# Numerical Zoom and Domain Decomposition

**http://www.ann.jussieu.fr/pironneau**

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# The Site of Bure

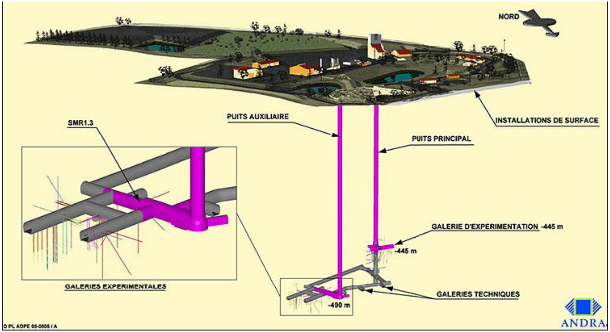



Figure: Schematic view of the Bure project (East of France)

Nuclear waste is cooled, processed, then buried safely for 1M years
Simulation requires a super computer, or does it really?

# The COUPLEX I Test Case

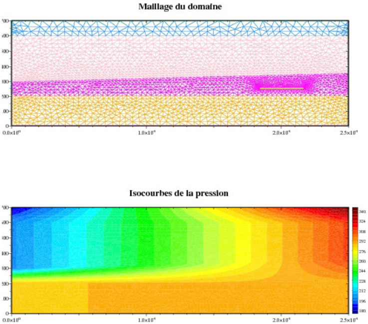



Figure: A 2D multilayered geometry 20km long, 500m high with permeability variations $\frac{K^+}{K^-} = O(10^9)$. Hydrostatic pressure by a FEM.

$$\nabla \cdot (K \nabla H) = 0, \quad H \text{ or } \frac{\partial H}{\partial n} \text{ given on } \Gamma$$

# COUPLEX I : Concentration of Radio-Nucleides

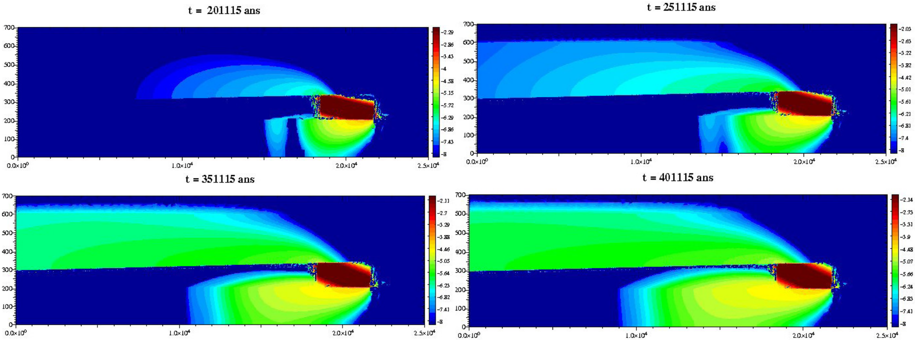
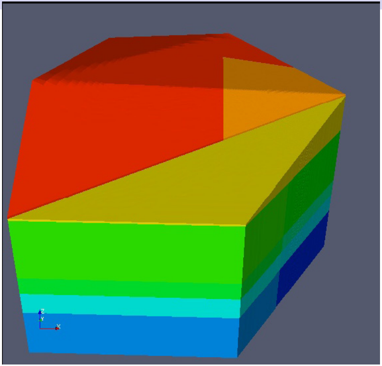



Figure: Concentration at 4 times with Discontinuous Galerkin FEM (Apoung-Despré).

$$r\partial_t c + \lambda c + u\nabla c - \nabla\cdot(K\nabla c) = q(t)\delta(x - x_R)$$

# Couplex II: Geological figures

| Layer | Permeability |
|---|---|
| Tithonien | $3.10^{-5}$ |
| Kimmeridgien I | $3.10^{-4}$ |
| Kimmeridgien II | $10^{-12}$ |
| Oxfordien I | $2.10^{-7}$ |
| Oxfordien II | $8.10^{-9}$ |
| Oxfordien III | $4.10^{-12}$ |
| Callovo-Oxfordien | $10^{-13}$ |
| Dogger | $2.510^{-6}$ |



Layer decomposition: $K^+\frac{\partial H^+}{\partial n} = K^-\frac{\partial H^-}{\partial n}$ implies that $\frac{\partial H^+}{\partial n} = O(\frac{K^-}{K^+})$.
So $\frac{\partial H}{\partial n}|_{KI-KII} \approx 0$ is a B.C. that decouples the top from the bottom.
Later $H^-|_{KII} = H^+$ is used as B.C for the bottom.
Note that the Callovo-Oxfordian+Oxfordian III have $H|_\Gamma$ given from top and bottom separate calculations.

