Konstrukcija jedne familije asocijacijskih shema s 5 klasa

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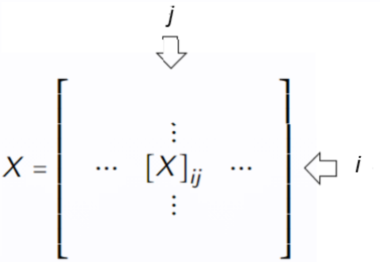
Sadržaj

Oznake

- O_n, I_n, J_n
- $$R_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$
- X blokmatrica



Digraf

Digraf je uređeni par $\Gamma = (V, E)$, $V \neq \emptyset$ konačan skup vrhova, $E \subseteq \{(x, y) | x, y \in V, x \neq y\}$ skup lukova.

Kažemo da x dominira y te da je y dominiran od x ako $(x, y) \in E$. Digraph je k-**regularan** ako svaki njegov vrh dominira i dominiran je od točno k vrhova.

Digraf je **asimetričan** ako $(x, y) \in E \Rightarrow (y, x) \notin E$.

Digraf

$\Gamma = (V, E)$, $|V| = v$.

Matrica incidencije $M = [m_{xy}]_{x,y \in V}$ digrafa Γ je $(v \times v)$ $(0,1)$-matrica kojoj su retci i stupci indeksirani vrhovima digrafa, tako da je $m_{xy} = 1$ ako $(x, y) \in E$, odnosno $m_{xy} = 0$ ako $(x, y) \notin E$.

- Digraf je k-regularan akko $MJ_v = M^T J_v = kJ_v$.
- Digraf asimetričan akko je $M + M^T$ $(0,1)$-matrica.

Digraf djeljivog dizajna

$\Gamma = (V, E)$ k-regularan asimetričan digraf, $|V| = v$.

Γ je **digraf djeljivog dizajna** s parametrima $(v, k, \lambda_1, \lambda_2, m, n)$ ako se V može particionirati u m klasa veličine n tako da:

- $\forall x, y \in V$, $x \neq y$, x i y iz iste klase, broj vrhova z koji dominiraju ili su dominirani i od x i od y je λ_1
- $\forall x, y \in V$, x i y iz različitih klasa, broj vrhova z koji dominiraju ili su dominirani i od x i od y je λ_2

Oznaka: $DDD(v, k, \lambda_1, \lambda_2, m, n)$

Digraf djeljivog dizajna

M matrica incidencije digrafa Γ.

Γ je $DDD(v, k, \lambda_1, \lambda_2, m, n)$ akko

- $M + M^T$ je $(0, 1)$-matrica
- $MM^T = M^T M = kI_v + \lambda_1 (I_m \otimes J_n - I_v) + \lambda_2 (J_v - I_m \otimes J_n)$

Balansirane generalizirane težinske matrice

G konačna multiplikativna grupa.

Balansirana generalizirana težinska matrica nad G s parametrima (v, k, λ) je matrica $W = \left[w_{ij}\right]$ s elementima iz $G \cup \{0\}$, reda v, takva da:

- svaki red od W sadrži točno k nenula elemenata,
- $\forall i, h \in \{1, 2, \ldots, v\}$, $i \neq h$, multiskup

$$\{\ w_{ij} w_{hj}^{-1} \mid 1 \leq j \leq v,\ \ w_{ij}, w_{hj} \neq 0\ \}$$

  sadrži točno $\frac{\lambda}{|G|}$ kopija svakog elementa iz G.

Oznaka: $BGW(v, k, \lambda)$

Kose balansirane generalizirane težinske matrice

G ciklička konačna multiplikativna grupa, $G = \langle U \rangle$.
U tom slučaju za elemente od W označavati ćemo sa $U^{w_{ij}}$.

Kosa balansirana generalizirana težinska matrica je matrica $W = [U^{w_{ij}}]$ koja je $BGW(n+1, n, n-1)$ nad cikličkom grupom $G = \langle U \rangle$ reda $2m$ sa dijagonalom 0 za koju vrijedi $U^{w_{ji}} = U^{w_{ij+m}}$, $\forall i \neq j$.

