Meshfree Adaptative Aitken-Schwarz
Domain Decomposition for Darcy flow

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Dedicated to Alain Bourgeat’s 60th birthday
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Partially founded by: GDR MOMAS, ANR-TL-07 LIBRAERO, ANR-CIS-07 MICAS
Objectives: make a Schwarz DDM that has:

- scalable properties
- Artificial condition independent of the parameter (even make convergent a divergent Schwarz method)
- can be used as "black box", no direct impact on the implementation of local solver.
1. The Dirichlet-Neumann Map
2. The Generalized Schwarz Alternating Method
3. The Aitken-Schwarz Method
4. Non separable operator, non regular mesh, adaptive Aitken-Schwarz
5. Aitken meshfree acceleration
Let $\Omega \subset \mathbb{R}^n$ a bounded domain with $\Gamma := \partial \Omega$ Lipschitz.

**The trace operator : $\gamma_0$**

$\forall u \in H^1(\Omega), \ \exists \gamma_0 u \in H^{1/2}(\Gamma)$ satisfying

$$\|\gamma_0 u\|_{H^{1/2}(\Gamma)} \leq c_T \|u\|_{H^1(\Omega)}. \ (1)$$

vice versa the bounded extension operator : $\varepsilon$

$\forall v \in H^{1/2}(\Gamma), \ \exists \varepsilon v \in H^1(\Omega)$ satisfying $\gamma_0 \varepsilon v = v$ and

$$\|\varepsilon v\|_{H^1(\Omega)} \leq c_{IT} \|v\|_{H^{1/2}(\Gamma)}. \ (2)$$
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Set \( L(x)u(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} [a_{ji}(x) \frac{\partial}{\partial x_i} u(x)], \quad a_{ji} \in L_{\infty}(\Omega) \)

\( L(.) \) is assumed to be uniformly elliptic,

\[
\sum_{i,j=1}^{n} a_{ji}(x) \xi_j \xi_i \geq c_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega
\]

The conormal derivative \( \gamma_1 \) is given by

\[
\gamma_1 u(x) := \sum_{i,j=1}^{n} n_j(x) [a_{ji}(x) \frac{\partial}{\partial x_i} u(x)], \quad \forall x \in \Gamma
\]

where \( n(x) \) is the exterior unit normal vector.

\[
a(u, v) = \sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial}{\partial x_j} v(x) a_{ji}(x) \frac{\partial}{\partial x_i} u(x)
\]

\[
= \int_{\Omega} Lu(x)v(x)dx + \int_{\Gamma} \gamma_1 u(x) \gamma_0 v(x) dS_x
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\[ = \int_{\Omega} Lu(x)v(x)dx + \int_{\Gamma} \gamma_1 u(x)v(x)dS_x \]
Necas Lem. ⇒ \exists! u = u_0 + \varepsilon g \in H^1(\Omega) sol. of Dirichlet Pb

\[ L(x)u(x) = f(x), \text{ for } x \in \Omega, \gamma_0 u(x) = g(x), \text{ for } x \in \Gamma (4) \]

Then defining the linear application \( \forall w \in H^{1/2}(\Gamma) \)

\[ l(w) = a(u, \varepsilon w) - \int_\Omega f(x)\varepsilon w(x)dx. \]

Riez thm : \( \exists \lambda \in H^{-1/2}(\Gamma) : \langle \lambda, w \rangle_{L^2(\Gamma)} = l(w) \ \forall w \in H^{1/2}(\Gamma). \)

Hence, the conormal derivative \( \lambda \in H^{-1/2}(\Gamma) \) satisfies

\[ \int_\Gamma \lambda w \ ds_x = a(u_0 + \varepsilon g, \varepsilon w) - \int_\Omega f \varepsilon w \ dx \ \forall w \in H^{1/2}(\Gamma). \]

⇒ \( f \) fixed, we have a DtoN map : \( g = \gamma_0 u \mapsto \lambda := \gamma_1 u \)

\[ \gamma_1 u(x) = Sg(x) - Nf(x), \ \forall w \in \Gamma \]
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$$L(x)u(x) = f(x), \text{ for } x \in \Omega, \gamma_0 u(x) = g(x), \text{ for } x \in \Gamma$$ (4)

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Hence, the conormal derivative $\lambda \in H^{-1/2}(\Gamma)$ satisfies

$$\int_{\Gamma} \lambda w ds_x = a(u_0 + \varepsilon g, \varepsilon w) - \int_{\Omega} f \varepsilon w dx \forall w \in H^{1/2}(\Gamma).$$

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The Generalized Schwarz Alternating Method (GSAM)


Consider $\Omega = \Omega_1 \cup \Omega_2$ with the two artificial boundaries $\Gamma_1, \Gamma_2$ intersecting $\partial \Omega$.