# COUPLEX II Hydrostatic Pressure

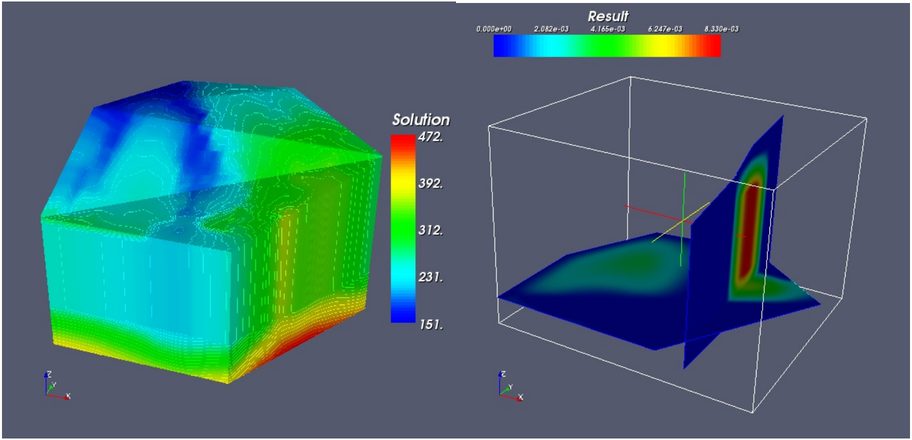



Figure: Final result and comparison with a global solution on a supercomputer (Apoung)

# The Clay Layer with the repository

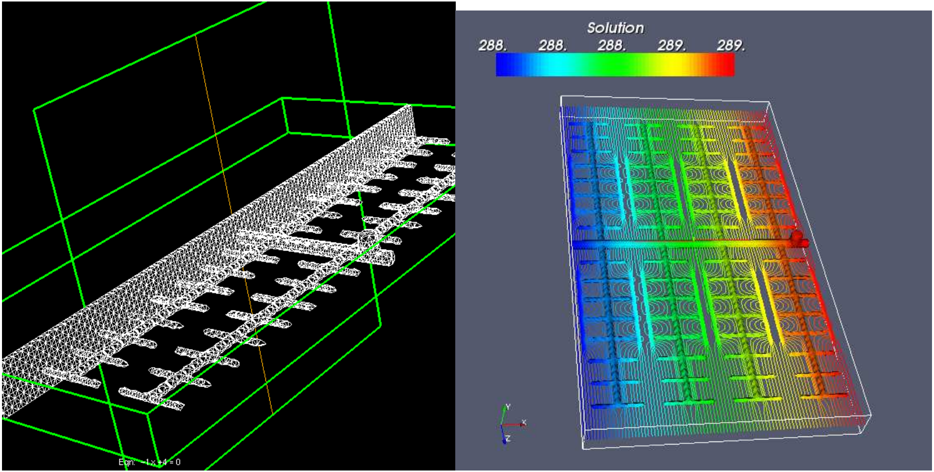



Figure: A computation within the clay layer only with Dirichlet B.C. from the surrounding layers (Apoung-Delpino). Left: a geometrical zoom

# First Numerical Zoom

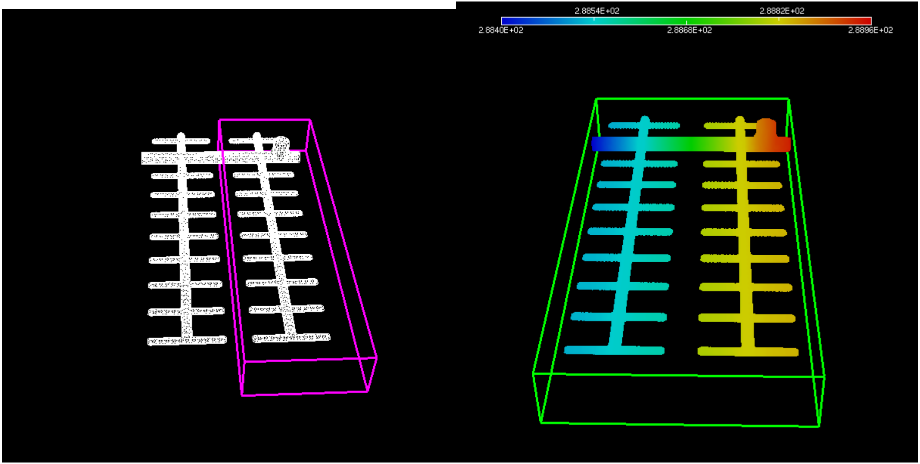



Figure: Mesh and Sol of Darcy's in a portion of the entire site.