Primjer

Primjer

Neka je $G = \langle \sigma \rangle$ *ciklička grupa reda 3.*

$$W = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \sigma & \sigma^2 \\ 1 & 1 & 0 & \sigma^2 & \sigma \\ 1 & \sigma & \sigma^2 & 0 & 1 \\ 1 & \sigma^2 & \sigma & 1 & 0 \end{bmatrix}.$$

W *je* $BGW(5,4,3)$ *nad* G.

Kosa balansirana generalizirana težinska matrica

Lema

[1] *Neka su* $r, m \in \mathbb{N}$ *tako da je* r *neparna potencija prostog broja, te* $\frac{r-1}{m}$ *neparan broj. Tada postoji kosa* $BGW(r+1, r, r-1)$ *sa dijagonalom* 0 *nad cikličkom grupom reda* m.

[1] Yury J. Ionin, H. Kharaghani, Doubly regular digraphs and symmetric designs, J. Comb. Theory, Ser. A 101 (2003) 35-48.

Asocijacijske sheme

$d \in \mathbb{N}$.
$X \neq \emptyset$ konačan skup, $|X| = n$.
$R_i \subseteq X \times X$, $R_i \neq \emptyset$.
A_i matrica incidencije grafa (X, R_i), $i = 0, 1, \ldots, d$.

Asocijacijska shema s d klasa je uređeni par $(X, \{R_i\}_{i=0}^{d})$, gdje matrice susjedstva $A_0, A_1, \ldots, A_d$ zadovoljavaju uvijete:

- $A_0 = I_n$,
- $\sum_{i=0}^{d} A_i = J_n$,
- $A_i^T \in \{A_1, A_2, \ldots, A_d\}$, $i = 1, 2, \ldots, d$,
- $\exists p_{ij}^k \in \mathbb{N}_0$ tako da je $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k, \forall i, j$.

Hadamardove matrice

Hadamardova matrica reda n je $(n \times n)$ matrica $H = \left[h_{ij}\right]$, $h_{ij} \in \{-1, 1\}$, za koju vrijedi $HH^T = H^T H = nI_n$.

Primjer

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Red Hadamardove matrice je 1, 2 ili $4k$, za neki $k \in \mathbb{N}$.

Hadamardove matrice

Hadamardova matrica je **normalizirana** ako $h_{1j} = h_{j1} = 1, \ \forall j$.

Svaka Hadamardova matrica može se transformirati u ekvivalentnu Hadamardovu matricu koristeći slijedeće operacije:

- zamjena dva retka ili dva stupca
- množenje retka ili stupca s -1

Koristeći navedene operacije, normalizirati se može bilo koja Hadamardova matrica.

Za normaliziranu Hadamardovu matricu reda ≥ 2, svaki redak i stupac osim prvih imaju $\frac{n}{2}$ elemenata -1, te $\frac{n}{2}$ elemenata 1.

Matrice C_i

H normalizirana Hadamardova matrica reda n, $n \geq 4$.

Označimo $(i+1)$-vi redak od H sa r_i, $i = 0, 1, 2, \ldots, n-1$.

$$r_i = \begin{bmatrix} r_{i1} & r_{i2} & \ldots & r_{in} \end{bmatrix}.$$

Za $n = 1, 2, \ldots, n-1$, definiramo $(n \times n)$ matrice C_i sa

$$C_i = r_i^T \cdot r_i = \begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{in} \end{bmatrix} \cdot \begin{bmatrix} r_{i1} & r_{i2} & \ldots & r_{in} \end{bmatrix} = \begin{bmatrix} r_{i1}^2 & r_{i1} r_{i2} & \ldots & r_{i1} r_{in} \\ r_{i2} r_{i1} & r_{i2}^2 & \ldots & r_{i2} r_{in} \\ \vdots & \vdots & \ddots & \vdots \\ r_{in} r_{i1} & r_{in} r_{i2} & \ldots & r_{in}^2 \end{bmatrix}.$$

