Homogenization of a Pseudoparabolic System

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Outline

Richards Equation

Two-phase flow through a partially-saturated porous medium with *porosity* $\phi(x)$, *permeability* $K(x)$, relative permeability $k_w(u)$ and *capillary pressure function* $P_c(u)$:

$$\phi(x)\frac{\partial u}{\partial t} - \nabla \cdot K(x)\frac{k_w(u)}{\mu_w}\nabla\left(P_c(u) + \rho G d(x)\right) = 0,$$

$u(x,t)$ denotes saturation, and gravitational effects depend on depth $d(x) = x_3$.

Dynamic Capillary Pressure

Experimental determination of $p = P_c(u)$ is based on the assumption that this is an instantaneous process. In reality it requires substantial time to approach an equilibrium before measurements can be taken.

Hassanizadeh-Gray (1993) model

$P_{c,dyn}(u) \equiv P_c(u) + \tau_H \frac{\partial u}{\partial t}$:

$$\phi(x)\frac{\partial u}{\partial t} - \nabla \cdot K(x)\frac{k_w(u)}{\mu_w}\nabla\left(P_c(u) + \rho G d(x)\right) - \nabla \cdot K(x)\frac{k_w(u)}{\mu_w}\nabla \tau_H \frac{\partial u}{\partial t} = 0\,.$$

pseudoparabolic equation

Linearize ... the *pseudoparabolic equation*

$$\frac{\partial}{\partial t}(\phi(x)u(t,x)) - \nabla \cdot \kappa(x)\nabla\left(u(t,x) + \tau(x)\frac{\partial}{\partial t}\phi(x)u(t,x)\right) = 0$$

is distinguished from the usual parabolic equation by $\tau(x) > 0$. Porous media applications require that we know how to homogenize such equations.

Bensoussan, Lions, and Papanicolaou briefly investigated the homogenization of pseudoparabolic equations as an example for which the limiting problem is of a different type, and perhaps *non-local*, not even a PDE. We shall see below that this occurs when certain variables are eliminated or *hidden*.

pseudoparabolic system

$$\frac{\partial}{\partial t}\left(\phi(x)u(t,x)\right) + \frac{1}{\tau(x)}\left(u(t,x) - v(t,x)\right) = 0,$$

$$-\nabla \cdot \left(\kappa(x)\nabla v(t,x)\right) + \frac{1}{\tau(x)}\left(v(t,x) - u(t,x)\right) = 0, \quad x \in \Omega,$$

$$v(t,s) = 0,\ s \in \partial\Omega,$$

$$\phi(x)u(0,x) = \phi(x)u_0(x),\ x \in \Omega.$$

Asymptotic Expansion

Let Y denote the unit cube in $\mathbb{R}^N$. Let the Y-periodic functions $\phi(y),\ \tau(y), \kappa(y)$ be given and define $\phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon}),\ \tau^\varepsilon(x) = \tau(\frac{x}{\varepsilon}),\ \kappa^\varepsilon(x) = \kappa(\frac{x}{\varepsilon})$.
The corresponding solution $u^\varepsilon,\ v^\varepsilon$ depends on ε.
We write these as formal asymptotic expansions

$$u^\varepsilon(t,x) = \sum_{p=0}^{\infty} \varepsilon^p u_p(t,x,y), \quad v^\varepsilon(t,x) = \sum_{p=0}^{\infty} \varepsilon^p v_p(t,x,y),$$

$$y = \frac{x}{\varepsilon},$$

with each $u_p(t,x,\cdot),\ v_p(t,x,\cdot)$ being Y-periodic.

Cell problem

The effective tensor κ^* is obtained in this calculation as $\kappa^*_{ij} = \int_Y \kappa(y)(\nabla_y \omega_i(y) + \mathbf{e}_i) \cdot (\nabla_y \omega_j(y) + \mathbf{e}_j)\, dy$, where

Periodic Cell Problem: ω_j is Y-periodic and

$$-\nabla_y \cdot \kappa(y)\,(\nabla_y \omega_j(y) + \mathbf{e}_j) = 0,\ j = 1 \ldots N.$$

partially-upscaled system

The leading terms in the expansion satisfy the pseudoparabolic system

$$\phi(y)\frac{\partial u_0(t,x,y)}{\partial t} + \frac{1}{\tau(y)}(u_0(t,x,y) - v_0(t,x)) = 0,$$

$$-\nabla \cdot \kappa^* \nabla v_0(t,x) + \int_Y \frac{1}{\tau(y)}(v_0(t,x) - u_0(t,x,y))dy = 0,$$

together with boundary and initial conditons,

$$v_0(t,s) = 0,\ s \in \partial\Omega, \quad u_0(0,x,y) = u_0(x).$$

Upscaled pseudoparabolic equation

Only if the product $\phi(\cdot)\,\tau(\cdot)$ is constant do we get $u_0(t,x,y) = u_0(t,x)$ independent of $y \in Y$, and in that case we can eliminate v_0 from the system:

$$\phi^* \frac{\partial u_0(t,x)}{\partial t} - \nabla \cdot \kappa^* \nabla u_0(t,x) - \nabla \cdot \kappa^* \nabla \phi^* \tau^* \frac{\partial u_0(t,x)}{\partial t} = 0\,.$$

