THE PLASTIC QUASIGROUPS

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ABSTRACT. Plastic quasigroups are idempotent medial quasigroups satisfying the identity $a \cdot (a \cdot ab)b = b$. Our main result is a one-to-one correspondence with G_2-quasigroups, studied in an earlier paper. A Toyoda-like representation theorem for plastic quasigroups is proved.

1. Introduction

An *IM-quasigroup* is a solvable and cancellative groupoid $(Q, \cdot)$ satisfying the identities of idempotency and mediality:

$$aa = a, \tag{1}$$

$$ab \cdot cd = ac \cdot bd. \tag{2}$$

Immediate consequences are the identities known as *elasticity*, *left* and *right distributivity*:

$$ab \cdot a = a \cdot ba, \tag{3}$$

$$a \cdot bc = ab \cdot ac, \tag{4}$$

$$ab \cdot c = ac \cdot bc. \tag{5}$$

Definition 1.1. *A* plastic quasigroup *is an IM-quasigroup in which the following identity holds:*

$$a \cdot (a \cdot ab)b = b. \tag{6}$$

Examples can be constructed from solutions of the cubic equation $p^3 - p - 1 = 0$. Take a real or complex vector space Q and define multiplication of its elements by $a \cdot b = (1-p)a + pb$. Then, $(Q, \cdot)$ is a plastic quasigroup.

This cubic equation has a real solution $p = \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}} \approx 1.3247$ and two conjugate complex solutions. The real solution p is the smallest Pisot-Vijayaraghavan number [2]. It was considered in [4] as the second member of a sequence (φ_n) that generalises the golden

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ratio $\varphi_1 = \frac{1+\sqrt{5}}{2}$. There is also an "upper" series of generalised golden ratios (ϕ_n), related to the "lower" series by $\phi_n = \varphi_n^n$.

Quasigroups associated with the second upper golden ratio $\phi_2 = p^2$ were studied in [5]. The analogy between these, so-called G_2-quasigroups, and the golden section quasigroups defined in [10] was emphasised. The analogy extends to the use of the golden ratio in architecture. Not all of the popular beliefs about the omnipresence of the golden ratio in famous works of architecture are true, cf. [6]. However, some architects deliberately used the golden ratio in their works, most notably Le Corbusier. There was also an architect who used the number p as an ideal ratio for spatial objects and called it the *plastic number*; namely, the Dutch architect and Benedictine monk Hans van der Laan (see [1] and [8]). We decided to call quasigroups related to the number p the *plastic quasigroups*, although g_2-quasigroups would also have been appropriate.

In this paper a one-to-one correspondence between plastic quasigroups and G_2-quasigroups is established. A special version of Toyoda's representation theorem for plastic quasigroups is proved.

2. From plastic quasigroups to G_2-quasigroups

The purpose of this section is to show that a G_2-quasigroup can be defined from an arbitrary plastic quasigroup $(Q, \cdot)$.

Lemma 2.1. *An IM-quasigroup satisfies identity* (6) *if and only if it satisfies the following identity:*

$$a \cdot (a \cdot ab)c = b \cdot bc. \tag{7}$$

Proof. In a plastic quasigroup we have:

$$\begin{aligned}(a \cdot (a \cdot ab)c)(a \cdot bc) &\overset{(4)}{=} a((a \cdot ab)c \cdot bc) \overset{(5)}{=} a((a \cdot ab)b \cdot c) \overset{(4)}{=} \\ &= (a \cdot (a \cdot ab)b) \cdot ac \overset{(6)}{=} b \cdot ac \overset{(4)}{=} ba \cdot bc \overset{(5)}{=} (b \cdot bc)(a \cdot bc).\end{aligned}$$

Identity (7) is obtained by cancelling $a \cdot bc$ from the right. Conversely, identity (6) follows from (7) by substituting $c = b$. □