Matrice C_i

Vrijedi:

- $C_i^T = C_i, \ \forall i$

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$$(C_i C_j)_{kl} = \sum_{s=1}^{n} r_{ik} r_{is} r_{jl} r_{js} = r_{ik} r_{jl} \sum_{s=1}^{n} r_{is} r_{js}$$

Matrice C_i

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$$(C_i C_i)_{kl} = \sum_{s=1}^{n} r_{ik} r_{is} r_{il} r_{is} = r_{ik} r_{il} \sum_{s=1}^{n} r_{is}^2 = r_{ik} r_{il} \cdot n$$

Matrice C_i

Vrijedi:

- $C_i^T = C_i, \ \forall i$
- $C_i \cdot C_j = O_n \ , \ i \neq j$
- $(C_i^2)_{kl} = \begin{cases} \underbrace{r_{ik}^2}_{=1} \cdot n = n, & k = l \\ r_{ik} r_{il} \cdot n & k \neq l \end{cases}$

Blok matrica D

Neka je D $(n^2 \times n^2)$ blok matrica, $[D]_{ij} = \begin{cases} O_n, & i = j \\ C_{j-i}, & i < j \\ -C_{n-(i-j)}, & i > j. \end{cases}$

$$D = \begin{bmatrix} O_n & C_1 & C_2 & \dots & C_{n-3} & C_{n-2} & C_{n-1} \\ -C_{n-1} & O_n & C_1 & \dots & C_{n-4} & C_{n-3} & C_{n-2} \\ -C_{n-2} & -C_{n-1} & O_n & \dots & C_{n-5} & C_{n-4} & C_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ -C_3 & -C_4 & -C_5 & \dots & O_n & C_1 & C_2 \\ -C_2 & -C_3 & -C_4 & \dots & -C_{n-1} & O_n & C_1 \\ -C_1 & -C_2 & -C_3 & \dots & -C_{n-2} & -C_{n-1} & O_n \end{bmatrix}.$$

Blok matrica D

Lema

$DD^T = nI_n \otimes (nI_n - J_n).$

Blok matrica D

Lema

$DD^T = nI_n \otimes (nI_n - J_n)$.

$$\begin{bmatrix} \begin{matrix} n(n-1) & -n & \dots & -n \\ -n & n(n-1) & \dots & -n \\ \vdots & \vdots & \ddots & \vdots \\ -n & -n & \dots & n(n-1) \end{matrix} & & O_n \\ & \ddots & \\ O_n & & \begin{matrix} n(n-1) & -n & \dots & -n \\ -n & n(n-1) & \dots & -n \\ \vdots & \vdots & \ddots & \vdots \\ -n & -n & \dots & n(n-1) \end{matrix} \end{bmatrix}$$

Blok matrica D

Lema

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i-ti redak od D :

$$\begin{bmatrix} -C_{n-i+1} & -C_{n-i+2} & \dots & -C_{n-1} & O_n & C_1 & C_2 & \dots & C_{n-i} \end{bmatrix}.$$

$$\Rightarrow \left[DD^T\right]_{ii} = O_n^2 + \sum_{s=1}^{n-1} C_s^2 = \sum_{s=1}^{n-1} C_s^2.$$

Blok matrica D

Lema

$DD^T = nI_n \otimes (nI_n - J_n)$.

$$
\begin{bmatrix}
\begin{matrix} n(n-1) & -n & \dots & -n \\ -n & n(n-1) & \dots & -n \\ \vdots & \vdots & \ddots & \vdots \\ -n & -n & \dots & n(n-1) \end{matrix} & & O_n \\
& \ddots & \\
O_n & & \begin{matrix} n(n-1) & -n & \dots & -n \\ -n & n(n-1) & \dots & -n \\ \vdots & \vdots & \ddots & \vdots \\ -n & -n & \dots & n(n-1) \end{matrix}
\end{bmatrix}
$$