NOTE: $\phi^* = \int_Y \phi(y)\,dy$ is the average
$\tau^* = \left(\int_Y \frac{1}{\tau(y)} dy\right)^{-1}$ is the harmonic average

Classical Bimodal Medium

Unit cube Y is given in open disjoint complementary parts, Y_1 and Y_2,
$\chi_j(y) =$ Y-periodic characteristic function of Y_j.
Corresponding ε-periodic characteristic functions are

$$\chi_j^\varepsilon(x) \equiv \chi_j\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N, \quad j = 1, 2,$$

and these partition the global domain Ω into two sub-domains, Ω_1^ε and Ω_2^ε by
$\Omega_j^\varepsilon \equiv \left\{x \in \Omega : \chi_j^\varepsilon(x) = 1\right\}, \, j = 1, 2.$

Coefficients

Given $\phi_j(\cdot,\cdot),\ \kappa_j(\cdot,\cdot),\ \tau_j(\cdot,\cdot) \in L^\infty(\Omega; C(\overline{Y_j}))$, define Y-periodic functions in $L^\infty(\Omega; L^2_\#(Y))$ by

$$\phi(x,y) \equiv \phi_j(x,y), y \in Y_j,\ j = 1,2, \quad x \in \Omega,$$

similarly $\kappa(x,y)$ and $\tau(x,y)$. Corresponding functions on Ω_j^ε are

$$\phi_j^\varepsilon(x) \equiv \phi_j\left(x, \frac{x}{\varepsilon}\right), \quad \kappa_j^\varepsilon(x) \equiv \kappa_j\left(x, \frac{x}{\varepsilon}\right), \quad \tau_j^\varepsilon(x) \equiv \tau_j\left(x, \frac{x}{\varepsilon}\right),$$

and coefficients for the pseudoparabolic system are

$$\begin{aligned}
\phi^\varepsilon(x) &\equiv \chi_1^\varepsilon(x)\phi_1^\varepsilon(x) + \chi_2^\varepsilon(x)\phi_2^\varepsilon(x), \\
\kappa^\varepsilon(x) &\equiv \chi_1^\varepsilon(x)\kappa_1^\varepsilon(x) + \chi_2^\varepsilon(x)\kappa_2^\varepsilon(x), \\
\tau^\varepsilon(x) &\equiv \chi_1^\varepsilon(x)\tau_1^\varepsilon(x) + \chi_2^\varepsilon(x)\tau_2^\varepsilon(x).
\end{aligned}$$

The ε- problem

$u^\varepsilon(\cdot) \in H^1((0,T);L^2(\Omega))$ and $v^\varepsilon(\cdot) \in L^2((0,T);H_0^1(\Omega))$

$$\phi^\varepsilon(x)\frac{\partial u^\varepsilon(t,x)}{\partial t} + \frac{1}{\tau^\varepsilon(x)}\left(u^\varepsilon(t,x) - v^\varepsilon(t,x)\right) = 0, \, x \in \Omega,$$

$$-\nabla \cdot \left(\kappa_1^\varepsilon(x)\nabla v^\varepsilon(t,x)\right) + \frac{1}{\tau_1^\varepsilon(x)}\left(v^\varepsilon(t,x) - u^\varepsilon(t,x)\right) = 0, \, x \in \Omega_1^\varepsilon,$$

$$-\nabla \cdot \left(\kappa_2^\varepsilon(x)\nabla v^\varepsilon(t,x)\right) + \frac{1}{\tau_2^\varepsilon(x)}\left(v^\varepsilon(t,x) - u^\varepsilon(t,x)\right) = 0, \, x \in \Omega_2^\varepsilon,$$

$$\gamma_1^\varepsilon v^\varepsilon(t,s) = \gamma_2^\varepsilon v^\varepsilon(t,s),$$

$$\kappa_1^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu = \kappa_2^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu, \, s \in \Gamma^\varepsilon,$$

boundary condition $v^\varepsilon(t,s) = 0, \; s \in \partial\Omega$, and the initial condition $u^\varepsilon(0,x) = u_0(x), \; x \in \Omega$, independent of ε.