# Second Zoom

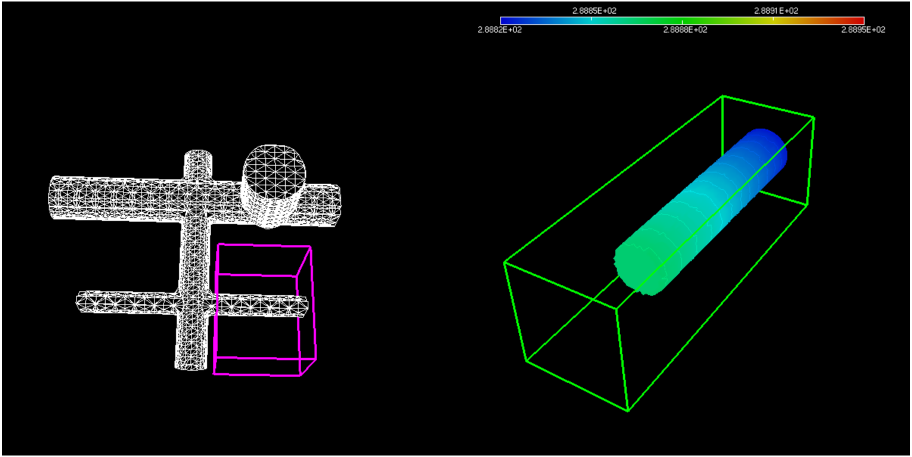
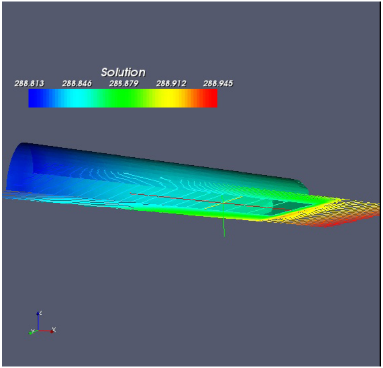
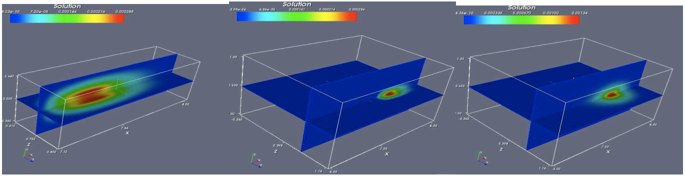



Figure: Mesh and Sol around a single gallery capable of evaluating the impact of a lining around the gallery.

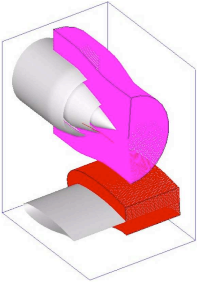
# Last Zoom and upscale comp. of the concentration

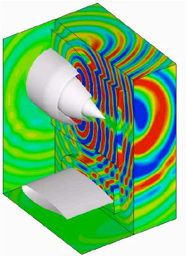


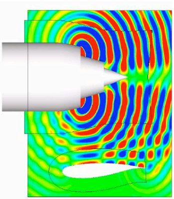



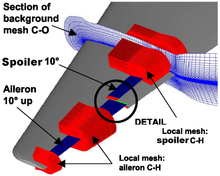

What are the errors in the end?

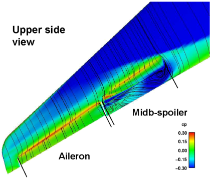
# Other Examples: What are the errors in the end?









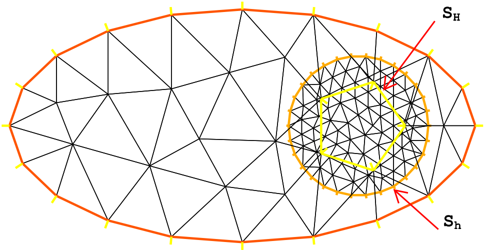
# Why Numerical Zoom

The dream is to combine graphical zoom and numerical zoom.

- Numerical zoom are needed when it is very expensive or impossible to solve the full problem
- For instance if the problem has multiple scales
- Improved precision may be found necessary a posteriori

- Numerical zoom methods exist:
  - Steger's Chimera method,
  - J.L. Lions's Hilbert space decomposition (HSD),
  - Glowinski-He-Rappaz-Wagner's Subspace correction methods (SCM), etc.
- The 3 methods are really the same: Schwarz-Hilbert Enrichment (SHE).
- We need error estimates .

