AS DDM
DTD

Meshfree Adaptative Aitken-Schwarz Domain Decomposition for Darcy flow

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Dedicated to Alain Bourgeat's 60^{th} birthday
Scaling Up and Modeling for Transport and Flow in Porous Media
Dubrovnik, Croatia, 13-16 October 2008
Partially founded by : GDR MOMAS, ANR-TL-07 LIBRAERO, ANR-CIS-07 MICAS

Objectives : make a Schwarz DDM that has :

- scalable properties
- Artificial condition independant of the parameter (even make convergent a divergent Schwarz method)
- can be used as ”black box”, no direct impact on the implementation of local solver.

Let $\Omega \subset \mathbf{R}^n$ a bounded domain with $\Gamma := \partial\Omega$ Lipschitz.

The trace operator : γ_0

$\forall u \in H^1(\Omega), \ \exists \gamma_0 u \in H^{1/2}(\Gamma)$ satisfying

$$||\gamma_0 u||_{H^{1/2}(\Gamma)} \leq c_T.||u||_{H^1(\Omega)}. \quad (1)$$

vice versa the bounded extension operator : ε

$\forall v \in H^{1/2}(\Gamma), \ \exists \varepsilon v \in H^1(\Omega)$ satisfying $\gamma_0 \varepsilon v = v$ and

$$||\varepsilon v||_{H^1(\Omega)} \leq c_{IT}.||v||_{H^{1/2}(\Gamma)}. \quad (2)$$

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$$\text{Set } L(x)u(x) = -\Sigma_{i,j=1}^{n} \frac{\partial}{\partial x_j}[a_{ji}(x)\frac{\partial}{\partial x_i}u(x)], \qquad a_{ji} \in L_\infty(\Omega) \tag{3}$$

$L(.)$ is assumed to be uniformly elliptic,

$$\Sigma_{i,j=1}^{n} a_{ji}(x)\xi_j\xi_l \geq c_0.|\xi|^2, \forall \xi \in \mathbf{R}^n, \forall x \in \Omega$$

The conormal derivative γ_1 is given by

$$\gamma_1 u(x) := \Sigma_{i,j=1}^{n} n_j(x)[a_{ji}(x)\frac{\partial}{\partial x_i}u(x)], \; \forall x \in \Gamma$$

where $n(x)$ is the exterior unit normal vector.

$$\begin{aligned} a(u,v) &= \sum_{i,j=1}^{n} \int_\Omega \frac{\partial}{\partial x_j} v(x) a_{ji}(x) \frac{\partial}{\partial x_i} u(x) \\ &= \int_\Omega Lu(x)v(x)dx + \int_\Gamma \gamma_1 u(x) \gamma_0 v(x) dS_x \end{aligned}$$

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Necas Lem. $\Rightarrow \exists! u = u_0 + \varepsilon g \in H^1(\Omega)$ sol. of Dirichlet Pb

$$L(x)u(x) = f(x), \text{for } x \in \Omega, \gamma_0 u(x) = g(x), \text{ for } x \in \Gamma \quad (4)$$

Then defining the linear application $\forall w \in H^{1/2}(\Gamma)$

$$l(w) = a(u, \varepsilon w) - \int_\Omega f(x)\varepsilon w(c)dx.$$

Riez thm : $\exists \lambda \in H^{-1/2}(\Gamma): \langle \lambda, w \rangle_{L_2(\Gamma)} = l(w)\ \forall w \in H^{1/2}(\Gamma)$.

Hence, the conormal derivative $\lambda \in H^{-1/2}(\Gamma)$ satisfies

$$\int_\Gamma \lambda\, w\, ds_x = a(u_0 + \varepsilon g, \varepsilon w) - \int_\Omega f\, \varepsilon w\, dx\ \forall w \in H^{1/2}(\Gamma).$$

$\Rightarrow$ f fixed, we have a DtoN map : $g = \gamma_0 u \mapsto \lambda := \gamma_1 u$

$$\gamma_1 u(x) = Sg(x) - Nf(x), \forall w \in \Gamma \quad (5)$$

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Outline

The Generalized Schwarz Alternating Method (GSAM)

B. Engquist and H.-K. Zhao, Appl. Numer. Math. **27** (1998), no. 4, 341–365.

Consider $\Omega = \Omega_1 \cup \Omega_2$ with the two artificial boundaries Γ_1, Γ_2 intersecting $\partial\Omega$.

Algorithm

$$
\begin{aligned}
L(x)u_1^{2n+1}(x) &= f(x),\ \forall x \in \Omega_1,\ u_1^{2n+1}(x) = g(x),\ \forall x \in \partial\Omega_1 \backslash \Gamma_1,\\
\Lambda_1 u_1^{2n+1} &+ \lambda_1 \frac{\partial u_1^{2n+1}(x)}{\partial n_1} = \Lambda_1 u_2^{2n} + \lambda_1 \frac{\partial u_2^{2n}(x)}{\partial n_1},\ \forall x \in \Gamma_1\\
L(x)u_2^{2n+2}(x) &= f(x),\ \forall x \in \Omega_2,\ u_2^{2n+2}(x) = g(x),\ \forall x \in \partial\Omega_2 \backslash \Gamma_2,\\
\Lambda_2 u_2^{2n+2} &+ \lambda_2 \frac{\partial u_2^{2n+2}(x)}{\partial n_2} = \Lambda_2 u_1^{2n+1} + \lambda_2 \frac{\partial u_1^{2n+1}(x)}{\partial n_2},\ \forall x \in \Gamma_2.
\end{aligned}
$$

where Λ_i's are some operators and λ_i's are constants. $(\Lambda_1 = I, \lambda_1 = 0, \Lambda_2 = 0, \lambda_2 = 1)$ Schwarz Neumann-Dirichlet Algorithm