**Algorithm**

\[
L(x)u^{2n+1}_1(x) = f(x), \quad \forall x \in \Omega_1, \quad u^{2n+1}_1(x) = g(x), \quad \forall x \in \partial \Omega_1 \setminus \Gamma_1,
\]

\[
\Lambda_1 u^{2n+1}_1 + \lambda_1 \frac{\partial u^{2n+1}_1(x)}{\partial n_1} = \Lambda_1 u^{2n}_2 + \lambda_1 \frac{\partial u^{2n}_2(x)}{\partial n_1}, \quad \forall x \in \Gamma_1,
\]

\[
L(x)u^{2n+2}_2(x) = f(x), \quad \forall x \in \Omega_2, \quad u^{2n+2}_2(x) = g(x), \quad \forall x \in \partial \Omega_2 \setminus \Gamma_2,
\]

\[
\Lambda_2 u^{2n+2}_2 + \lambda_2 \frac{\partial u^{2n+2}_2(x)}{\partial n_2} = \Lambda_2 u^{2n+1}_1 + \lambda_2 \frac{\partial u^{2n+1}_1(x)}{\partial n_2}, \quad \forall x \in \Gamma_2.
\]

where $\Lambda_i$'s are some operators and $\lambda_i$'s are constants. 

($\Lambda_1 = I, \lambda_1 = 0, \Lambda_2 = 0, \lambda_2 = 1$) Schwarz Neumann-Dirichlet Algorithm
If $\lambda_1 = 1$ and $\Lambda_1$ is the DtoN operator at $\Gamma_1$ associated to the homogeneous PDE in $\Omega_2$ with homogeneous boundary condition on $\partial \Omega_2 \cap \partial \Omega$ then GSAM converge in two steps.

**Proof** Let $e^n_i = u - u^n$, $i = 1, 2$, then

\[
L(x)e_1^1(x) = 0, \ \forall x \in \Omega_1, \ e_1^1(x) = 0, \ \forall x \in \partial \Omega_1 \setminus \Gamma_1,
\]

\[
\Lambda_1 e_1^1 + \left. \frac{\partial e_1^1(x)}{\partial n_1} \right|_{\partial \Omega_2} = \Lambda_1 e_2^0 + \left. \frac{\partial e_2^0(x)}{\partial n_1} \right|_{\partial \Omega_2}, \ \forall x \in \Gamma_1
\]

since $\Lambda_1$ is the DtoN operator at $\Gamma_1$ in $\Omega_2$

\[
\left. \frac{\partial e_2^0}{\partial n_1} \right|_{\partial \Omega_2} + \Lambda_1 e_2^0 = - \left. \frac{\partial e_2^0}{\partial n_2} \right|_{\partial \Omega_2} + \left. \frac{\partial e_2^0}{\partial n_2} \right|_{\partial \Omega_2} = 0, \ \Rightarrow e_1^1 = 0 \text{in } \Omega_1
\]

Hence we get the exact solution in two steps.
Pb : $\Lambda_i$ DtoN operators are global operators (linking all the subdomains when $> 3$).

In practice, the algebraical approximations of this operators are used (see Nataf, Gander).

On the other hand, the convergence property of the Schwarz Alternating methodology is used to define the Aitken-Schwarz methodology.
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In practice, the algebraical approximations of this operators are used (see Nataf, Gander).

On the other hand, the convergence property of the Schwarz Alternating methodology is used to define the Aitken-Schwarz methodology.
Let $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_{12} = \Omega_1 \cap \Omega_2$, $\Omega_{ii} = \Omega_i \setminus \Omega_{12}$

$e_i^n = u - u_i^n$ in $\Omega_i$ satisfies:

$$(\Lambda_1 + \lambda_1 S_1) R_1 e_1^{2n+1} = (\Lambda_1 - \lambda_1 S_{22}) R_{22} P_2 e_2^{2n}$$
$$(\Lambda_2 + \lambda_2 S_2) R_2 e_2^{2n+2} = (\Lambda_2 - \lambda_2 S_{22}) R_{11} P_1 e_1^{2n+1}$$

with

- $P_i : H^1(\Omega_i) \rightarrow H^1(\Omega_{ii})$
- $S_i (S_{ii})$ the DtoN map operator in $\Omega_i$ ($\Omega_{ii}$) on $\Gamma_i$ ($\Gamma_{mod(i,2)+1}$).
- $R_i : H^1(\Omega_i) \rightarrow H^{1/2}(\Gamma_i)$, $R_{ii} : H^1(\Omega_{ii}) \rightarrow H^{1/2}(\Gamma_{mod(i,2)+1})$,
- $R_i^* : R_i R_i^* = I$,
- $\forall g \in H^{1/2}(\Gamma_i)$, $L(x) R_i^* g = 0$, $R_i^* g = g$ on $\Gamma_i$, $R_i^* g = 0$ on $\partial \Omega_i \setminus \Gamma_i$

Thus the convergence of GSAM is purely linear! Aitken-Schwarz DDM uses this property to accelerate the convergence:
Consequently, no direct approximation of the DtoN map is used, but an approximation of the operator of error linked to this DtoN map is performed.
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Acceleration of Schwarz Method for Elliptic Problems

M. Garbey and D. Tromeur-Dervout: *On some Aitken like acceleration of the Schwarz method*,


- **additive Schwarz algorithm:**
  - \( L[u_1^{n+1}] = f \) in \( \Omega_1 \), \( u_1^{n+1}|_{\Gamma_1} = u_2^n|\Gamma_1 \),
  - \( L[u_2^{n+1}] = f \) in \( \Omega_2 \), \( u_2^{n+1}|_{\Gamma_2} = u_1^n|\Gamma_2 \).