# The Schwarz-Zoom Method




Find $u_H^{m+1} \in V_H,\ u_h^{m+1} \in V_h$, such that $\forall w_H \in V_{0H},\ \ \forall w_h \in V_{0h}$

$$a_H(u_H^{m+1}, w_H) = (f, w_H), \quad u_H^{m+1}|_{S_H} = \gamma_H u_h^m, \quad u_H^{m+1}|_{\Gamma_H} = g_H,$$
$$a_h(u_h^{m+1}, w_h) = (f, w_h), \quad u_h^{m+1}|_{S_h} = \gamma_h u_H^m, \quad u_h^{m+1}|_{\Gamma_h} = g_h$$

where $\gamma_H$ (resp $\gamma_h$) is the interpolation operator on $V_H$ (resp $V_h$), where $S_H$and $\Gamma_H$ are the polygonal approximation of $S_1$ and $\Gamma_1$ and similarly for $S_h, \Gamma_h$ with $S_2, \Gamma_2$.

# Convergence of Discrete Schwarz-Zoom Method

**Hypothesis 1** Assume that the maximum principle holds for each system independently and that the solution $\nu_H \in V_H$ of

$$a_H(\nu_H, w_H) = 0, \ \ \forall w_H \in V_{0H}, \ \ \nu_H|_{S_H} = 1, \ \ \nu_H|_{\Gamma_H} = 0$$

satisfies $|\nu_H|_{\infty,S_h} := \lambda < 1$.

**Theorem** Then the discrete Schwarz algorithm converges to:

$$a_H(u_H^*, w_H) = (f, w_H), \ \ \forall w_H \in V_{0H}, \ \ u_H^*|_{S_H} = \gamma_H u_h^*, \ \ u_H^*|_{\Gamma_H} = g_H$$
$$a_h(u_h^*, w_h) = (f, w_h), \ \ \forall w_h \in V_{0h}, \ \ u_h^*|_{S_h} = \gamma_h u_H^*$$

and

$$\max(||u_H^* - u||_{\infty,\Omega_H}, ||u_h^* - u||_{\infty,\Omega_h})$$
$$\leq C(H^2 \log \frac{1}{H} ||u||_{H^{2,\infty}(\Omega_H)} + h^2 \log \frac{1}{h} ||u||_{H^{2,\infty}(\Omega_h)}) \quad (1)$$

see also X.C. Cai and M. Dryja and M. Sarkis (SIAM 99)

# Proof of Convergence

By the maximum principle and the fact that $\gamma_H$ and $\gamma_h$ decrease the $L^\infty$ norms, problems of the type: find $v_H \in V_H,\ v_h \in V_h$

$$a_H(v_H, w_H) = 0,\ \ \forall w_H \in V_{0H},\ \ v_H|_{S_H} = \gamma_H u_h,\ \ v_H^{m+1}|_{\Gamma_H} = 0$$
$$a_h(v_h, w_h) = 0,\ \ \forall w_h \in V_{0h},\ \ v_h^{m+1}|_{S_h} = \gamma_h v_H$$

satisfy

$$\|v_H\|_\infty \le \|u_h\|_{\infty,S_H}, \quad \|v_h\|_\infty \le \|v_H\|_{\infty,S_h}.$$

Combining this with the estimate on the solution of (1) we obtain

$$\|v_h\|_\infty \le \|v_H\|_{\infty,S_h} \le \lambda\|v_H\|_\infty \le \lambda\|u_h\|_\infty.$$

# Proof of Error estimate (I of II)

The solution $u$ to the continuous problem satisfies $u|_\Gamma = g$ and

$$a_H(u,w) = (f,w) \ \ \forall w \in H^1_0(\Omega_H), \ \ u = \gamma_H u + (u - \gamma_H u) \text{ on } S_H,$$
$$a_h(u,w) = (f,w) \ \ \forall w \in H^1_0(\Omega_h), \ \ u = \gamma_h u + (u - \gamma_h u) \text{ on } S_h$$

Let $e = u^*_H - u$ and $\varepsilon = u^*_h - u$. Setting $w = w_H$ in the first equation and $w = w_h$ in the second, we have

$$a_H(e,w_H) = 0 \ \ \forall w_H \in V_{0H}, \ \ e = \gamma_H \varepsilon - (u - \gamma_H u) \text{ on } S_H, \ \ e|_\Gamma = g_H - g$$
$$a_h(\varepsilon,w_h) = 0 \ \ \forall w_h \in V_{0h}, \ \ \varepsilon = \gamma_h e - (u - \gamma_h u) \text{ on } S_h$$

Let $\Pi_H u \in V_H$ and $\Pi_h u \in V_h$ be the solutions of

$$a_H(\Pi_H u, w_H) = a_H(u, w_H) \ \ \forall w_H \in V_{0H}, \ \ \Pi_H u = \gamma_H u \text{ on } S_H, \ \Pi_H u|_\Gamma = g_H$$
$$a_h(\Pi_h u, w_h) = a_h(u, w_h) \ \ \forall w_h \in V_{0h}, \ \ \Pi_h u = \gamma_h u \text{ on } S_h$$