If $\lambda_1 = 1$ and Λ_1 is the DtoN operator at Γ_1 associated to the homogeneous PDE in Ω_2 with homogeneous boundary condition on $\partial\Omega_2 \cap \partial\Omega$ then GSAM converge in two steps.
proof Let $e_i^n = u - u^n, i = 1, 2, $, then

$$\begin{aligned} L(x)e_1^1(x) &= 0,\ \forall x \in \Omega_1,\ e_1^1(x) = 0,\ \forall x \in \partial\Omega_1 \backslash \Gamma_1, \\ \Lambda_1 e_1^1 &+ \frac{\partial e_1^1(x)}{\partial n_1} = \Lambda_1 e_2^0 + \frac{\partial e_2^0(x)}{\partial n_1},\ \forall x \in \Gamma_1 \end{aligned}$$

since Λ_1 is the DtoN operator at Γ_1 in Ω_2

$$\frac{\partial e_2^0}{\partial n_1} + \Lambda_1 e_2^0 = -\frac{\partial e_2^0}{\partial n_2} + \frac{\partial e_2^0}{\partial n_2} = 0,\ \Rightarrow e_1^1 = 0 \text{in } \Omega_1$$

Hence we get the exact solution in two steps []

- Pb : Λ_i DtoN operators are global operators (linking all the subdomains when > 3).
- In practice, the algebraical approximations of this operators are used (see Nataf, Gander).
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Let $\Omega = \Omega_1 \cup \Omega_2,\ \Omega_{12} = \Omega_1 \cap \Omega_2,\ \Omega_{ii} = \Omega_i \backslash \Omega_{12}$
$e_i^n = u - u_i^n$ in Ω_i satisfies :

$$\begin{array}{rcl} (\Lambda_1 + \lambda_1 S_1) R_1 e_1^{2n+1} & = & (\Lambda_1 - \lambda_1 S_{22}) R_{22} P_2 e_2^{2n} \\ (\Lambda_2 + \lambda_2 S_2) R_2 e_2^{2n+2} & = & (\Lambda_2 - \lambda_2 S_{22}) R_{11} P_1 e_1^{2n+1} \end{array}$$

with

- $P_i : H^1(\Omega_i) \rightarrow H^1(\Omega_{ii})$
- S_i (S_{ii}) the DtoN map operator in Ω_i (Ω_{ii}) on Γ_i ($\Gamma_{mod(i,2)+1}$).
- $R_i : H^1(\Omega_i) \rightarrow H^{1/2}(\Gamma_i)$, $R_{ii} : H^1(\Omega_{ii}) \rightarrow H^{1/2}(\Gamma_{mod(i,2)+1})$,
- $R_i^* :\ R_i R_i^* = I$,
  $\forall g \in H^{1/2}(\Gamma_i),\ L(x) R_i^* g = 0, R_i^* g = g\ on \Gamma_i, R_i^* g =$
  $0\ on\ \partial\Omega_i \backslash \Gamma_i$

Thus the convergence of GSAM is purely linear ! ! Aitken-Schwarz DDM uses this property to accelerate the convergence :

- Consequently, no direct approximation of the DtoN map is used, but an approximation of the operator of error linked to this DtoN map is performed.

Outline

M.Garbey and D.Tromeur-Dervout : *On some Aitken like acceleration of the Schwarz method*,
Int. J. for Numerical Methods in Fluids, 40(12) :1493-1513,2002

- additive Schwarz algorithm :
  - $L[u_1^{n+1}] = f \; in \; \Omega_1, \; u_{1|\Gamma_1}^{n+1} = u_{2|\Gamma_1}^{n},$
  - $L[u_2^{n+1}] = f \; in \; \Omega_2, \; u_{2|\Gamma_2}^{n+1} = u_{1|\Gamma_2}^{n}.$
- the interface error operator T is **linear**, i.e
  - $u_{1|\Gamma_2}^{n+1} - U_{|\Gamma_2} = \delta_1(u_{2|\Gamma_1}^{n} - U_{|\Gamma_1}),$
  - $u_{2|\Gamma_1}^{n+1} - U_{|\Gamma_1} = \delta_2(u_{1|\Gamma_2}^{n} - U_{|\Gamma_2}).$
- Consequently
  - $u_{1|\Gamma_2}^{2} - u_{1|\Gamma_2}^{1} = \delta_1(u_{2|\Gamma_1}^{1} - u_{2|\Gamma_1}^{0}),$
  - $u_{2|\Gamma_1}^{2} - u_{2|\Gamma_1}^{1} = \delta_2(u_{1|\Gamma_2}^{1} - u_{1|\Gamma_2}^{0}),$
- Computation of $\delta_{1/2}$:
  $L[v_{1/2}] = 0 \; in \; \Omega_{1/2}, \; v_{\Gamma_{1/2}} = 1.$ thus $\delta_{1/2} = v_{\Gamma_{2/1}}.$
- iff $\delta \neq 1$ **Aitken-Schwarz** gives the solution with exactly **3** iterations and possibly **2** in the analytical case.