- **the interface error operator** \( T \) is **linear**, i.e.
  - \( u_1^{n+1} - U|\Gamma_2 = \delta_1 (u_2^n|\Gamma_1 - U|\Gamma_1) \),
  - \( u_2^{n+1} - U|\Gamma_1 = \delta_2 (u_1^n|\Gamma_2 - U|\Gamma_2) \).

- Consequently
  - \( u_1^2|\Gamma_2 - u_1^1|\Gamma_2 = \delta_1 (u_2^1|\Gamma_1 - u_2^0|\Gamma_1) \),
  - \( u_2^2|\Gamma_1 - u_2^1|\Gamma_1 = \delta_2 (u_1^1|\Gamma_2 - u_1^0|\Gamma_2) \).

- **Computation of** \( \delta_{1/2} \):
  - \( L[v_{1/2}] = 0 \) in \( \Omega_{1/2} \), \( v_{\Gamma_{1/2}} = 1 \). Thus \( \delta_{1/2} = v_{\Gamma_{2/1}} \).

- **iff** \( \delta \neq 1 \) **Aitken-Schwarz** gives the solution with exactly 3 iterations and possibly 2 in the analytical case.
Acceleration of Schwarz Method for Elliptic Problems


- **additive Schwarz** algorithm:
  
  - \( L[u_1^{n+1}] = f \text{ in } \Omega_1, \quad u_1^{n+1}|_{\Gamma_1} = u_2^n|_{\Gamma_1}, \)
  
  - \( L[u_2^{n+1}] = f \text{ in } \Omega_2, \quad u_2^{n+1}|_{\Gamma_2} = u_1^n|_{\Gamma_2}. \)

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- **iff** \( \delta \neq 1 \) **Aitken-Schwarz** gives the solution with **exactly 3** iterations and possibly **2** in the analytical case.
Example on a toy problem: Darcy-Stokes coupling

\[
\begin{align*}
\begin{cases}
-\nabla \cdot T(u_1, p_1) &= f_1, \text{ in } \Omega_1 \\
\nabla \cdot u_1 &= 0, \text{ in } \Omega_1 \\
T &:= -p_1 I + 2\mu D(u_1), \\
D(u_1) &:= \frac{1}{2} \nabla u_1 + \frac{1}{2} \nabla u_1^T \\
\mu u_2 + K^2 \nabla p_2 &= 0, \text{ in } \Omega_2 \\
\nabla \cdot u_2 &= f_2, \text{ in } \Omega_2
\end{cases}
\end{align*}
\]

\[
B.C. : \quad u_1 = 0, \text{ on } \partial \Omega_1 \setminus \Gamma, \quad p_2 = 0 \text{ on } \partial \Omega_2 \setminus \Gamma
\]

Beavers-Joseph-Saffman boundary condition on \(\Gamma\)

\[
- n_1 \cdot T(u_1, p_1) \cdot \tau_1 = \frac{\alpha}{K} u_1 \cdot \tau_1, \text{ on } \Gamma
\]

Transmission conditions to close the system:

\[
\begin{align*}
u_1 \cdot n_1 &= u_2 \cdot n_1, \text{ on } \Gamma \\
- n_1 \cdot T(u_1, p_1) \cdot n_1 &= p_2, \text{ on } \Gamma.
\end{align*}
\]
Example on a toy problem: Darcy-Stokes coupling

\[ u_{i1}(x, y) = \sum \hat{u}_{i1,k}(x) \cos(ky), \]
\[ u_{i2}(x, y) = \sum \hat{u}_{i2,k}(x) \cos(ky), \]
\[ p_i(x, y) = \sum \hat{p}_{i,k}(x) \sin(ky). \]

Schwarz errors \( e_{i1}, e_{i2}, e_{ip} \) for each mode \( k \) in \( \Omega_i \) satisfy:

\[
\begin{align*}
\mu \frac{\partial^2}{\partial x^2} e_{11}^n(x) - \mu k^2 e_{11}^n(x) - \frac{\partial}{\partial x} e_{1p}^n(x) &= 0, \forall x \in ]0, \gamma[, \\
\mu \frac{\partial^2}{\partial x^2} e_{12}^n(x) - \mu k^2 e_{12}^n(x) - ke_{1p}^n(x) &= 0, \forall x \in ]0, \gamma[, \\
\frac{\partial}{\partial x} e_{11}^n(x) - ke_{12}^n(x) &= 0, \forall x \in ]0, \gamma[ \\
\mu ke_{11}^n(\gamma) - mu \frac{\partial}{\partial x} e_{12}^n(\gamma) - \frac{\alpha}{K} e_{12}^n(\gamma) &= 0 \\
e_{11}^n(0) &= e_{12}^n(0) = 0 \\
e_{1p}^n(\gamma) - 2\mu \frac{\partial}{\partial x} e_{11}^n(\gamma) &= \eta^n = e_{2p}^{n-1/2}(\gamma) \\
\frac{\partial}{\partial x} e_{21}^{n+1/2}(x) - ke_{22}^{n+1/2}(x) &= 0, \forall x \in ]\gamma, 1[ \\
e_{2p}^{n+1/2}(1) &= 0 \\
e_{21}^{n+1/2}(\gamma) &= x \xi^{n+1/2} = e_{11}^n(\gamma)
\end{align*}
\]
\[
\begin{pmatrix}
\eta^{n+1} \\
\xi^{n+1/2}
\end{pmatrix} = \begin{pmatrix}
0 & \rho_1 \\
\rho_2 & 0
\end{pmatrix} \begin{pmatrix}
\eta^n \\
\xi^{n-1/2}
\end{pmatrix}
\]

\[
\rho_1 = \frac{\mu \tanh(k(1 - \gamma))}{kK^2}
\]

\[
\rho_2 = \frac{-4\alpha \sinh(k\gamma) + 2\mu kK(e^{-2k\gamma} - e^{2k\gamma} + 4k\gamma) + 4k^2\gamma^2\alpha}{-2k\alpha(e^{-2k\gamma} - 2 + e^{2k\gamma}) \mu}
\]

- convergence (eventually divergence) depends on parameters value but not of the iteration and not of the solution.
- each mode can be accelerated by the Aitken process
- even with \(\rho_1 \rho_2\) very closed to 1.

\[
\begin{array}{c}
\rho_1 \rho_2
\end{array}
\]

with \(\alpha = 100, K^2 = 0.01, \mu = 1, \gamma = 0.5\).
Example of linear convergence for the Schwarz Neumann-Dirichlet algo.

\[ [\alpha, \Gamma_1] \cup [\Gamma_1, \Gamma_2] \cup [\Gamma_2, \beta], \Gamma_1 < \Gamma_2. \text{ Schwarz writes:} \]

\[
\begin{align*}
\Delta u_1^{(j)} &= f \text{ on } [\alpha, \Gamma_1] \\
 u_1^{(j)}(\alpha) &= 0 \\
 u_1^{(j)}(\Gamma_1) &= u_1^{(j-\frac{1}{2})}(\Gamma_2)
\end{align*}
, \quad
\begin{align*}
\Delta u_2^{(j+\frac{1}{2})} &= f \text{ on } [\Gamma_1, \Gamma_2] \\
 \frac{\partial u_2^{(j+\frac{1}{2})}(\Gamma_1)}{\partial n} &= \frac{\partial u_1^{(j)}(\Gamma_1)}{\partial n} \\
 u_2^{(j+\frac{1}{2})}(\Gamma_2) &= u_3^{(j)}(\Gamma_2)
\end{align*}
\]

\[
\begin{align*}
\Delta u_3^{(j)} &= f \text{ on } [\Gamma_2, \beta] \\
 \frac{\partial u_3^{(j)}(\Gamma_2)}{\partial n} &= \frac{\partial u_2^{(j-\frac{1}{2})}(\Gamma_2)}{\partial n} \\
 u_3^{(j)}(\beta) &= 0
\end{align*}
\]

The error on subdomain \( i \) writes \( e_i(x) = c_i x + d_i \).

\[
e_1^{(j)}(x) = e_2^{(j-\frac{1}{2})}(\Gamma_1) \frac{(\alpha - x)}{\alpha - \Gamma_1}, \quad e_3^{(j)}(x) = \frac{\partial}{\partial n} e_2^{(j-\frac{1}{2})}(\Gamma_2)(x - \beta)
\]
Num. analysis for the Neumann-Dirichlet algo. (3 subdomains)

Error on the second subdomain satisfies

$$e_{2}^{(j+\frac{1}{2})}(x) = \frac{\partial}{\partial n}e_{1}^{(j)}(\Gamma_{1})(x - \Gamma_{2}) + e_{3}^{(j)}(\Gamma_{2})$$  (8)

Replacing $e_{3}^{(j)}(\Gamma_{2})$ and $\frac{\partial}{\partial n}e_{1}^{(j)}(\Gamma_{1})$, $e_{2}^{(j+\frac{1}{2})}(x)$ writes :

$$e_{2}^{(j+\frac{1}{2})}(x) = -\frac{x - \Gamma_{2}}{\alpha - \Gamma_{1}}e_{2}^{(j-\frac{1}{2})}(\Gamma_{1}) + (\Gamma_{2} - \beta)\frac{\partial}{\partial n}e_{2}^{(j-\frac{1}{2})}(\Gamma_{2})$$  (9)

