 21 40	5 10 14 22 41
5 11 12 23 39	5 15 19 26 33
5 16 20 24 34	5 17 18 25 35
6 8 23 27 35	6 7 21 28 33
7 8 22 29 34	6 13 14 26 39
7 12 14 24 40	8 12 13 25 41
6 17 19 20 40	7 15 18 20 41
8 16 18 19 39	6 16 25 36 37
7 17 26 37 38	8 15 24 36 38
9 11 16 28 38	9 10 17 29 36
10 11 15 27 37	9 20 25 30 32
10 18 26 30 31	11 19 24 31 32
9 22 24 26 35	10 23 24 25 33
11 21 25 26 34	9 27 34 39 41
10 28 35 39 40	11 29 33 40 41
12 18 28 29 32	13 19 27 29 30
14 20 27 28 31	12 20 33 35 37
13 18 33 34 38	14 19 34 35 36
12 21 22 31 36	13 22 23 32 37
14 21 23 30 38	15 17 21 32 39
15 16 22 30 40	16 17 23 31 41

$\mathcal{D}_{13}$	
1 2 3 4 5	1 6 9 12 15
1 7 10 13 16	1 8 11 14 17
1 18 21 24 27	1 19 22 25 28
1 20 23 26 29	1 30 33 36 39
1 31 34 37 40	1 32 35 38 41
2 6 11 18 22	2 7 9 19 23
2 8 10 20 21	2 12 16 30 35
2 13 17 31 33	2 14 15 32 34
2 24 29 37 41	2 25 27 38 39
2 26 28 36 40	3 6 10 32 33
3 7 11 30 34	3 8 9 31 35
3 12 17 26 27	3 13 15 24 28
3 14 16 25 29	3 18 23 38 40
3 19 21 36 41	3 20 22 37 39
4 6 26 30 37	4 7 24 31 38
4 8 25 32 36	4 9 14 18 39
4 10 12 19 40	4 11 13 20 41
4 15 23 27 33	4 16 21 28 34
4 17 22 29 35	5 6 21 29 31
5 7 22 27 32	5 8 23 28 30
5 9 13 25 37	5 10 14 26 38
5 11 12 24 36	5 15 20 35 40
5 16 18 33 41	5 17 19 34 39
6 8 27 34 41	6 7 28 35 39
7 8 29 33 40	6 13 14 23 36
7 12 14 21 37	8 12 13 22 38
6 16 19 20 38	7 17 18 20 36
8 15 18 19 37	6 17 24 25 40
7 15 25 26 41	8 16 24 26 39
9 11 16 27 40	9 10 17 28 41
10 11 15 29 39	9 20 24 30 32
10 18 25 30 31	11 19 26 31 32
9 21 22 26 33	10 22 23 24 34
11 21 23 25 35	9 29 34 36 38
10 27 35 36 37	11 28 33 37 38
12 20 25 33 34	13 18 26 34 35
14 19 24 33 35	12 18 28 29 32
13 19 27 29 30	14 20 27 28 31
12 23 31 39 41	13 21 32 39 40
14 22 30 40 41	15 17 21 30 38
15 16 22 31 36	16 17 23 32 37

$\mathcal{D}_{14}$	
1 2 3 4 5	1 6 27 31 36
1 7 12 30 37	1 8 11 15 32
1 9 14 19 25	1 10 21 33 39
1 13 20 26 35	1 16 22 34 40
1 17 24 28 29	1 18 23 38 41
2 6 29 34 35	2 7 15 20 41
2 8 19 23 37	2 9 11 28 30
2 10 13 32 38	2 12 17 26 40
2 14 22 36 39	2 16 25 31 33
2 18 21 24 27	3 6 30 38 40
3 7 22 24 32	3 8 33 35 36
3 9 12 15 21	3 10 16 19 28
3 11 17 23 31	3 13 25 27 37
3 14 18 20 34	3 26 29 39 41
4 6 28 37 39	4 7 13 31 34
4 8 25 29 40	4 9 18 32 35
4 10 22 23 27	4 11 16 21 41
4 12 20 24 36	4 14 17 30 33
4 15 19 26 38	5 6 32 33 41
5 7 18 25 28	5 8 14 24 38
5 9 22 26 31	5 10 17 20 37
5 11 19 34 36	5 12 16 23 35
5 13 21 29 30	5 15 27 39 40
6 7 8 9 10	6 11 12 13 14
7 11 26 27 33	6 15 16 17 18
11 18 22 29 37	7 16 29 36 38
6 19 20 21 22	11 20 25 38 39
7 17 19 35 39	6 23 24 25 26
10 11 24 35 40	7 14 21 23 40
8 21 26 28 34	10 12 25 34 41
14 27 28 35 41	23 30 32 34 39
19 24 30 31 41	20 28 31 32 40
15 24 33 34 37	9 36 37 40 41
13 15 23 28 36	9 17 27 34 38
8 13 17 22 41	12 22 28 33 38
9 20 23 29 33	9 13 16 24 39
13 18 19 33 40	12 19 27 29 32
8 16 20 27 30	8 12 18 31 39
10 14 15 29 31	14 16 26 32 37
10 18 26 30 36	15 22 25 30 35
21 31 35 37 38	17 21 25 32 36