Blok matrica N

Neka je N $(n^2 \times n^2)$ blok matrica,

$$N = \begin{bmatrix} O_n & I_n & O_n & \cdots & O_n \\ O_n & O_n & I_n & \cdots & O_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_n & O_n & O_n & \cdots & I_n \\ -I_n & O_n & O_n & \cdots & O_n \end{bmatrix}.$$

Blok matrica N

Vrijedi:

- $NN^T = I_{n^2} \quad \Rightarrow \quad N^T = N^{-1}$
- $N^2 = \begin{bmatrix} O_n & O_n & I_n & \cdots & O_n \\ O_n & O_n & O_n & \cdots & O_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I_n & O_n & O_n & \cdots & O_n \\ O_n & -I_n & O_n & \cdots & O_n \end{bmatrix}, \ldots,$

$$\mathrm{N}^n = \begin{bmatrix} -I_n & O_n & O_n & \cdots & O_n \\ O_n & -I_n & O_n & \cdots & O_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_n & O_n & O_n & \cdots & O_n \\ O_n & -O_n & O_n & \cdots & -I_n \end{bmatrix} = -I_{n^2} \quad \Rightarrow \quad N^{2n} = I_{n^2}$$

Blok matrica R

R $(n^2 \times n^2)$ blok matrica,

$$R = R_n \otimes I_n = \begin{bmatrix} O_n & O_n & \dots & O_n & I_n \\ O_n & O_n & \dots & I_n & O_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_n & I_n & \dots & O_n & O_n \\ I_n & O_n & \dots & O_n & O_n \end{bmatrix}.$$

Vrijedi:

- $R = R^T$
- $R \cdot R^T = I_{n^2}$

kosa BGW

Neka je $p = 2qn + 1$ potencija prostog broja, $q = \frac{p-1}{2n}$ neparan cijeli broj.

Lema

Neka su $r, m \in \mathbb{N}$ tako da je r neparna potencija prostog broja, te $\frac{r-1}{m}$ neparan broj. Tada postoji kosa $BGW(r+1, r, r-1)$ sa dijagonalom 0 nad cikličkom grupom reda m.

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$\Rightarrow \quad \exists$ kosa $BGW(p+1, p, p-1)$, $W = [N^{w_{ij}}]$, s dijagonalom 0 nad $\langle N \rangle$

Blok matrica B

Neka je B $((p+1)n^2 \times (p+1)n^2)$ blok matrica tako da je $(n^2 \times n^2)$ (i,j)-ti blok od B

$$[B]_{ij} = \begin{cases} O_{n^2} & i = j \\ DN^{w_{ij}}R & i \neq j \end{cases}.$$

Vrijedi: $B^T = -B$

(i, j)-ti blok od B^T:

$$\left[B^T\right]_{ij} = [B]_{ji}^T = \left(DN^{w_{ji}}R\right)^T$$

Vrijedi: $B^T = -B$

(i,j)-ti blok od B^T:

$$\left[B^T\right]_{ij} = [B]_{ji}^T = \left(DN^{w_{ji}}R\right)^T$$
$$= \left(D\left(-N^{w_{ij}}\right)R\right)^T = R^T\left(-N^{w_{ij}}\right)^T D^T$$

- $W = \left[N^{wij}\right]$ kosa BGW nad $\langle N\rangle$ reda $2n$
  $\Rightarrow \quad N^{w_{ji}} = N^{w_{ij}+n} = N^{w_{ij}} \cdot \left(-I_{n^2}\right) = -N^{w_{ij}}$

Vrijedi: $B^T = -B$

(i,j)-ti blok od B^T:

$$\begin{aligned}\left[B^T\right]_{ij} &= [B]_{ji}^T = (DN^{w_{ji}}R)^T \\ &= (D(-N^{w_{ij}})R)^T = R^T(-N^{w_{ij}})^T D^T \\ &= -RN^{-w_{ij}}D^T.\end{aligned}$$

- $W = \left[N^{wij}\right]$ kosa BGW nad $\langle N\rangle$ reda $2n$
  $\Rightarrow \quad N^{w_{ji}} = N^{w_{ij}+n} = N^{w_{ij}} \cdot (-I_{n^2}) = -N^{w_{ij}}$
- $R^T = R,\ N^T = N^{-1} \Rightarrow \ (N^{w_{ij}})^T = N^{-w_{ij}}$