Consequently, the following identity holds :

$$\begin{pmatrix}
e_{2}^{(j)}(\Gamma_{1}) \\
\frac{\partial}{\partial n}e_{2}^{(j)}(\Gamma_{2})
\end{pmatrix} = \begin{pmatrix}
\Gamma_{2} - \Gamma_{1} \\
\alpha - \Gamma_{1}
\end{pmatrix} \begin{pmatrix}
\Gamma_{2} - \beta \\
-1
\end{pmatrix} \begin{pmatrix}
e_{2}^{(j-1)}(\Gamma_{1}) \\
\frac{\partial}{\partial n}e_{2}^{(j-1)}(\Gamma_{2})
\end{pmatrix}$$  (10)

Consequently the matrix do not depends of the solution, neither of the iteration, but only of the operator and the shape of the domain.
Num. analysis for the Neumann-Dirichlet algo. (3 subdomains)

Cvg for 1D Poisson pb with 3 non-overlapping subdomains $\alpha = 0, \beta = 1, \Gamma_1 = 0.44, \Gamma_2 = 0.7$
Aitken acceleration of convergence in n-D

\[ \vec{x}_{i+1} - \vec{\xi} = P(\vec{x}_i - \vec{\xi}), \quad i = 0, 1, \ldots \]

\[
\begin{pmatrix}
\vec{x}_{N+1} - \vec{x}_N \\
\vdots \\
\vec{x}_2 - \vec{x}_1
\end{pmatrix}
= P \begin{pmatrix}
\vec{x}_N - \vec{x}_{N-1} \\
\vdots \\
\vec{x}_1 - \vec{x}_0
\end{pmatrix}
\]

Thus if \(( \vec{x}_N - \vec{x}_{N-1} \ldots \vec{x}_1 - \vec{x}_0 )\) is non singular then \(P = \begin{pmatrix}
\vec{x}_{N+1} - \vec{x}_N \\
\vdots \\
\vec{x}_2 - \vec{x}_1
\end{pmatrix} ( \vec{x}_N - \vec{x}_{N-1} \ldots \vec{x}_1 - \vec{x}_0 )^{-1}\)

If \(\|P\| < 1\) then \(\vec{\xi} = (Id - P)^{-1}(\vec{x}_{N+1} - P\vec{x}_N)\)

The construction of \(P\) claims at least \(N + 1\) iterates if the error components are linked together. \(\Rightarrow\)

- write the solution in a functional basis were the components error are decoupled
- Construct an approximation of \(P\)
The Aitken-Schwarz Adaptive Aitken meshfree acceleration of convergence in n-D

\[ \vec{x}_{i+1} - \vec{\xi} = P(\vec{x}_i - \vec{\xi}), \quad i = 0, 1, \ldots \]

\[ \begin{pmatrix} \vec{x}_{N+1} - \vec{x}_N \\ \vdots \\ \vec{x}_2 - \vec{x}_1 \end{pmatrix} = P \begin{pmatrix} \vec{x}_N - \vec{x}_{N-1} \\ \vdots \\ \vec{x}_1 - \vec{x}_0 \end{pmatrix} \]

Thus if \( \begin{pmatrix} \vec{x}_N - \vec{x}_{N-1} \\ \vdots \\ \vec{x}_1 - \vec{x}_0 \end{pmatrix} \) is non singular then

\[ P = \begin{pmatrix} \vec{x}_{N+1} - \vec{x}_N \\ \vdots \\ \vec{x}_2 - \vec{x}_1 \end{pmatrix} (\begin{pmatrix} \vec{x}_N - \vec{x}_{N-1} \\ \vdots \\ \vec{x}_1 - \vec{x}_0 \end{pmatrix})^{-1} \]

If \( ||P|| < 1 \) then

\[ \vec{\xi} = (Id - P)^{-1}(\vec{x}_{N+1} - P\vec{x}_N) \]

The construction of \( P \) claims at least \( N + 1 \) iterates if the error components are linked together. \( \Rightarrow \)

- write the solution in a functional basis were the components error are decoupled
- Construct an approximation of \( P \)
Aitken acceleration of convergence in n-D

\[ \vec{x}_{i+1} - \vec{\xi} = P(\vec{x}_i - \vec{\xi}), \ i = 0, 1, \ldots \]

\[
\begin{pmatrix}
\vec{x}_{N+1} - \vec{x}_N \\
\vdots \\
\vec{x}_2 - \vec{x}_1
\end{pmatrix}
= \begin{pmatrix}
P(\vec{x}_N - \vec{x}_{N-1}) \\
\vdots \\
P(\vec{x}_1 - \vec{x}_0)
\end{pmatrix}
\]

Thus if \((\vec{x}_N - \vec{x}_{N-1}) \ldots (\vec{x}_1 - \vec{x}_0)\) is non singular then

\[ P = \begin{pmatrix}
\vec{x}_{N+1} - \vec{x}_N \\
\vdots \\
\vec{x}_2 - \vec{x}_1
\end{pmatrix}
(\vec{x}_N - \vec{x}_{N-1}) \ldots (\vec{x}_1 - \vec{x}_0)^{-1}
\]