By Schatz& Wahlbin, we have

$$||\Pi_H u - u||_{\infty,\Omega_H} \leq H^2 \log\frac{1}{H} ||u||_{H^{2,\infty}(\Omega_H)},$$
$$||\Pi_h - u||_{\infty,\Omega_h} \leq h^2 \log\frac{1}{h} ||u||_{H^{2,\infty}(\Omega_h)}.$$

# Proof of Error estimate (II)

Finally let

$$\varepsilon_H = u_H - \Pi_H u = e + u - \Pi_H u, \quad \varepsilon_h = u_h - \Pi_h u = \varepsilon + u - \Pi_h u$$

Then $\varepsilon_H \in V_H$, $\varepsilon_h \in V_h$ and

$$a_H(\varepsilon_H, w_H) = 0 \ \ \forall w_H \in V_{0H}, \ \ \varepsilon_H = \gamma_H(\varepsilon_h + \Pi_h u - u) \text{ on } S_H, \ \ \varepsilon_H|_\Gamma = 0$$
$$a_h(\varepsilon_h, w_h) = 0 \ \ \forall w_h \in V_{0h}, \ \ \varepsilon_h = \gamma_h(\varepsilon_H + \Pi_H u - u) \text{ on } S_h$$

The maximum principle (like in (2) and (2)) again yields

$$\|\varepsilon_H\|_\infty \le \|\Pi_h u - u\|_{\infty,S_H} + \|\varepsilon_h\|_{\infty,S_H},$$
$$\|\varepsilon_h\|_\infty \le \|\Pi_H u - u\|_{\infty,S_h} + \|\varepsilon_H\|_{\infty,S_h},$$
$$\|\varepsilon_H\|_{\infty,S_h} \le \lambda\|\varepsilon_H\|_\infty$$

Therefore

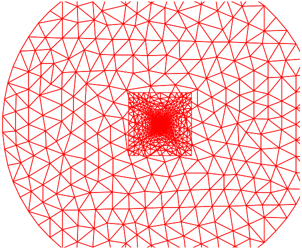
$$\max(\|\varepsilon_h\|_\infty, \|\varepsilon_H\|_\infty) \le \frac{1}{1-\lambda}(\|\Pi_H u - u\|_{\infty,\Omega_H} + \|\Pi_h u - u\|_{\infty,\Omega_h})$$

All would be well if Schwarz didn't require to dig a hole in the zoom.

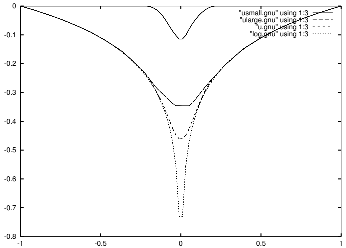
$$u \in V \;:\; a(u,v) = < f|v > \quad \forall v \in V$$

If $V_H$ is not rich enough, use $V_H + V_h$ and solve $u_H \in V_H,\; u_h \in V_h$ :

$$a(u_H + u_h, v_H + v_h) = < f|v_H + v_h > \quad \forall v_H \in V_H,\; v_h \in V_h$$



$f = 1 + \delta_0$




If solved iteratively, it is similar to Schwarz'DDM or Steger's Chimera at the continuous level: when $\Omega_1 \cup \Omega_2 = \Omega$, $\Omega_1 \cap \Omega_2 \neq \emptyset$.

# Discretization and Proof of Uniqueness (Brezzi)

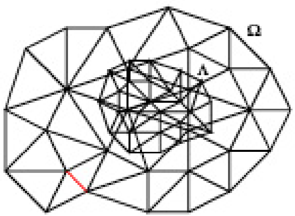
Find $U_H \in V_{0H} \approx H_0^1(\Omega)$, $u_h \in V_{0h} \approx H_0^1(\Lambda)$

$$a(U_H + u_h, W_H + w_h) = < f | W_H + w_h > \quad \forall W_H \in V_{0H} \;\; \forall w_h \in V_{0h}$$

**Theorem** The solution is unique if no vertex belong to both triangulations.

**Proof**

If $u_h = U_H$ on $\Lambda$ then they are linear on $\Lambda$ because $\Delta u_h = \Delta U_H$ and each is a distribution on the edges. The only singularity, if any, are at the intersection of both set of edges (which are points), but being in $H^{-1}$ it cannot be singular at isolated points. So $\Delta u_h = \Delta U_H|_\Lambda = 0$