Vrijedi: $B^T = -B$

(i, j)-ti blok od B^T:

$$\begin{aligned}\left[B^T\right]_{ij} &= [B]_{ji}^T = \left(DN^{w_{ji}}R\right)^T \\ &= \left(D\left(-N^{w_{ij}}\right)R\right)^T = R^T\left(-N^{w_{ij}}\right)^T D^T \\ &= -RN^{-w_{ij}}D^T. \\ &= -N^{w_{ij}}RD^T = -N^{w_{ij}}DR = -DN^{w_{ij}}R = -\left[B\right]_{ij}\end{aligned}$$

- $W = \left[N^{wij}\right]$ kosa BGW nad $\langle N\rangle$ reda $2n$
  $\Rightarrow \quad N^{w_{ji}} = N^{w_{ij}+n} = N^{w_{ij}} \cdot \left(-I_{n^2}\right) = -N^{w_{ij}}$
- $R^T = R,\ N^T = N^{-1} \Rightarrow \ \left(N^{w_{ij}}\right)^T = N^{-w_{ij}}$
- $RN = N^{-1}R,\ RD^T = DR,\ ND = DN$

$B = A_1 - A_2$

A_1, A_2 disjunktne $(0,1)$-matrice takve da je $B = A_1 - A_2$.

Vrijedi: $A_1^T = A_2$.

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$$\begin{aligned} B = A_1 - A_2 \quad &\Rightarrow \quad B^T = A_1^T - A_2^T \\ &\Rightarrow \quad -B = A_2 - A_1 \end{aligned}$$

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$$\begin{aligned} &\Rightarrow \quad A_1^T - A_2^T = A_2 - A_1 \\ &\Rightarrow \quad A_1^T + A_1 = A_2 + A_2^T \end{aligned}$$

DDD

Teorem

Matrice A_1 i A_2 su matrice susjedstva od $DDD\left(n^2(p+1), \frac{(n^2-n)p}{2}, \frac{(n^2-2n)p}{4}, \frac{(n-1)^2(p-1)}{2}, p+1, n^2\right)$.

Γ je $DDD(v, k, \lambda_1, \lambda_2, m, n)$ akko

- $M + M^T$ je $(0,1)$-matrica
- $MM^T = M^T M = kI_v + \lambda_1 (I_m \otimes J_n - I_v) + \lambda_2 (J_v - I_m \otimes J_n)$

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Matrice A_1 i A_2 su matrice susjedstva od $DDD\left(n^2(p+1), \frac{(n^2-n)p}{2}, \frac{(n^2-2n)p}{4}, \frac{(n-1)^2(p-1)}{2}, p+1, n^2\right)$.

- $A_1 + A_1^T$ je $(0,1)$-matrica
- $$\begin{aligned} A_1A_1^T = A_1^TA_1 = &\frac{n(n-1)p}{2} I_{n^2(p+1)} + \frac{n(n-2)p}{4}\left(I_{p+1}\otimes J_{n^2} - I_{n^2(p+1)}\right) \\ &+ \frac{(n-1)^2(p-1)}{2}\left(J_{n^2(p+1)} - I_{p+1}\otimes J_{n^2}\right) \\ &= \frac{n^2p}{4} I_{p+1}\otimes I_{n^2} + \frac{n(n-2)p}{4} I_{p+1}\otimes J_{n^2} \\ &+ \frac{(n-1)^2(p-1)}{4}\left(J_{p+1} - I_{p+1}\right)\otimes J_{n^2} \end{aligned}$$

BB^T

i-ti redak od B:

$$\left[DN^{w_{i1}}R \quad DN^{w_{i2}}R \quad \ldots O_{n^2} \quad DN^{w_{i\,i+1}}R \quad \ldots DN^{w_{i\,p+1}}R\right]$$