Acceleration of Schwarz Method for Elliptic Problems

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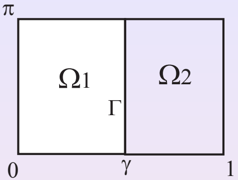
Example on a toy problem : Darcy-Stokes coupling

$$\left\{\begin{array}{lcl} -\nabla.T(u_1,p_1) & = & f_1, \text{ in } \Omega_1 \\ \nabla.u_1 & = & 0, \text{ in } \Omega_1 \\ T := -p_1 I & + & 2\mu D(u_1), \\ D(u_1) := \frac{1}{2}\nabla u_1 & + & \frac{1}{2}\nabla u_1^T \end{array}\right.$$

$$\left\{\begin{array}{lcl} \mu u_2 + K^2\nabla p_2 & = & 0, \text{ in } \Omega_2 \\ \nabla.u_2 & = & f_2, \text{ in } \Omega_2 \end{array}\right.$$



$$B.C.: \; u_1 = 0, \text{ on } \partial\Omega_1\backslash\Gamma, \; p_2 = 0 \text{ on } \partial\Omega_2\backslash\Gamma$$

Beavers-Joseph-Saffman boundary condition on Γ

$$-n_1.T(u_1,p_1).\tau_1 = \frac{\alpha}{K} u_1.\tau_1, \text{ on } \Gamma$$

Transmission conditions to close the system :

$$u_1.n_1 = u_2.n_1, \text{ on } \Gamma$$

$$-n_1.T(u_1,p_1).n_1 = p_2, \text{ on } \Gamma.$$

Example on a toy problem : Darcy-Stokes coupling

$$
\begin{aligned}
u_{i1}(x,y) &= \Sigma \hat{u}_{i1,k}(x) cos(ky), \\
u_{i2}(x,y) &= \Sigma \hat{u}_{i2,k}(x) cos(ky), \\
p_i(x,y) &= \Sigma \hat{p}_{i,k}(x) sin(ky).
\end{aligned}
$$

Schwarz errors e_{i1}, e_{i2}, e_{ip} for each mode k in Ω_i satisfy ,

$$
\left\{
\begin{array}{l}
\mu \frac{\partial^2}{\partial x^2} e^n_{11}(x) - \mu k^2 e^n_{11}(x) - \frac{\partial}{\partial x} e^n_{1p}(x) = 0, \forall x \in]0,\gamma[, \\
\mu \frac{\partial^2}{\partial x^2} e^n_{12}(x) - \mu k^2 e^n_{12}(x) - k e^n_{1p}(x) = 0, \forall x \in]0,\gamma[, \\
\frac{\partial}{\partial x} e^n_{11}(x) - k e^n_{12}(x) = 0, \forall x \in]0,\gamma[\\
\mu k e^n_{11}(\gamma) - mu \frac{\partial}{\partial x} e^n_{12}(\gamma) - \frac{\alpha}{K} e^n_{12}(\gamma) = 0 \\
e^n_{11}(0) = e^n_{12}(0) = 0 \\
e^n_{1p}(\gamma) - 2\mu \frac{\partial}{\partial x} e^n_{11}(\gamma) = \eta^n = e^{n-1/2}_{2p}(\gamma)
\end{array}
\right.
$$

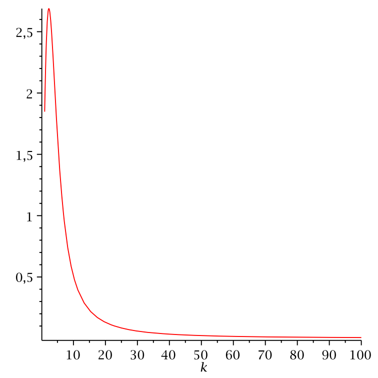
$$
\left\{
\begin{array}{l}
\frac{\partial}{\partial x} e^{n+1/2}_{21}(x) - k e^{n+1/2}_{22}(x) = 0, \ \forall x \in]\gamma, 1[\\
e^{n+1/2}_{2p}(1) = 0 \\
e^{n+1/2}_{21}(\gamma) = xi^{n+1/2} = e^n_{11}(\gamma)
\end{array}
\right.
$$

$$\begin{pmatrix} \eta^{n+1} \\ \xi^{n+1/2} \end{pmatrix} = \begin{pmatrix} 0 & \rho_1 \\ \rho_2 & 0 \end{pmatrix} \begin{pmatrix} \eta^{n} \\ \xi^{n-1/2} \end{pmatrix}$$

$$\rho_1 = \frac{\mu \tanh(k(1-\gamma))}{kK^2}$$

$$\rho_2 = \frac{-4\alpha \sinh(k\gamma) + 2\,\mu\, kK(e^{-2\,k\gamma} - e^{2\,k\gamma} + 4\,k\gamma) + 4\,k^2\gamma^2\alpha}{-2\,k\alpha\,\left(e^{-2\,k\gamma} - 2 + e^{2\,k\gamma}\right)\mu}$$

- convergence (eventually divergence) depends on parameters value but not of the iteration and not of the solution.
- each mode can be accelerated by the Aitken process
- even with $\rho_1\rho_2$ very closed to 1.



$\rho_1\rho_2$

with $\alpha = 100$, $K^2 = 0.01$, $\mu = 1$, $\gamma = 0.5$

Example of linear convergence for the Schwarz Neumann-Dirichlet algo.