# Subspace Correction Method (SCM)

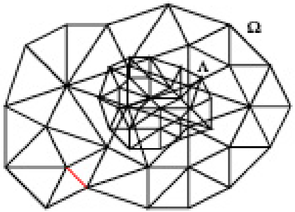
Find $U_H \in V_{0H} \approx H_0^1(\Omega)$, $u_h \in V_{0h} \approx H_0^1(\Lambda)$

$$a(U_H + u_h, W_H + w_h) = < f | W_H + w_h > \quad \forall W_H \in V_{0H} \;\; \forall w_h \in V_{0h}$$

**Theorem (Lozinski et al)**

If $u_H$ is computed with FEM of degree r and $u_h$ with FEM of degree s, then with $q = \max\{r, s\} + 1$,

$$\|u_H + u_h - u\|_1 \leq c(H^r \|u\|_{H^q(\Omega\backslash\Lambda)} + h^s \|u\|_{H^q(\Lambda)})$$




Iterative process? Inexact quadrature?

# Hilbert Space Decomposition with Inexact Quadrature

$$a_h(u_1+u_2,w_1+w_2)=a_h(u_1,w_1)+a_h(u_2,w_2)+a_h(u_1,w_2)+a_h(u_2,w_1)$$

2 grids: $\{T_k^1\}$ $\{T_k^2\}$ $a_h(u,v)=\sum_k\sum_{j=1..3}\frac{|T_k^1|}{3}\frac{\nabla u\cdot\nabla v}{I_{\Omega^1}+I_{\Omega^2}}|_{\xi_{jk}^1}$ + id with $T_k^2$

The gradients are computed on their native grids at vertices $\xi$.

Proposition *When vertices of $\mathcal{T}^i$ are strictly inside the $T^j$ the discrete Solution is unique and* $\|u_h^1+u_h^2-u\|_1\le\frac{c}{C}h(\|u^1\|_2+\|u^2\|_2)$

| | | $u-(u_1+u_2)$ | | |
|---|---|---|---|---|
| $N1$ | $L^2$ error | rate | $\nabla L^2$ error | rate |
| 10 | $1.696E-02$ | – | $2.394E-01$ | – |
| 20 | $5.044E-03$ | 1.75 | $1.204E-01$ | 0.99 |
| 40 | $1.129E-03$ | 2.16 | $5.596E-02$ | 1.10 |

Table: Numerical $L^2$ and $H^1$ errors, and convergence rate. Results are sensitive to rotation and translation of inner mesh

# Harmonic Patch Iterator for Speed-up (Lozinski)

Proximity of vertices could lead to drastically slow convergence $\Rightarrow$

**for** $n = 1...N$ **do**

Find $\lambda_H^n \in V_H^0 = \{v_H \in V_{0H} : \mathrm{supp}\ v_H \subset \Lambda\}$ such that

$$a(\lambda_H^n, \mu) = \langle f|v\rangle - a(u_h^{n-1}, \mu), \quad \forall \mu \in V_{0H}$$

Find $u_H^n \in V_{0H}$ such that

$$a(u_H^n, v) = \langle f|v\rangle - a(u_h^{n-1}, v) - a(\lambda_H^n, v), \quad \forall v \in V_{0H}$$

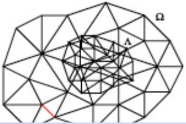
Find $u_h^n \in V_{0h}$ such that

$$a(u_h^n, v) = \langle f|v\rangle - a(u_H^{n-1}, v), \quad \forall v \in V_{0h}$$

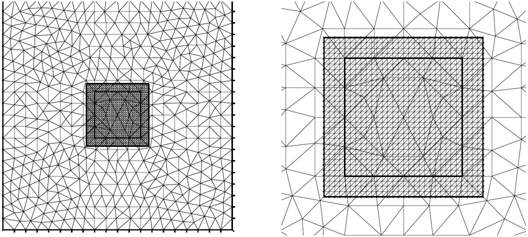
Set $u_{Hh}^n = u_H^n + u_h^n$

**end for**

Note: with $\tilde{u}_h^{n-1} = u_h^{n-1} + \lambda_H^n$ is it Schwarz?