BB^T

i-ti redak od B:

$$\begin{bmatrix} DN^{w_{i1}}R & DN^{w_{i2}}R & \dots O_{n^2} & DN^{w_{i\ i+1}}R & \dots DN^{w_{i\ p+1}}R \end{bmatrix}$$

j-ti stupac od B^T:

$$\begin{bmatrix} (DN^{w_{j1}}R)^T \\ (DN^{w_{j2}}R)^T \\ \vdots \\ O_{n^2} \\ (DN^{w_{j\ j+1}}R)^T \\ \vdots \\ (DN^{w_{j\ p+1}}R)^T \end{bmatrix}$$

BB^T

i-ti redak od B:

$$\left[DN^{w_{i1}}R \quad DN^{w_{i2}}R \quad \dots O_{n^2} \quad DN^{w_{i\ \ i+1}}R \quad \dots DN^{w_{i\ \ p+1}}R\right]$$

j-ti stupac od B^T:

$$\begin{bmatrix} \left(DN^{w_{j1}}R\right)^T \\ \left(DN^{w_{j2}}R\right)^T \\ \vdots \\ O_{n^2} \\ \left(DN^{w_{j\ \ j+1}}R\right)^T \\ \vdots \\ \left(DN^{w_{j\ \ p+1}}R\right)^T \end{bmatrix}$$

(i,j)-ti blok od BB^T:

$$\left[BB^T\right]_{ij} = \sum_{k=1}^{p+1} DN^{w_{ik}}R\left(DN^{w_{jk}}R\right)^T \cdot (1-\delta_{ik})(1-\delta_{jk})$$

BB^T

$$\left[BB^T\right]_{ij} = \sum_{k=1}^{p+1} (1-\delta_{ik})(1-\delta_{jk}) \cdot DN^{w_{ik}} R \left(DN^{w_{jk}} R\right)^T$$

$$= \sum_{k=1}^{p+1} (1-\delta_{ik})(1-\delta_{jk}) \cdot DN^{w_{ik}} \underbrace{RR^T}_{=I_{n^2}} N^{-w_{jk}} D^T$$

$$= \sum_{k=1}^{p+1} (1-\delta_{ik})(1-\delta_{jk}) DN^{w_{ik}-w_{jk}} D^T$$

BB^T

$$\left[BB^T\right]_{ij} = \sum_{k=1}^{p+1}(1-\delta_{ik})(1-\delta_{jk})DN^{w_{ik}-w_{jk}}D^T$$

- $i = j$

$$\begin{aligned}\left[BB^T\right]_{ii} &= \sum_{k=1}^{p+1}(1-\delta_{ik})^2 D\underbrace{N^{w_{ik}-w_{ik}}}_{=N^0=I_{n^2}}D^T = DD^T\sum_{k=1}^{p+1}(1-\delta_{ik})^2 \\ &= DD^T\sum_{k=1,k\neq i}^{p+1}1 = DD^T\cdot(p+1-1) = p\cdot DD^T \\ &= pnI_n\otimes(nI_n-J_n)\end{aligned}$$

BB^T

$\left[BB^T\right]_{ij} = \sum_{k=1}^{p+1}(1-\delta_{ik})(1-\delta_{jk})DN^{w_{ik}-w_{jk}}D^T$

- $i \neq j$

$$\begin{aligned}
\left[BB^T\right]_{ij} &= \sum_{k=1}^{p+1}(1-\delta_{ik})(1-\delta_{jk})DN^{w_{ik}-w_{jk}}D^T \\
&= D\left(\sum_{k=1}^{p+1}(1-\delta_{ik})(1-\delta_{jk})N^{w_{ik}-w_{jk}}\right)D^T \\
&= D\left(\sum_{k=1,k\neq i,j}^{p+1}N^{w_{ik}-w_{jk}}\right)D^T = D\left(\frac{p-1}{2n}\underbrace{\sum_{l=0}^{2n-1}N^l}_{=O_{n^2}}\right)D^T = O_{n^2}
\end{aligned}$$

- $i = j \;\Rightarrow\; \left[BB^T\right]_{ii} = pnI_n \otimes (nI_n - J_n)$
- $i \neq j \;\Rightarrow\; \left[BB^T\right]_{ij} = O_{n^2}$