By putting $c = a$ in (7) and using elasticity we get

$$a \cdot (a \cdot ab)a = a(a \cdot ab) \cdot a = b \cdot ba. \tag{8}$$

Lemma 2.2. *The following identity holds in any plastic quasigroup:*

$$a \cdot a(b \cdot ba) = b(b \cdot ba). \tag{9}$$

Proof. We have:

$$(a \cdot a(b \cdot ba)) \cdot ba \overset{(2)}{=} ab \cdot (a(b \cdot ba) \cdot a) \overset{(3)}{=} ab \cdot (a \cdot (b \cdot ba)a) \overset{(4)}{=}$$
$$= a(b \cdot (b \cdot ba)a) \overset{(6)}{=} aa \overset{(1)}{=} a \overset{(6)}{=} b \cdot (b \cdot ba)a \overset{(4)}{=} b(b \cdot ba) \cdot ba.$$

Identity (9) is obtained by cancelling ba from the right. □

Proposition 2.3. *In a plastic quasigroup, the following identity holds:*

$$b(b \cdot b(b \cdot ba)) \cdot a = b. \tag{10}$$

Proof. We have:

$$(b(b \cdot b(b \cdot ba)) \cdot a) \cdot ba \overset{(5)}{=} (b(b \cdot b(b \cdot ba)) \cdot b)a \overset{(3)}{=}$$
$$= (b \cdot b(b(b \cdot ba) \cdot b))a \overset{(8)}{=} (b \cdot b(a \cdot ab))a \overset{(5)}{=}$$
$$= ba \cdot (ba \cdot (a \cdot ab)a) \overset{(2)}{=} (b \cdot ba)(a \cdot (a \cdot ab)a) \overset{(8),(1)}{=} b \cdot ba.$$

Cancelling ba from the right yields (10). □

The defining identity for plastic quasigroups (6) is, in fact, a formula for left division using multiplication:

$$a \setminus b = (a \cdot ab)b.$$

Identity (10) gives rise to a formula for right division:

$$b \,/\, a = b(b \cdot b(b \cdot ba)).$$

Now we define a new binary operation $a * b = a \cdot ab$ in a plastic quasigroup $(Q, \cdot)$. This new operation is obviously idempotent. It is also medial and mutually medial with the old operation [11, Theorem 1]:

$$(a * b) * (c * d) = (a * c) * (b * d),$$
$$(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d).$$

Identifying $c = d$ and $a = c$, we get the following versions of distributivity:

$$ab * c = (a * c)(b * c), \tag{11}$$

$$a(b * d) = ab * ad. \tag{12}$$

Here and in the sequel, it is assumed that the old binary operation $\cdot$ has precedence over the new one $*$.

Theorem 2.4. *If $(Q, \cdot)$ is a plastic quasigroup and $a * b = a \cdot ab$, then $(Q, *)$ is a G_2-quasigroup.*

Proof. Because of [5, Proposition 2.2] and elasticity, it suffices to verify the single identity

$$a*((b*a)*a)=b. \tag{13}$$

The groupoid $(G,*)$ is then necessarily cancellative and solvable and is therefore a G_2-quasigroup. Let us multiply the left-hand side by b from the left:

$$\begin{aligned}
b(a*((b*a)*a)) &= b(a\cdot a((b\cdot ba)\cdot(b\cdot ba)a)) \overset{(4)}{=}\\
&= b((a\cdot a(b\cdot ba))\cdot a(a\cdot(b\cdot ba)a)) \overset{(3)}{=}\\
&= b((a\cdot a(b\cdot ba))\cdot(a\cdot a(b\cdot ba))a) \overset{(9)}{=}\\
&= b(b(b\cdot ba)\cdot(b(b\cdot ba)\cdot a)) \overset{(2)}{=}\\
&= b((b\cdot b(b\cdot ba))\cdot(b\cdot ba)a) \overset{(4)}{=}\\
&= b(b\cdot b(b\cdot ba))\cdot(b\cdot(b\cdot ba)a) \overset{(6)}{=}\\
&= b(b\cdot b(b\cdot ba))\cdot a \overset{(10)}{=} b \overset{(1)}{=} bb.
\end{aligned}$$

Cancelling the leftmost b yields (13). ☐

Theorem 2.5. *Let $(Q,\cdot)$ be a plastic quasigroup and $a*b=a\cdot ab$. Then, $((b*a)*a)*b=ab$.*