If \(\|P\| < 1\) then \(\vec{\xi} = (Id - P)^{-1}(\vec{x}_{N+1} - P\vec{x}_N)\)

The construction of \(P\) claims at least \(N + 1\) iterates if the error components are linked together. ⇒

- write the solution in a functional basis were the components error are decoupled
- Construct an approximation of \(P\)
Aitken acceleration of convergence in n-D

\[ \vec{x}_{i+1} - \vec{\xi} = P(\vec{x}_i - \vec{\xi}), \quad i = 0, 1, \ldots \]

\[ ( \vec{x}_{N+1} - \vec{x}_N \quad \ldots \quad \vec{x}_2 - \vec{x}_1 ) = P( \vec{x}_N - \vec{x}_{N-1} \quad \ldots \quad \vec{x}_1 - \vec{x}_0 ) \]

Thus if \(( \vec{x}_N - \vec{x}_{N-1} \quad \ldots \quad \vec{x}_1 - \vec{x}_0 )\) is non singular then \(P = \frac{( \vec{x}_{N+1} - \vec{x}_N \quad \ldots \quad \vec{x}_2 - \vec{x}_1 )}{( \vec{x}_N - \vec{x}_{N-1} \quad \ldots \quad \vec{x}_1 - \vec{x}_0 )^{-1}}\)

If \(\|P\| < 1\) then \(\vec{\xi} = (Id - P)^{-1}(\vec{x}_{N+1} - P\vec{x}_N)\)

The construction of \(P\) claims at least \(N + 1\) iterates if the error components are linked together. \(\Rightarrow\)

- write the solution in a functional basis were the components error are decoupled
- Construct an approximation of \(P\)
For GSAM with two subdomains, errors $e_{\Gamma h}^{i} = U_{\Gamma h}^{i+1} - U_{\Gamma h}^{i}$ satisfy

$$
\begin{pmatrix}
  e_{\Gamma_1 h}^{i+1} \\
  e_{\Gamma_2 h}^{i+1}
\end{pmatrix}
= P
\begin{pmatrix}
  e_{\Gamma_1 h}^{i} \\
  e_{\Gamma_2 h}^{i}
\end{pmatrix}
$$

(11)

- $\Gamma_h^j$ a discretisation of the interfaces
- $\Gamma_h$ to be the coarsest discretisation in the sense that it produces $V$ the smallest set of orthonormal vectors $\Phi_k$ that belong to $\Gamma_h$ with respect to a discrete hermitian form $[[\cdot, \cdot]]$.

- Let $U_{\Gamma h}$ be the decomposition of $U_{\Gamma}$ with respect to the orthogonal basis $V$.
  
  $U_{\Gamma h} = \sum_{k=0}^{N} \alpha_k \Phi_k$

- The $\alpha_k$ represents the "Fourier" coefficients of the solution with respect to the basis $V$. 
  
  The orthogonality $\Rightarrow \alpha_k = [[U_{\Gamma}, \Phi_k]]$

- Then

$$
\begin{pmatrix}
  \beta_{\Gamma_1 h}^{i+1} \\
  \beta_{\Gamma_2 h}^{i+1}
\end{pmatrix}
= P[[\cdot]]
\begin{pmatrix}
  \beta_{\Gamma_1 h}^{i} \\
  \beta_{\Gamma_2 h}^{i}
\end{pmatrix}
$$

(12)
For a separable operator in 2D or 3D and regular step size mesh

No coupling between the modes thus the operator P for the speed up is a block diagonal matrix and n-D is analogous to the 1-D

1. for Schwarz each wave has its own linear rate of convergence and high frequencies are damped first.

2. for high modes the matrix \( P \) can be approximate with neglecting far Macro-Domains interactions.

- **step1**: build \( P \) analytically or numerically from data given by two Schwarz iterates
- **step2**: apply one Jacobi Schwarz iterate to the differential problem with block solver of choice i.e. multigrids, FFT etc...

- **step3**: exchange boundary information:
step 4: compute the Fourier expansion $\hat{u}^n_{j|\Gamma_i}, n = 0, 1$ of the traces on the artificial interface $\Gamma_i, i = 1..nd$ for the initial boundary condition $u^0_{|\Gamma_i}$ and the Schwarz iterate solution $u^1_{|\Gamma_i}$.

step 5: apply generalized Aitken acceleration based on

$$\hat{u}^\infty = (Id - P)^{-1}(\hat{u}^1 - P\hat{u}^0)$$

in order to get $\hat{u}^\infty_{|\Gamma_i}$. 
3D DDM: Scalability of 1D AS (with PDC3D as inner solver)

- 3 Crays with 1280 procs (2 Germany, 1 USA),
- $732 \times 10^6$ unknowns Pb solved in less than 60s with $\|\epsilon\|_\infty < 10^{-8}$
- network 3-5 Mb/s (communication between 17s and 23s)

Explicit building of $P_{[[\ldots]]}$

uses how basis $\Phi_k$ are modified by the Schwarz iterate.