Općenito: $BB^T = pnI_{p+1} \otimes I_n \otimes (nI_n - J_n)$

$$\begin{aligned} npI_{p+1} \otimes I_n \otimes (nI_n - J_n) &= (A_1 - A_2)\left(A_1^T - A_2^T\right) \\ &= A_1A_1^T - A_1A_2^T - A_2A_1^T + A_2A_2^T \end{aligned}$$

$$A_1A_1^T - A_1A_2^T - A_2A_1^T + A_2A_2^T = (A_1 - A_2)\left(A_1^T - A_2^T\right) = npI_{p+1} \otimes I_n \otimes (nI_n - J_n) \tag{1}$$

$$\begin{aligned} A_1A_1^T + A_1A_2^T + A_2A_1^T + A_2A_2^T &= (A_1 + A_2)\left(A_1^T + A_2^T\right) \\ &= npI_{p+1} \otimes (I_n \otimes J_n + (n-2)J_{n^2}) \\ &+ (n-1)^2(p-1)\,(J_{p+1} - I_{p+1}) \otimes J_{n^2}. \end{aligned} \tag{2}$$

$$A_1A_1^T - A_1A_2^T + A_2A_1^T - A_2A_2^T = (A_1 + A_2)\left(A_1^T - A_2^T\right) = O_{(p+1)n^2}. \tag{3}$$

$$A_1A_1^T + A_1A_2^T - A_2A_1^T - A_2A_2^T = (A_1 - A_2)\left(A_1^T + A_2^T\right) = O_{(p+1)n^2}. \tag{4}$$

(1)+(2)+(3)+(4):

$$
\begin{aligned}
4A_1A_1^T &= npI_{p+1} \otimes I_n \otimes (nI_n - J_n) + npI_{p+1} \otimes (I_n \otimes J_n + (n-2)J_{n^2}) \\
&\quad + (n-1)^2(p-1)(J_{p+1} - I_{p+1}) \otimes J_{n^2} \\
&= npI_{p+1} \otimes \underbrace{I_n \otimes nI_n}_{nI_{n^2}} - npI_{p+1} \otimes I_n \otimes J_n + npI_{p+1} \otimes I_n \otimes J_n \\
&\quad + npI_{p+1} \otimes (n-2)J_{n^2} + (n-1)^2(p-1)(J_{p+1} - I_{p+1}) \otimes J_{n^2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad A_1A_1^T &= \frac{n^2p}{4} I_{p+1} \otimes I_{n^2} + \frac{n(n-2)p}{4} I_{p+1} \otimes J_{n^2} \\
&\quad + \frac{(n-1)^2(p-1)}{4}(J_{p+1} - I_{p+1}) \otimes J_{n^2}
\end{aligned}
$$

- (1)+(2)-(3)-(4):

$$A_2A_2^T = \frac{n^2p}{4}I_{p+1}\otimes I_{n^2} + \frac{n(n-2)p}{4}I_{p+1}\otimes J_{n^2} + \frac{(n-1)^2(p-1)}{4}\left(J_{p+1}-I_{p+1}\right)\otimes J_{n^2}$$

- -(1)+(2)-(3)+(4):

$$A_1A_2^T = \frac{-n^2p}{4}I_{p+1}\otimes I_{n^2} + \frac{np}{2}I_{p+1}\otimes I_n\otimes J_n + \frac{n(n-2)p}{4}I_{p+1}\otimes J_{n^2} + \frac{(n-1)^2(p-1)}{4}\left(J_{p+1}-I_{p+1}\right)\otimes J_{n^2}$$

- -(1)+(2)+(3)-(4):

$$A_2A_1^T = \frac{-n^2p}{4}I_{p+1}\otimes I_{n^2} + \frac{np}{2}I_{p+1}\otimes I_n\otimes J_n + \frac{n(n-2)p}{4}I_{p+1}\otimes J_{n^2} + \frac{(n-1)^2(p-1)}{4}\left(J_{p+1}-I_{p+1}\right)\otimes J_{n^2}$$