Proof. We first show that the following identity holds:

$$(b*a)*a=b*(ab*b). \tag{14}$$

Multiplying the right-hand side in $(G,*)$ by a from the left, we get

$$\begin{aligned}
&a*(b*(ab*b)) \overset{(11)}{=} a*(b*(a*b)b)=a\cdot a(b*(a*b)b) \overset{(12)}{=}\\
&\quad = a(ab*(a\cdot(a*b)b)) \overset{(6)}{=} a(ab*b) \overset{(11)}{=} a\cdot(a*b)b \overset{(6)}{=} b.
\end{aligned}$$

Therefore, $a*((b*a)*a) \overset{(13)}{=} b = a*(b*(ab*b))$ and (14) holds because $(G,*)$ is left cancellative. Multiplying this relation in $(G,*)$ by b from the right we see that $((b*a)*a)*b=(b*(ab*b))*b \overset{(3),(13)}{=} ab$. ☐

3. From G_2-quasigroups to plastic quasigroups

Now we change the notation and denote an arbitrary G_2-quasigroup by $(Q,\cdot)$. This is an IM-quasigroup satisfying the equivalent identities

$$(ab\cdot a)a=b, \tag{15}$$

$$a(ba\cdot a)=b. \tag{16}$$

Just like in plastic quasigroups, right and left division in G_2-quasigroups can be expressed by multiplication:

$$b \,/\, a = ab \cdot a,$$
$$a \setminus b = ba \cdot a.$$

We define a new binary operation by

$$a * b = (ba \cdot a)b = (a \setminus b)b = ab \setminus b. \tag{17}$$

Our goal is to show that $(Q, *)$ is a plastic quasigroup. The new operation $*$ is obviously idempotent. From [11] we know that it is medial and mutually medial with the original operation. Hence, distributivities (11), (12) and their respective left and right counterparts hold. Later on we shall use

$$a * bc = (a * b)(a * c). \tag{18}$$

In [11] it was also proved that $\cdot$ is left and right distributive over $\setminus$ and $/$, for example

$$(a \setminus b)c = ac \setminus bc. \tag{19}$$

Unlike in the proof of Theorem 2.4, where it was sufficient to verify a single identity, we now also have to prove that $(Q, *)$ is a quasigroup. For example, the binary operation defined by $a \circ b = b$ is idempotent, medial and satisfies (6), but $(Q, \circ)$ is not a quasigroup. The operation $*$ is easily seen to be left solvable and right cancellative:

Lemma 3.1. *Let $(Q, \cdot)$ be a G_2-quasigroup and $a * b = (ba \cdot a)b$. Then, $x * a = b$ if and only if $x = (a \,/\, b) \,/\, a$.*

Proof. $x * a = b \overset{(17)}{\iff} xa \setminus a = b \iff a = xa \cdot b \iff xa = a \,/\, b \iff x = (a \,/\, b) \,/\, a$. □

The next result is similar to Theorem 2.5.

Theorem 3.2. *Let $(Q, \cdot)$ be a G_2-quasigroup and $a * b = (ba \cdot a)b$. Then, $a * (a * b) = ab$.*

Proof. By repeated application of (17) we have:

$$\begin{aligned} a * (a * b) &= a(a * b) \setminus (a * b) = (a \cdot (ba \cdot a)b) \setminus (a * b) \overset{(4)}{=} \\ &= (a(ba \cdot a) \cdot ab) \setminus (a * b) \overset{(16)}{=} (b \cdot ab) \setminus (a * b) \overset{(3)}{=} \\ &= (ba \cdot b) \setminus (ba \cdot a)b \overset{(19)}{=} (ba \setminus (ba \cdot a))b = ab. \end{aligned}$$

□

Now the operation $*$ is also seen to be left cancellative:

$$a * x = a * y \Rightarrow a * (a * x) = a * (a * y) \Rightarrow ax = ay \Rightarrow x = y.$$

Finally, we can prove

Theorem 3.3. *If $(Q, \cdot)$ is a G_2-quasigroup and $a * b = (ba \cdot a)b$, then $(Q, *)$ is a plastic quasigroup.*