Steps to build the $P_{[[\ldots]]}$ matrix

a. starts from the the basis function $\Phi_k$ and get its value on interface in the physical space

b. performs two schwarz iterates with zeros local right hand sides and homogeneous boundary condition on $\partial \Omega = \partial (\Omega_1 \cap \Omega_2)$

c. decomposes the trace solution on the interface in the basis $V$. We then obtains the column $k$ of the matrix $P_{[[\ldots]]}$
Explicit building of $P_{[[\ldots]]}$

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c. decomposes the trace solution on the interface in the basis $V$. We then obtains the column $k$ of the matrix $P_{[[\ldots]]}$
\( P_{[\ldots,\ldots]} \) can be computed in parallel, (\# local subdomain solve = \# interface points, and the number of columns computed during the Schwarz iterates can be set according to the computer architecture.

- Its adaptive computation is required to save computing.
- The Fourier mode convergence gives a tool to select the Fourier modes that slow the convergence.
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- Its adaptive computation is required to save computing.
- The Fourier mode convergence gives a tool to select the Fourier modes that slow the convergence.
1. The Dirichlet-Neumann Map

2. The Generalized Schwarz Alternating Method

3. The Aitken-Schwarz Method

4. Non separable operator, non regular mesh, adaptive Aitken-Schwarz

5. Aitken meshfree acceleration
Adaptive building of the non diagonal matrix $P_{[[..]]}$ (non separable pb/non uniform mesh)

A. Frullone & DTD : Adaptive acceleration of the Aitken-Schwarz Domain Decomposition on nonuniform nonmatching grids submitted (Non Uniform Fourier basis orthogonal with respect to a numerical hermitian form)

- Select Fourier modes higher than a fixed tolerance. Index = array containing the list of selected modes.
- Take the subset $\tilde{v}$ of Fourier modes from 1 to max(Index).
- Approximate $P_{[[..]]}$ with $P^{*}_{[[..]]}$ using only $\tilde{v}$.
- Accelerate $\tilde{v}$ through the equation:

$$\tilde{v}^\infty = (Id - P^{*}_{[[..]]})^{-1}(\tilde{v}^{n+1} - P^{*}_{[[..]]}\tilde{v}^n)$$

Other modes are not accelerated.
Adaptive building of the non diagonal matrix $P_{[[.,.]]}$ \(\text{ (non separable pb/non uniform mesh)}\)

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  - Accelerate $\tilde{\mathbf{v}}$ through the equation:
    \[
    \tilde{\mathbf{v}}^\infty = (\mathbf{I}d - P^*_{[[.,.]]})^{-1}(\tilde{\mathbf{v}}^{n+1} - P^*_{[[.,.]]}\tilde{\mathbf{v}}^n)
    \]

Other modes are not accelerated.
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- Approximate $P_{[.,.]}$ with $P^*_{[.,.]}$ using only $\tilde{v}$.
- Accelerate $\tilde{v}$ through the equation:

$$\tilde{v}^\infty = (\text{Id} - P^*_{[.,.]} )^{-1}(\tilde{v}^{n+1} - P^*_{[.,.]}\tilde{v}^n)$$

Other modes are not accelerated.
AS-DDM on a strongly non separable operator and irregular matching grids

\[
\begin{aligned}
\begin{cases}
\nabla \cdot (a(x, y) \nabla) u(x, y) &= f(x, y), \quad \text{on } \Omega = ]0, 1[^2 \\
u(x, y) &= 0, \quad (x, y) \in \partial \Omega \\
a(x, y) &= a_0 + (1 - a_0)(1 + tanh((x - (3h \ast y + 1/2 - h))/\mu))/2,
\end{cases}
\end{aligned}
\]

and \(a_0 = 10^1, \mu = 10^{-2}\).
Numerical results

**Fig.**: adaptive acceleration using sub-blocks of $P_{\ldots\ldots}$, with 100 points on the interface, overlap $= 1$, $\epsilon = h_u/8$ and Fourier modes tolerance $= \|\hat{u}^k\|_\infty/10^i$ for $i = 1.5$ and 3 for 1st iteration and $i = 4$ for successive iterations.
1. The Dirichlet-Neumann Map
2. The Generalized Schwarz Alternating Method
3. The Aitken-Schwarz Method
4. Non separable operator, non regular mesh, adaptive Aitken-Schwarz
5. Aitken meshfree acceleration
The two salient features of the Aitken-Schwarz methodology

- Have a representation in a basis of the Boundary condition. This basis having some orthogonality property in order to separate the coefficient associated to a vector of this basis.
- Have a decreasing of the coefficients of this representation of the BC in this basis, in order to select only the mode of interest in the Aitken acceleration process.

⇒ Singular value Decomposition (or Proper orthogonal Decomposition) have these properties.