$[\alpha,\Gamma_1]\cup[\Gamma_1,\Gamma_2]\cup[\Gamma_2,\beta]$, $\Gamma_1<\Gamma_2$. Schwarz writes :

$$\left\{\begin{array}{l}\Delta u_1^{(j)}=f \text{ on } [\alpha,\Gamma_1]\\ u_1^{(j)}(\alpha)=0\\ u_1^{(j)}(\Gamma_1)=u_1^{(j-\frac{1}{2})}(\Gamma_2)\end{array}\right.,\quad \left\{\begin{array}{l}\Delta u_2^{(j+\frac{1}{2})}=f \text{ on } [\Gamma_1,\Gamma_2]\\ \dfrac{\partial u_2^{(j+\frac{1}{2})}(\Gamma_1)}{\partial n}=\dfrac{\partial u_1^{(j)}(\Gamma_1)}{\partial n}\\ u_2^{(j+\frac{1}{2})}(\Gamma_2)=u_3^{(j)}(\Gamma_2)\end{array}\right.,\tag{6}$$

$$\left\{\begin{array}{l}\Delta u_3^{(j)}=f \text{ on } [\Gamma_2,\beta]\\ \dfrac{\partial u_3^{(j)}(\Gamma_2)}{\partial n}=\dfrac{\partial u_2^{(j-\frac{1}{2})}(\Gamma_2)}{\partial n}\\ u_3^{(j)}(\beta)=0\end{array}\right..$$

The error on subdomain i writes $e_i(x)=c_ix+d_i$.

$$e_1^{(j)}(x)=e_2^{(j-\frac{1}{2})}(\Gamma_1)\frac{(\alpha-x)}{\alpha-\Gamma_1},\ e_3^{(j)}(x)=\frac{\partial}{\partial n}e_2^{(j-\frac{1}{2})}(\Gamma_2)(x-\beta)\tag{7}$$

Error on the second subdomain satisfies

$$e_2^{(j+\frac{1}{2})}(x) = \frac{\partial}{\partial n} e_1^{(j)}(\Gamma_1)(x - \Gamma_2) + e_3^{(j)}(\Gamma_2) \tag{8}$$

Replacing $e_3^{(j)}(\Gamma_2)$ and $\frac{\partial}{\partial n} e_1^{(j)}(\Gamma_1)$, $e_2^{(j+\frac{1}{2})}(x)$ writes :

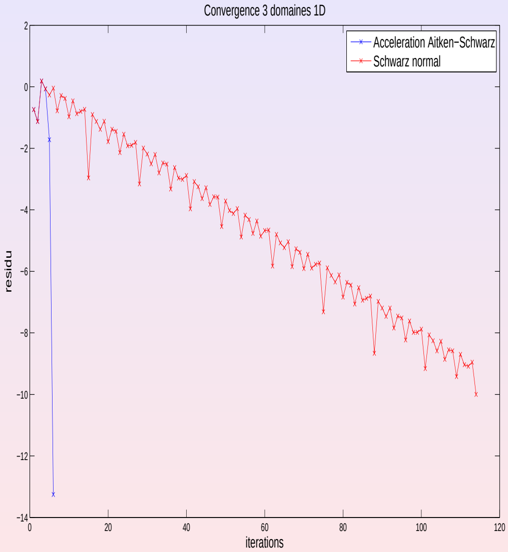
$$e_2^{(j+\frac{1}{2})}(x) = -\frac{x - \Gamma_2}{\alpha - \Gamma_1} e_2^{(j-\frac{1}{2})}(\Gamma_1) + (\Gamma_2 - \beta)\frac{\partial}{\partial n} e_2^{(j-\frac{1}{2})}(\Gamma_2) \tag{9}$$

Consequently, the following identity holds :

$$\begin{pmatrix} e_2^{(j)}(\Gamma_1) \\ \frac{\partial}{\partial n} e_2^{(j)}(\Gamma_2) \end{pmatrix} = \begin{pmatrix} \frac{\Gamma_2 - \Gamma_1}{\alpha - \Gamma_1} & \Gamma_2 - \beta \\ \frac{-1}{\alpha - \Gamma_1} & 0 \end{pmatrix} \begin{pmatrix} e_2^{(j-1)}(\Gamma_1) \\ \frac{\partial}{\partial n} e_2^{(j-1)}(\Gamma_2) \end{pmatrix} \tag{10}$$

Consequently the matrix do not depends of the solution, neither of the iteration, but only of the operator and the shape of the domain.

Num. analysis for the Neumann-Dirichlet algo. (3 subdomains)



Cvg for 1D Poisson pb with 3 non-overlapping subdomains $\alpha = 0$, $\beta = 1$, $\Gamma_1 = 0.44$,$\Gamma_2 = 0.7$

Aitken acceleration of convergence in n-D

- $\vec{x}_{i+1} - \vec{\xi} = P(\vec{x}_i - \vec{\xi}),\ i = 0, 1, \ldots$
- $\left(\ \vec{x}_{N+1} - \vec{x}_N \quad \ldots \quad \vec{x}_2 - \vec{x}_1\ \right) = P\left(\ \vec{x}_N - \vec{x}_{N-1} \quad \ldots \quad \vec{x}_1 - \vec{x}_0\ \right)$
- Thus if $\left(\ \vec{x}_N - \vec{x}_{N-1} \quad \ldots \quad \vec{x}_1 - \vec{x}_0\ \right)$ is non singular then $P = \left(\ \vec{x}_{N+1} - \vec{x}_N \quad \ldots \quad \vec{x}_2 - \vec{x}_1\ \right)\left(\ \vec{x}_N - \vec{x}_{N-1} \quad \ldots \quad \vec{x}_1 - \vec{x}_0\ \right)^{-1}$
  If $||P|| < 1$ then $\vec{\xi} = (Id - P)^{-1}(\vec{x}_{N+1} - P\vec{x}_N)$
- The construction of P claims at least $N+1$ iterates if the error components are linked together. $\Rightarrow$
  - write the solution in a functional basis were the components error are decoupled
  - Construct an approximation of P