two-scale limit

LEMMA 1: For each $\varepsilon > 0$, let $u^\varepsilon(\cdot),\ v^\varepsilon(\cdot)$ denote the unique solution to the pseudoparabolic ε-problem. There exist

(i) a function U in $L^2\big((0,T)\times\Omega; L^2_\#(Y)\big)$,

(ii) a function v in $L^2\big((0,T); H^1_0(\Omega)\big)$,

(ii) a function V in $L^2\big((0,T)\times\Omega; H^1_\#(Y)/\mathbb{R}\big)$,

and a subsequence which two-scale converges

$$u^\varepsilon \xrightarrow{2} U(t,x,y),$$

$$v^\varepsilon \xrightarrow{2} v(t,x),$$

$$\nabla v^\varepsilon \xrightarrow{2} \nabla v(t,x) + \nabla_y V(t,x,y).$$

The *effective tensor* κ^* is given by

$$\kappa^*_{ij}(x) = \int_Y \kappa(x,y)(\nabla_y \omega_i(x,y) + \mathbf{e}_i) \cdot (\nabla_y \omega_j(x,y) + \mathbf{e}_j)\, dy.$$

where each ω_k is the solution of the periodic cell problem

$$\omega_k \in L^2(\Omega; H^1_\#(Y)) :$$
$$\int_Y \kappa(x,y)\big(\nabla_y \omega_k(x,y) + \mathbf{e}_k\big) \cdot \nabla_y \Psi(x,y)\, dy = 0$$
$$\text{for all } \Psi \in L^2(\Omega; H^1_\#(Y)).$$

(Let's ask that $\int_Y \omega_k(x,y)\, dy = 0$ to fix the constant.)

THEOREM 1: The limits $U,\ v$ in Lemma 1 are the solution of the partially homogenized pseudoparabolic system

$$\phi(x,y)\frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau(x,y)}\big(U(t,x,y) - v(t,x)\big) = 0,$$

$$\int_Y \frac{1}{\tau(x,y)}\big(v(t,x) - U(t,x,y)\big)\,dy - \nabla \cdot \kappa^* \nabla v(t,x) = 0,$$

with boundary conditions $v(t,s) = 0,\ s \in \partial\Omega$,
initial condition $U(0,x,y) = u_0(x)$.

upscaled bimodal case

If each of $\phi_j,\ \tau_j \in L^\infty(\Omega)$ is independent of $y \in Y_j$, then

$$U(t,x,y) \equiv \begin{cases} U_1(t,x), & y \in Y_1\,, \\ U_2(t,x), & y \in Y_2\,, \end{cases}$$

and we have the homogenized bimodal system

$$|Y_1|\phi_1(x)\frac{\partial U_1(t,x)}{\partial t} + \frac{|Y_1|}{\tau_1(x)}\big(U_1(t,x) - v(t,x)\big) = 0\,,$$

$$|Y_2|\phi_2(x)\frac{\partial U_2(t,x)}{\partial t} + \frac{|Y_2|}{\tau_2(x)}\big(U_2(t,x) - v(t,x)\big) = 0\,,$$

$$\frac{|Y_1|}{\tau_1(x)}\big(v(t,x) - U_1(t,x)\big) + \frac{|Y_2|}{\tau_2(x)}\big(v(t,x) - U_2(t,x)\big) - \nabla\cdot\kappa^*\nabla v(t,x) = 0\,.$$

The Highly-Heterogeneous Case

The permeability is scaled by ε^2 in the second region Ω_2^ε, so the flux is given by $-\varepsilon^2\kappa_2\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon$ in Ω_2^ε:

$$\kappa^\varepsilon(x) \equiv \chi_1^\varepsilon(x)\kappa_1^\varepsilon(x) + \varepsilon^2\chi_2^\varepsilon(x)\kappa_2^\varepsilon(x).$$

The ε-problem

$$\phi^\varepsilon(x)\frac{\partial u^\varepsilon(t,x)}{\partial t} + \frac{1}{\tau^\varepsilon(x)}\left(u^\varepsilon(t,x) - v^\varepsilon(t,x)\right) = 0,\ x \in \Omega,$$

$$-\nabla\cdot\left(\kappa_1^\varepsilon(x)\nabla v^\varepsilon(t,x)\right) + \frac{1}{\tau_1^\varepsilon(x)}\left(v^\varepsilon(t,x) - u^\varepsilon(t,x)\right) = 0,\ x \in \Omega_1^\varepsilon,$$

$$-\nabla\cdot\left(\varepsilon^2\kappa_2^\varepsilon(x)\nabla v^\varepsilon(t,x)\right) + \frac{1}{\tau_2^\varepsilon(x)}\left(v^\varepsilon(t,x) - u^\varepsilon(t,x)\right) = 0,\ x \in \Omega_2^\varepsilon,$$

$$\gamma_1^\varepsilon v^\varepsilon(t,s) = \gamma_2^\varepsilon v^\varepsilon(t,s),$$

$$\kappa_1^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu = \varepsilon^2\kappa_2^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu,\ s \in \Gamma^\varepsilon.$$

boundary condition $v^\varepsilon(t,s) = 0,\ s \in \partial\Omega$, and the initial condition $u^\varepsilon(0,x) = u_0(x),\ x \in \Omega$, independent of ε.