AS s 5 klasa

$$A_0 = I_{n^2(p+1)}$$
$$A_3 = (J_{p+1} - I_{p+1}) \otimes J_{n^2} - A_1 - A_2$$
$$A_4 = I_{p+1} \otimes I_n \otimes (J_n - I_n)$$
$$A_5 = I_{p+1} \otimes (J_n - I_n) \otimes J_n$$

Teorem

Skup matrica $\{A_0, A_1, A_2, A_3, A_4, A_5\}$ je komutativna asocijacijska shema s 5 klasa.

$$\mathcal{A} = span_{\mathbb{C}}\{A_0, A_1, A_2, A_3, A_4, A_5\}$$

$$A_3 = J_{p+1} \otimes J_{n^2} - I_{p+1} \otimes J_{n^2} - A_1 - A_2 = J_{n^2(p+1)} - I_{p+1} J_{n^2} - A_1 - A_2,$$
$$A_4 = I_{p+1} \otimes I_n \otimes J_n - I_{p+1} \otimes I_n \otimes I_n = I_{p+1} \otimes I_n \otimes J_n - A_0,$$
$$A_5 = I_{p+1} \otimes J_n \otimes J_n - I_{p+1} \otimes I_n \otimes J_n = I_{p+1} \otimes J_{n^2} - I_{p+1} \otimes I_n \otimes J_n.$$

$$\Rightarrow \quad A_3 + A_4 + A_5 = J_{n^2(p+1)} - A_1 - A_2 - A_0$$
$$\Rightarrow \quad A_0 + A_1 + A_2 + A_3 + A_4 + A_5 = J_{n^2(p+1)}$$

$$A_1^T = A_2 \in \mathcal{A}$$
$$A_2^T = A_1 \in \mathcal{A}$$

$$\begin{aligned} A_3^T &= \left((J_{p+1} - I_{p+1}) \otimes J_{n^2} - A_1 - A_2 \right)^T \\ &= \left((J_{p+1} - I_{p+1}) \otimes J_{n^2} \right)^T - A_1^T - A_2^T \\ &= (J_{p+1} - I_{p+1}) \otimes J_{n^2} - A_2 - A_1 = A_3 \in \mathcal{A}, \end{aligned}$$
$$A_4^T = A_4 \in \mathcal{A}$$
$$A_5^T = A_5 \in \mathcal{A}$$

$$A_1A_2 = A_2A_1 = \frac{n^2p}{4}A_0 + \frac{n(n-2)p}{4}(A_5 + A_4 + A_0) + \frac{(n-1)^2(p-1)}{4}(A_3 + A_1 + A_2) \in \mathcal{A},$$

$$A_1A_1 = A_2A_2 = -\frac{n^2p}{4}A_0 + \frac{np}{2}(A_4 + A_0) + \frac{n(n-2)p}{4}(A_5 + A_4 + A_0) + \frac{(n-1)^2(p-1)}{4}(A_3 + A_1 + A_2) \in \mathcal{A}.$$

$$A_1 A_4 = A_1 \left(I_{p+1} \otimes I_n \otimes J_n \right) - A_1 = \frac{n}{2} \left(A_1 + A_2 \right) - A_1 \in \mathcal{A}$$

$$\Rightarrow \quad A_4 A_2 = \left(A_1 A_4 \right)^T = \frac{n}{2} \left(A_1 + A_2 \right)^T - A_1^T = \frac{n}{2} \left(A_2 + A_1 \right) - A_2 \in \mathcal{A}$$

$$A_4 A_5 = (n-1) \left[I_{p+1} \otimes \left(J_n - I_n \right) \otimes J_n \right] = (n-1) A_5 \in \mathcal{A}$$

$$\Rightarrow \quad A_5 A_4 = A_5^T A_4^T = \left(A_4 A_5 \right)^T = \left((n-1) A_5 \right)^T = (n-1) A_5^T = (n-1) A_5 \in \mathcal{A}$$

Hvala na pažnji! 🙂