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For GSAM with two subdomains, errors $e^i_{\Gamma^j_h} = U^{i+1}_{\Gamma^j_h} - U^i_{\Gamma^j_h}$ satisfy

$$\begin{pmatrix} e^{i+1}_{\Gamma^1_h} \\ e^{i+1}_{\Gamma^2_h} \end{pmatrix} = P \begin{pmatrix} e^{i}_{\Gamma^1_h} \\ e^{i}_{\Gamma^2_h} \end{pmatrix} \tag{11}$$

- Γ^j_h a discretisation of the interfaces
  Γ_h to be the coarsest discretisation in the sense that it produces V the smallest set of orthonormal vectors Φ_k that belong to Γ_h with respect to a discrete hermitian form $[[.,.]]$.
- Let U_{Γ_h} be the decomposition of U_Γ with respect to the orthogonal basis V.
  $U_{\Gamma_h} = \sum_{k=0}^{N} \alpha_k \Phi_k$
- The α_k represents the "Fourier" coefficients of the solution with respect to the basis V.
  The orthogonality $\Rightarrow \alpha_k = [[U_\Gamma, \Phi_k]]$
- Then

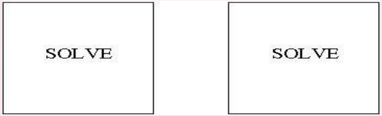
$$\begin{pmatrix} \beta^{i+1}_{\Gamma^1_h} \\ \beta^{i+1}_{\Gamma^2_h} \end{pmatrix} = P_{[[.,.]]} \begin{pmatrix} \beta^{i}_{\Gamma^1_h} \\ \beta^{i}_{\Gamma^2_h} \end{pmatrix} \tag{12}$$

For a separable operator in 2D or 3D and regular step size mesh

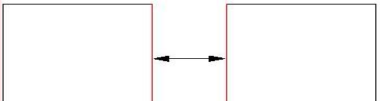
No coupling between the modes thus the operator P for the speed up is a block diagonal matrix and n-D is analogous to the 1-D

1. for Schwarz each wave has is own linear rate of convergence and high frequencies are damped first.
2. for high modes the matrix P can be approximate with neglecting far Macro-Domains interactions.

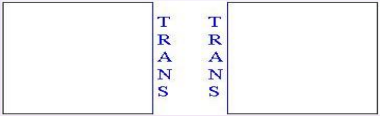
- step1 : build P analytically or numerically from data given by two Schwarz iterates
- step2 : apply one Jacobi Schwarz iterate to the differential problem with block solver of choice i.e multigrids, FFT etc...



- step3 : exchange boundary information :



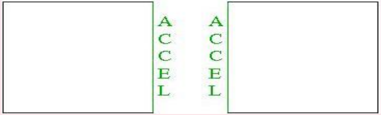
- step4 : compute the Fourier expansion $\hat{u}^n_{j|\Gamma_i}, n = 0, 1$ of **the traces on the artificial interface** $\Gamma_i, i = 1..nd$ for the initial boundary condition $u^0_{|\Gamma_i}$ and the Schwarz iterate solution $u^1_{|\Gamma_i}$.



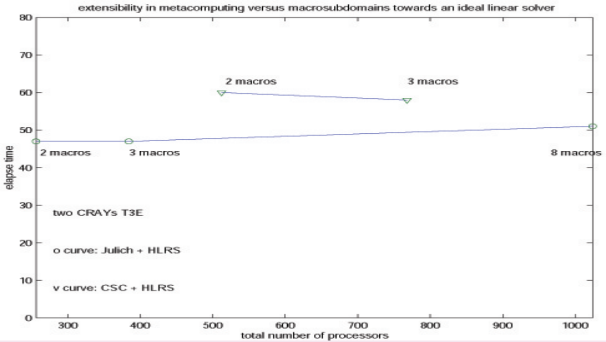
- step5 : apply generalized Aitken acceleration based on

$$\hat{u}^\infty = (Id - P)^{-1}(\hat{u}^1 - P\hat{u}^0)$$

in order to get $\hat{u}^\infty_{|\Gamma_i}$.



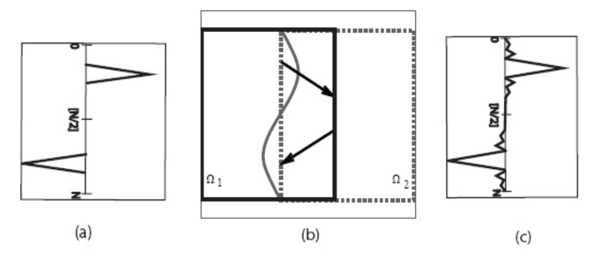
3D DDM : Scalability of 1D AS (with PDC3D as inner solver)



- 3 Crays with 1280 procs (2 Germany, 1 USA) ,
- 732 10^6 unknowns Pb solved in less than 60s with $||e||_\infty < 10^{-8}$
- network 3-5 Mb/s (communication between 17s and 23s)
- *Barberou, Garbey, Hess, Resch, Rossi, Toivanen and Tromeur-Dervout, J. of Parallel and Distributed Computing, special issue on Grid computing, 63(5) :564-577, 2003*

Explicit building of $P_{[[.,.]]}$

uses how basis Φ_k are modified by the Schwarz iterate.