The two-scale limit

LEMMA 2: There exist $U \in L^2\big((0,T) \times \Omega; L^2_\#(Y)\big)$,
$v_1 \in L^2\big((0,T); H^1_0(\Omega)\big)$,
$V_j \in L^2\big((0,T) \times \Omega; H^1_\#(Y_j)/\mathbb{R}\big)$, $j = 1, 2$,
and a two-scale convergent subsequence

$$u^\varepsilon(t,x) \xrightarrow{2} U(t,x,y),$$

$$\chi_1^\varepsilon v^\varepsilon \xrightarrow{2} \chi_1(y) v_1(t,x),$$

$$\chi_1^\varepsilon \nabla v^\varepsilon \xrightarrow{2} \chi_1(y)[\nabla v_1(t,x) + \nabla_y V_1(t,x,y)],$$

$$\chi_2^\varepsilon v^\varepsilon \xrightarrow{2} \chi_2(y) V_2(t,x,y),$$

$$\varepsilon \chi_2^\varepsilon \nabla v^\varepsilon \xrightarrow{2} \chi_2(y) \nabla_y V_2(t,x,y).$$

The Cell Problem

Define $\omega_k(x, y)$ by

$$\omega_k \in L^2(\Omega; H^1_\#(Y_1)) : \quad \int_{Y_1} \omega_k(x,y)\,dy = 0,$$

$$\int_{Y_1} \kappa_1(x,y)\big(\nabla_y \omega_k(x,y) + \mathbf{e}_k\big) \cdot \nabla_y \Psi_1(x,y)\,dy = 0$$

$$\text{for all } \Psi_1 \in L^2(\Omega; H^1_\#(Y_1)).$$

The effective tensor κ^* is given by

$$\kappa^*_{ij}(x) = \int_{Y_1} \kappa_1(x,y)(\nabla_y \omega_i(x,y) + \mathbf{e}_i) \cdot (\nabla_y \omega_j(x\ ,y) + \mathbf{e}_j)\,dy.$$

THEOREM 2: The limits U, v_1, V_2 satisfy the partially homogenized pseudoparabolic system

$$\phi_1(x,y)\frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau_1(x,y)}\big(U(t,x,y) - v_1(t,x)\big) = 0,\ y \in Y_1,$$

$$\int_{Y_1} \frac{1}{\tau_1(x,y)}\big(v_1(t,x) - U(t,x,y)\big)\,dy - \nabla\cdot\kappa^*\nabla v_1(t,x)$$

$$+ \int_\Gamma \kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot\nu\,dS = 0,$$

and for each $x \in \Omega$ and $y \in Y_2$

$$\phi_2(x,y)\frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau_2(x,y)}\big(U(t,x,y) - V_2(t,x,y)\big) = 0,$$

$$\frac{1}{\tau_2(x,y)}\big(V_2(t,x,y) - U(t,x,y)\big) - \nabla_y\cdot\kappa_2(x,y)\nabla_y V_2(t,x,y) = 0,$$

with $\gamma V_2(t,x,y) = v_1(t,x),\ y \in \Gamma.$

The Macro-Micro model

If $\phi_1(x)$, $\tau_1(x)$, then $u(t,x) \equiv U(t,x,y),\ y \in Y_1$,

$$\phi_1(x)\frac{\partial u(t,x)}{\partial t} + \frac{1}{\tau_1(x)}\left(u(t,x) - v_1(t,x)\right) = 0\,,$$

$$\frac{1}{\tau_1(x)}\left(v_1(t,x) - u(t,x)\right) - \frac{1}{|Y_1|}\nabla \cdot \kappa^* \nabla v_1(t,x)$$

$$+\frac{1}{|Y_1|}\int_\Gamma \kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot \nu\, dS = 0,$$

$$\phi_2(x,y)\frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau_2(x,y)}\left(U(t,x,y) - V_2(t,x,y)\right) = 0,$$

$$\frac{1}{\tau_2(x,y)}\left(V_2(t,x,y) - U(t,x,y)\right) - \nabla_y \cdot \kappa_2(x,y)\nabla_y V_2(t,x,y) = 0,$$

for $y \in Y_2$, with $\gamma V_2(t,x,y) = v_1(t,x),\ y \in \Gamma.$