We can use the SVD of the BC values in order to build $P$ and to accelerate the convergence to the right BC.
Let $X_1^q = [x_1, \ldots, x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X_1^q = USV$ the singular value decomposition of $X$. ($U' \ast U = I, V'V = I$)

Schwarz : $X_2^{q+2} - X_2^{q+1} = P(X_2^{q+1} - X_1^q)$

Then $U'(X_2^{q+2} - X_2^{q+1})(U'(X_2^{q+1} - X_1^q))^{-1} = U'PU = \tilde{P}$

$x_\infty = U((I - \tilde{P})^{-1}(U'x_{q+2} - \tilde{P}U'x_{q+1})$

Subject to numerical problem in the inverting
Let $X_1^q = [x_1, ..., x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X_1^q = USV$ the singular value decomposition of $X$. ($U' * U = I$, $V' V = I$)

Schwarz: $X_3^{q+2} - X_2^{q+1} = P(X_2^{q+1} - X_1^q)$

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Subject to numerical problem in the inverting
Let $X_1^q = [x_1, \ldots, x_q]$, be the traces of the $q$ Schwarz iterates.

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Schwarz : $X_3^{q+2} - X_2^{q+1} = P(X_2^{q+1} - X_1^q)$

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$x_\infty = U((I - \tilde{P})^{-1}(U'x_{q+2} - \tilde{P}U'x_{q+1})$ Subject to numerical problem in the inverting
Let $X^q_1 = [x_1, \ldots, x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X^q_1 = USV$ the singular value decomposition of $X$. 
($U' \ast U = I, V'V = I$)

Select the modes that be involved in the acceleration based on the singular value

Applied one Schwarz on the basis functions $U^*$ to determine columns of $\tilde{P}^*$

then $x^*_\infty = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*))$

Complete with the last iterate components.

no inverting, more accurate
Let $X_1^q = [x_1, \ldots, x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X_1^q = USV$ the singular value decomposition of $X$. ($U' \ast U = I, V'V = I$)

Select the modes that be involved in the acceleration based on the singular value

Applied one Schwarz on the basis functions $U^*$ to determine columns of $\tilde{P}^*$

then $x_\infty^* = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*))$

Complete with the last iterate components.

no inverting, more accurate
Let $X_1^q = [x_1, \ldots, x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X_1^q = USV$ the singular value decomposition of $X$. 
($U' \ast U = I, V' V = I$)

Select the modes that be involved in the acceleration based on the singular value.

Applied one Schwarz on the basis functions $U^*$ to determine columns of $\tilde{P}^*$

then $x_\infty^* = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*))$

Complete with the last iterate components.

no inverting, more accurate
Let $X^q_1 = [x_1, \ldots, x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X^q_1 = USV$ the singular value decomposition of $X$.

$(U' \ast U = I, V' V = I)$

Select the modes that be involved in the acceleration based on the singular value

Applied one Schwarz on the basis functions $U^*$ to determine columns of $\tilde{P}^*$

then $x^*_\infty = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*)$ Complete with the last iterate components.

no inverting, more accurate
Let $X^q_1 = [x_1, ..., x_q]$, be the traces of the $q$ Schwarz iterates.

Let $X^q_1 = USV$ the singular value decomposition of $X$. ($U' \ast U = I, V'V = I$)

Select the modes that be involved in the acceleration based on the singular value.

Applied one Schwarz on the basis functions $U^*$ to determine columns of $\tilde{P}^*$

then $x^*_{\infty} = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*)$

Complete with the last iterate components.

no inverting, more accurate
\[ \nabla \cdot (K(x, y) \nabla u) = f, \quad \text{on} \Omega, \quad u = 0, \quad \text{on} \partial \Omega \text{ in random porous media} \]

Exponential covariance: 
\[ C_Y(x, y) = \sigma_Y^2 \exp \left( - \left[ \left( \frac{x}{\lambda_x} \right)^2 + \left( \frac{y}{\lambda_y} \right)^2 \right]^{1/2} \right) \]

\( \lambda_x \) (\( \lambda_y \)) is the directional \( \ln(K) \) correlation length scales

\( \sigma^2 \) is the variance of \( \ln(K) \)

\[ \log_{10}(K) \in [-7.28, 7.69] \text{ distribution } \lambda_x = \lambda_y = 5, \sigma^2 = 4 \]
Schwarz DDM: random distribution of $K$ along the interfaces
Singular values of the SVD of the Schwarz iterates on $\Gamma_1$
Basis $U$ of the SVD of the Schwarz iterates on $\Gamma_1$
Coefficients of the traces in the basis U

log10(abs(S^V'))

coefficient in basis U
16 modes are used in the acceleration process only.
Convergence of AS with acceleration based on SVD

K permeability with lognormal random distribution ($\lambda=5$, $\sigma^2=6$)

Aitken-Schwarz-SVD convergence for ($\lambda=5$, $\sigma^2=6$), overlap=5h
The two main features for Aitken acceleration are orthogonal basis with decreasing coefficients for the representation of the traces in this basis.

- It works very well when this basis link to the mesh on interfacial interface is available.
- SVD decomposition as the right properties without the drawback to be link to the underlying mesh.
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