Steps to build the $P_{[[.,.]]}$ matrix

a starts from the the basis function Φ_k and get its value on interface in the physical space

b performs two schwarz iterates with zeros local right hand sides and homogeneous boundary condition on $\partial\Omega = \partial(\Omega_1 \cap \Omega_2)$

c decomposes the trace solution on the interface in the basis V. We then obtains the column k of the matrix $P_{[[.,.]]}$

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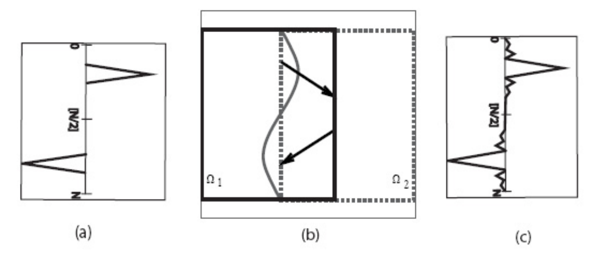

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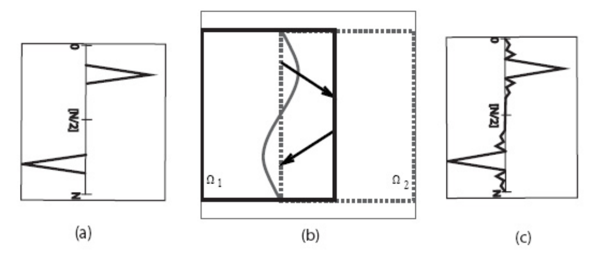

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- Its adaptive computation is required to save computing.
- The Fourier mode convergence gives a tool to select the Fourier modes that slow the convergence.

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Outline

Adaptive building of the non diagonal matrix $P_{[[.,.]]}$ (non separable pb/non uniform mesh)

A. Frullone & DTD :Adaptive acceleration of the Aitken-Schwarz Domain Decomposition on nonuniform nonmatching grids submitted (Non Uniform Fourier basis ortogonal with respect to a numerical hermitian form)

- Select Fourier modes higher than a fixed tolerance. Index = array containing the list of selected modes.
- Take the subset $\tilde{v}$ of Fourier modes from 1 to max(Index).
- Approximate $P_{[[.,.]]}$ with $P^*_{[[.,.]]}$ using only $\tilde{v}$.
- Accelerate $\tilde{v}$ through the equation :

$$\tilde{v}^\infty = (Id - P^*_{[[.,.]]})^{-1}(\tilde{v}^{n+1} - P^*_{[[.,.]]}\tilde{v}^n)$$

Other modes are not accelerated.

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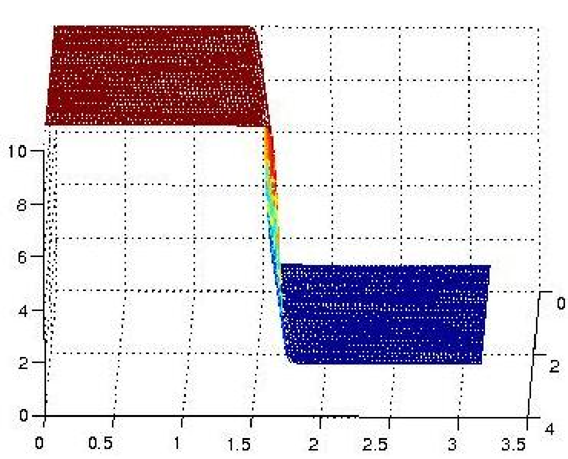
FINANCÉ PAR ANR

AS-DDM on a strongly non separable operator and irregular matching grids

$$\begin{cases} \nabla.(a(x,y)\nabla)u(x,y) = f(x,y), & \text{on } \Omega =]0,1[^2 \\ u(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}$$

$a(x,y) = a_0 + (1-a_0)(1 + tanh((x - (3h * y + 1/2 - h))/\mu))/2$, and $a_0 = 10^1, \mu = 10^{-2}$.



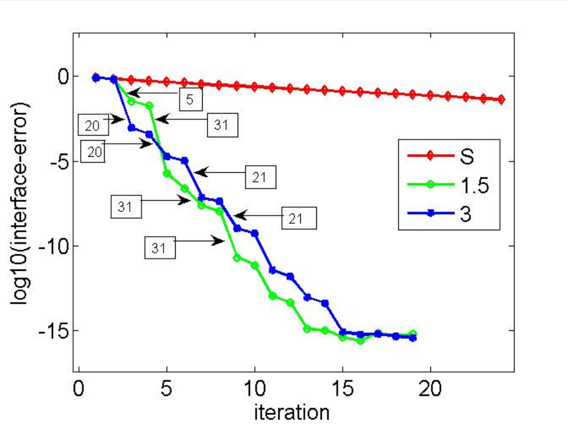


FIG.: adaptive acceleration using sub-blocks of $P_{[[.,.]]}$, with 100 points on the interface, overlap$=1$, $\epsilon = h_u/8$ and Fourier modes tolerance = $||\hat{u}^k||_\infty/10^i$ for $i = 1.5$ and 3 for 1st iteration and $i = 4$ for successive iterations.

Outline

FINANCÉ PAR ANR

The two salient features of the Aitken-Schwarz methodology

- Have a representation in a basis of the Boundary condition. This basis having some orthogonality property in order to separate the coefficient associated to a vector of this basis.
- Have a decreasing of the coefficients of this representation of the BC in this basis, in order to select only the mode of interest in the Aitken acceleration process.
- $\Rightarrow$ Singular value Decomposition (or Proper orthogonal Decomposition) have these properties.

We can use the SVD of the BC values in order to build P and to accelerate the convergence to the right BC.