# Harmonic Patches



| 1/10 | | 1/20 | | 1/40 | |
|---|---|---|---|---|---|
| $H^1$ | $L^2$ | $H^1$ | $L^2$ | $H^1$ | $L^2$ |
| 1.00 | 8.50E−1 | 9.98E−1 | 8.48E−1 | 9.98E−1 | 8.49E−1 |
| 1.03E−2 | 6.18E−3 | 2.18E−3 | 1.25E−3 | 6.08E−4 | 4.05E−4 |
| 1.01E−2 | 5.22E−3 | 2.36E−3 | 1.27E−3 | 6.42E−4 | 4.36E−4 |
| 8.93E−3 | 4.79E−3 | 2.10E−3 | 1.11E−3 | 5.56E−4 | 3.01E−4 |
| 9.40E−3 | 5.09E−3 | 2.16E−3 | 1.17E−3 | 5.91E−4 | 3.72E−4 |
| 8.72E−3 | 4.89E−3 | 2.09E−3 | 1.09E−3 | 5.51E−4 | 2.87E−4 |
| 11 | | 4 | | 3 | |
| 0.8236 | | 0.9339 | | 0.9698 | |

# Discrete one way Schwarz

If the $\Lambda_h$ is a submesh of $\Omega_H$ then the same algorithm is:

1: **for** $n = 1...N$ **do**

2: Find $u_H^n - g_H \in V_{0H}$ such that

$$a(u_H^n, v) = \langle f|v\rangle - a_h(w_h^{n-1}, v) + a_\Lambda(u_H^{n-1}, v), \quad \forall v \in V_{0H}$$

3: Find $w_h^n \in V_h$ such that ($r_h$ is a trace interpolation operator)

$$a(w_h^n, v) = \langle f|v\rangle, \quad \forall v \in V_{0h}, \quad w_h^n|_{\partial\Lambda} = r_h u_H^n|_{\partial\Lambda}$$

4: **end for**

5: Set

$$u_{Hh}^n = \begin{cases} w_h^n, & \text{in } \Lambda \\ u_H^n, & \text{outside } \Lambda \end{cases}$$

# Implementation in 2D with freefem++ (F. Hecht)

http://www.freefem.org

*// embedded meshes with keyword splitmesh*

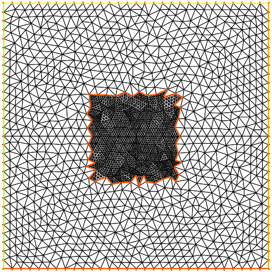
```
int n=10, m=4;
real x0=0.33,y0=0.33,x1=0.66,y1=0.66;
mesh TH=square(n,n);
mesh Th = splitmesh(TH,(x>x0 && x<x1 && y>y0 && y<y1)*m);
mesh THh=splitmesh(TH,1+(x>x0&&x<x1&&y>y0&& y<y1)*(m-1));

solve aH(U,V) = int2d(TH)(K*(dx(U)*dx(V)+dy(U)*dy(V)))
+ int2d(Th)(K*(dx(u)*dx(V)+dy(u)*dy(V)))
- int2d(THh)(K*(dx(Uold)*dx(V)+dy(Uold)*dy(V)))
- int2d(TH)(f*V) + on(dOmega, U=g);
```

# 2D Academic case

$K = 1$ except in a Disk 0.1 in the center where $K = 100$:

$$u = y - \frac{1}{2}, \text{ in the disk } = -\frac{1+K}{4} - \frac{(1-K)\delta^2}{4(x^2+y^2)} \text{ elsewhere} \qquad (2)$$



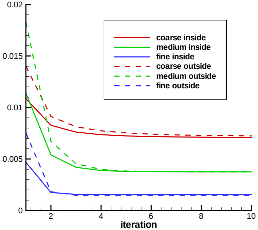



Figure: The initial mesh $\Omega_H$ is is divided 4 times in the zoom. Convergence history for 3 different initial meshes of the unit square: a coarse, medium (documented in the text) and fine mesh. 3 curves correspond to the errors on the mesh $H$ and 3 for the mesh $h$.