Proof. We already know that $*$ is idempotent, medial, left and right cancellative and left solvable. It suffices to show that it satisfies (6), since right solvability follows directly from this identity. The previous theorem reduces (6) to $a * (ab * b) = b$, which we now prove:

$$\begin{aligned} a * (ab * b) &\overset{(17)}{=} a * ((b \cdot ab) \cdot ab)b \overset{(5)}{=} a * ((b \cdot ab)b \cdot (ab \cdot b)) \overset{(3),(15)}{=} \\ &= a * a(ab \cdot b) \overset{(18)}{=} a(a * (ab \cdot b)) \overset{(17)}{=} a \cdot ((ab \cdot b)a \cdot a)(ab \cdot b) \overset{(2)}{=} \\ &= a \cdot ((ab \cdot b)a \cdot ab)(ab) \overset{(5)}{=} a \cdot (((ab \cdot a) \cdot ba) \cdot ab)(ab) \overset{(2)}{=} \\ &= a \cdot ((ab \cdot a)a \cdot (ba \cdot b))(ab) \overset{(15)}{=} a(b(ba \cdot b) \cdot ab) \overset{(2)}{=} \\ &= a(ba \cdot (ba \cdot b)b) \overset{(15)}{=} a(ba \cdot a) \overset{(16)}{=} b. \end{aligned}$$

□

We have now established a one-to-one correspondence between plastic quasigroups and G_2-quasigroups on the same set Q. More precisely, let $\mathcal{P}$ be the set of all plastic quasigroups and $\mathcal{G}$ the set of all G_2-quasigroups, both with underlying set Q. Define $\Phi : \mathcal{P} \to \mathcal{G}$ by $\Phi(Q, \cdot) = (Q, *)$, where $a * b = a \cdot ab$. By Theorem 2.4, the mapping Φ is well defined, i.e. $(Q, *)$ is a G_2-quasigroup. Furthermore, define $\Psi : \mathcal{G} \to \mathcal{P}$ by $\Psi(Q, \cdot) = (Q, *)$, where $a * b = (ba \cdot a)b$. This mapping is well defined because of Theorem 3.3. From Theorem 2.5 it follows that $\Psi \circ \Phi = \mathbf{1}_{\mathcal{P}}$, and from Theorem 3.2 it follows that $\Phi \circ \Psi = \mathbf{1}_{\mathcal{G}}$. Using this correspondence all geometric concepts and results proved in [5] for G_2-quasigroups can be transferred to plastic quasigroups. We illustrate this for the representation theorems of [5, Section 4].

It is known that every medial quasigroup is an isotope of an Abelian group. This result is usually referred to as Toyoda's theorem [9] and is sometimes also attributed to Bruck [3] and to Murdoch [7]. A special version of this theorem for G_2-quasigroups was proved in [5]. Here is the corresponding result for plastic quasigroups.

Theorem 3.4. *A groupoid $(Q, \cdot)$ is a plastic quasigroup if and only if there is an Abelian group $(Q, +)$ with an automorphism ψ such that $\psi^3 - \psi - \mathbf{1}_Q = 0$ and $a \cdot b = a + \psi(b - a)$, $\forall a, b \in Q$.*

Proof. If we take an Abelian group $(Q,+)$ with automorphism ψ and define $a \cdot b = a + \psi(b-a)$, then $(Q,\cdot)$ is obviously an IM-quasigroup. Identity (6) follows from the property $\psi^3 - \psi - \mathbf{1}_Q = 0$. Conversely, let $(Q,\cdot)$ be an arbitrary plastic quasigroup. There is a G_2-quasigroup $(Q,*)$ such that $a * b = a \cdot ab$ and $a \cdot b = ((b*a)*a)*b$. According to [5], Theorems 4.1, 4.2 and 4.3, there is an Abelian group $(Q,+)$ with automorphism φ such that $\varphi^3 - 2\varphi^2 + \varphi - \mathbf{1}_Q = 0$ and $a * b = a + \varphi(b-a)$. By direct computation we see that $a \cdot b = ((b*a)*a)*b = b + (\varphi^3 - 3\varphi^2 + 2\varphi)(a-b) = a + (\varphi^2 - \varphi)(b-a)$. It is also easily verified that $\psi = \varphi^2 - \varphi$ is an automorphism of $(Q,+)$ satisfying $\psi^3 - \psi - \mathbf{1}_Q = 0$. □

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