FINANCÉ PAR ANR

- Let $X_1^q = [x_1, ..., x_q]$, be the traces of the q Schwarz iterates.
- Let $X_1^q = USV$ the singular value decomposition of X. $(U' * U = I, V'V = I)$
- Schwarz : $X_3^{q+2} - X_2^{q+1} = P(X_2^{q+1} - X_1^q)$
- Then $U'(X_3^{q+2} - X_2^{q+1})(U'(X_2^{q+1} - X_1^q))^{-1} = U'PU = \tilde{P}$
- $x_\infty = U((I - \tilde{P})^{-1}(U'x_{q+2} - \tilde{P}U'x_{q+1})$

Subject to numerical problem in the inverting

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- Select the modes that be involved in the acceleration based on the singular value
- Applied one Schwarz on the basis functions U^* to determine columns of $\tilde{P}^*$
- then $x_\infty^* = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*)$
- Complete with the last iterate components.

no inverting, more accurate

Aitken-Schwarz SVD : version 2

- Let $X_1^q = [x_1, ..., x_q]$, be the traces of the q Schwarz iterates.
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no inverting, more accurate

Aitken-Schwarz SVD : version 2

- Let $X_1^q = [x_1, ..., x_q]$, be the traces of the q Schwarz iterates.
- Let $X_1^q = USV$ the singular value decomposition of X. $(U' * U = I, V'V = I)$
- Select the modes that be involved in the acceleration based on the singular value
- Applied one Schwarz on the basis functions U^* to determine columns of $\tilde{P}^*$
- then $x_\infty^* = U^*((I - \tilde{P}^*)^{-1}((U'x_{q+2})^* - \tilde{P}^*(U'x_{q+1})^*)$
- Complete with the last iterate components.

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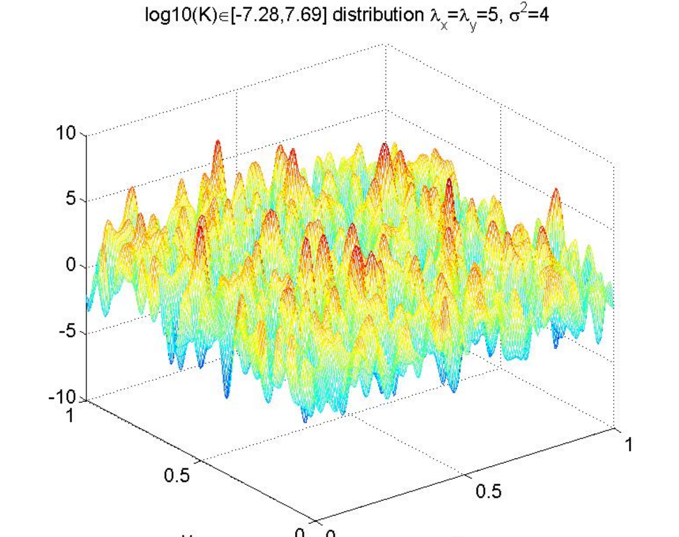
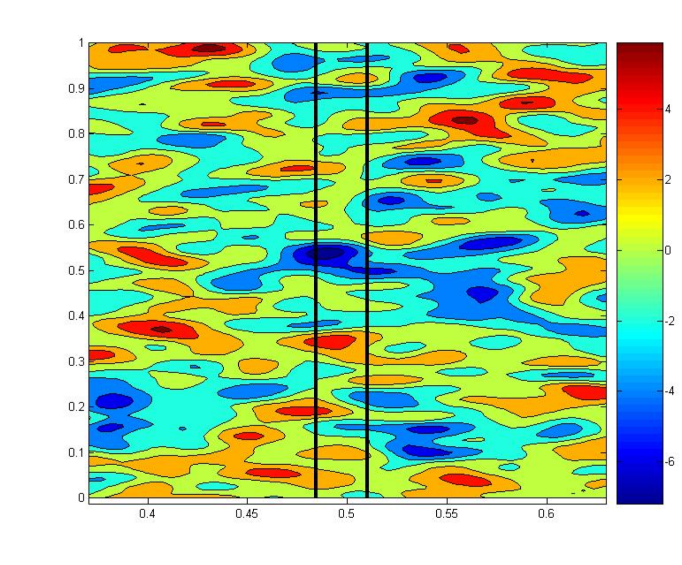
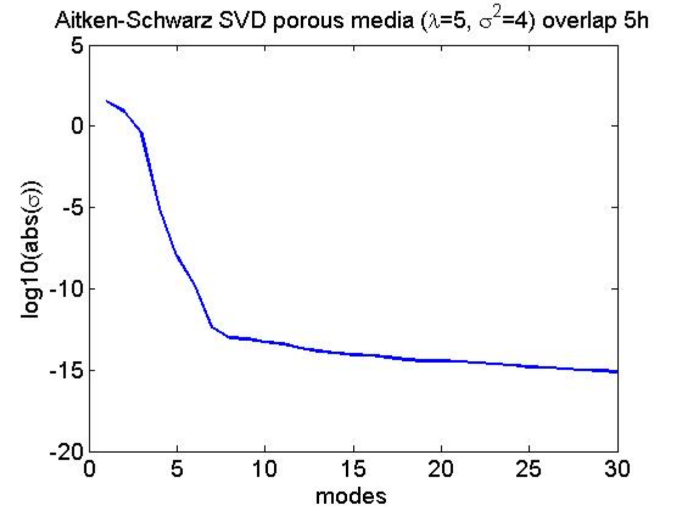
Exponential covariance : $C_Y(x,y) = \sigma_Y^2 exp(-[(\frac{x}{\lambda_x})^2 + (\frac{y}{\lambda_y})^2]^{\frac{1}{2}})$
λ_x (λ_y) is the directional $ln(K)$ correlation length scales
σ^2 is the variance of ln(K)



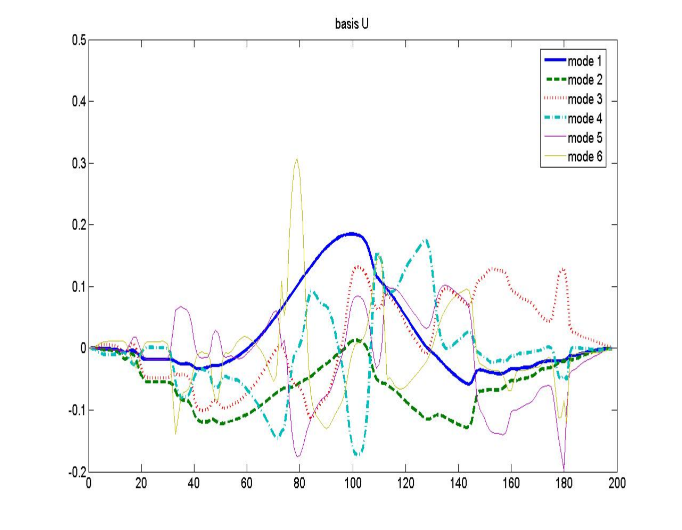
Schwarz DDM : random distribution of K along the interfaces

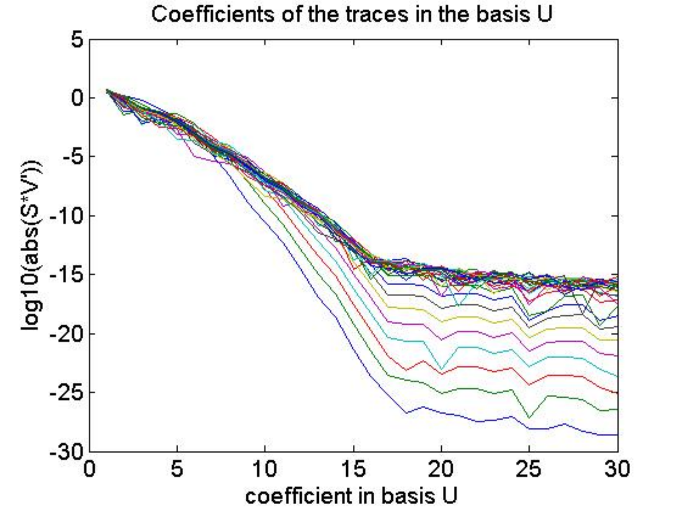


Singular values of the SVD of the Schwarz iterates on Γ_1

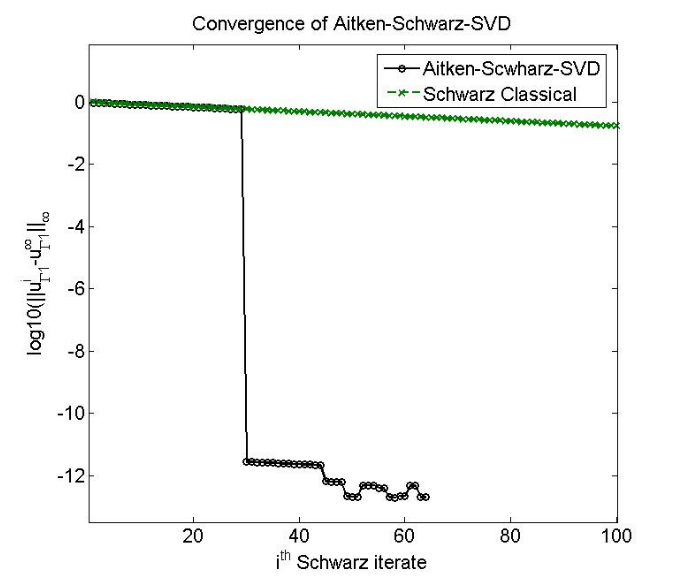


Basis U of the SVD of the Schwarz iterates on Γ_1




Coefficients of the traces in the basis U
log10(abs(S*V'))
coefficient in basis U
5
0
-5
-10
-15
-20
-25
-30
0
5
10
15
20
25
30

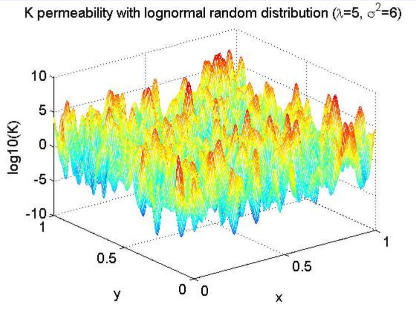
Convergence of the Aitken-Schwarz SVD

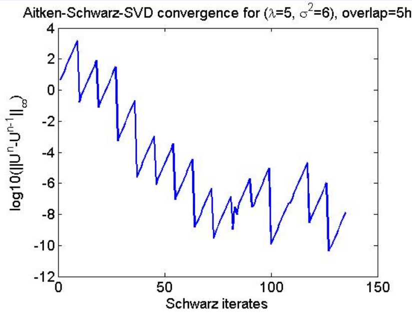


only

16 modes are used in the acceleration process

Convergence of AS with acceleration based on SVD





Conclusions

- The two main features for Aitken acceleration are orthogonal basis with decreasing coefficients for the representation of the traces in this basis.
- It works very well when this basis link to the mesh on interfacial interface is available
- SVD decomposition as the right properties without the drawback to be link to the underlying mesh.
- Parallel implementation of Aitken-Schwarz with SVD is under progress in the framework of MICAS project for large computational domain.

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