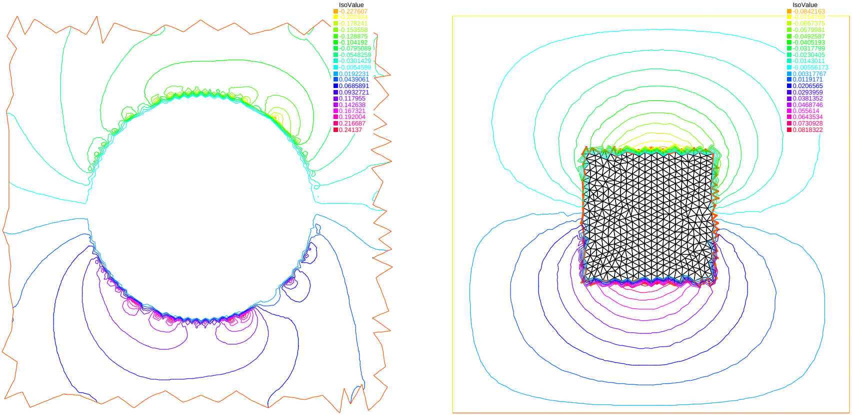
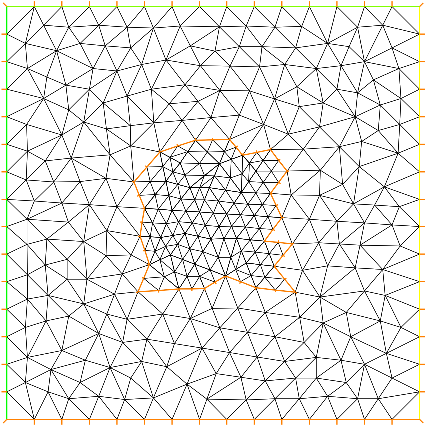
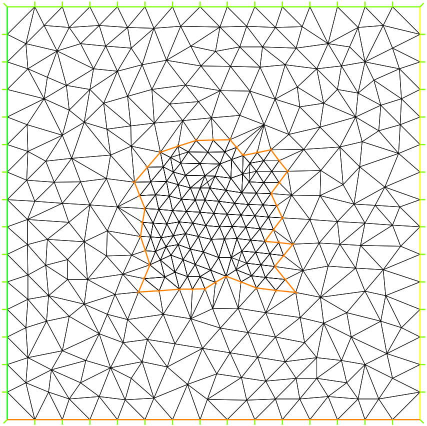



Figure: Error at each point for the converge solution in $\Lambda$ (left) and outside (right) $\Lambda$ on the fine mesh of Fig. The color scales from -0.23 to 0.24 on the left and from -0.08 to 0.08 on the right.

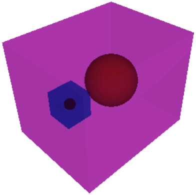
# Embedded Meshes: Relation with Schwarz' DDM

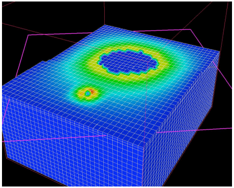


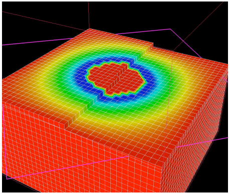


Left: Divide the Triangles which have a vertex in $(.33, .66)^2 \Rightarrow$ not a valid mesh. Right: a valid mesh is obtained by joining the hanging vertices to their opposite vertex.

# Comparison with Schwarz: 3D academic case







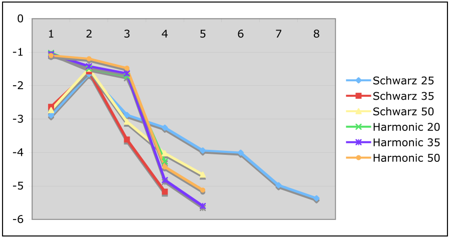



Figure: Zoom around the small sphere, view of the solution and zoom

# freefem3d (S. Delpino) + medit (P. Frey)

```
vector n = (50,50,50);
vector a = (-2,-2,-2), b = (3,2,2), c = (2.2,-0.3,-0.3), d = (1.7,0.3,0.3);
scene S = pov("test.pov");
mesh M = structured(n,a,b);
domain O = domain(S,outside(<1,0,0>)and outside(<0,1,0>));
mesh L = structured(n,c,d);
domain P = domain(S,outside(<0,1,0>));
femfunction u(M)=0, v(L)=0, uold(L)=0;
double err;
do{
  solve(u) in O by M{
  pde(u)  - div(grad(u)) =0; u = 0 on M; u = 1 on <1,0,0>; u = v on <0,1,0>;
  };
  solve(v) in P by L{
   pde(v)     - div(grad(v)) = 0;     v = -1 on <0,1,0>;     v = u on L;
  };
  err = int[L] ((u-uold)^2); uold =u;
}while{err>3e-5);
```

Table: Convergence error on the zoom variable for Couplex

| Schwarz 25 | Schwarz 35 | Schwarz 50 | SHE 20 | SHE 35 | SHE 50 |
|---|---|---|---|---|---|
| 1.297E-3 | 2.319E-3 | 1.890E-3 | 9.477E-2 | 8.766E-2 | 7.928E-2 |
| 2.209E-2 | 2.653E-2 | 3.189E-2 | 3.225E-02 | 3.782E-02 | 6.345E-02 |
| 1.321E-3 | 2.441E-4 | 8.320E-4 | 1.899E-2 | 2.309E-3 | 3.316E-2 |
| 5.519E-4 | 6.745E-06 | 9.425E-05 | 5.403E-05 | 1.504E-05 | 3.723E-05 |
| 1.146E-4 | | 2.184E-05 | | 2.521E-06 | 7.525E-06 |
| 9.885E-05 | | | | | |
| 1.055E-05 | | | | | |

# Comparison with Schwarz for Couplex

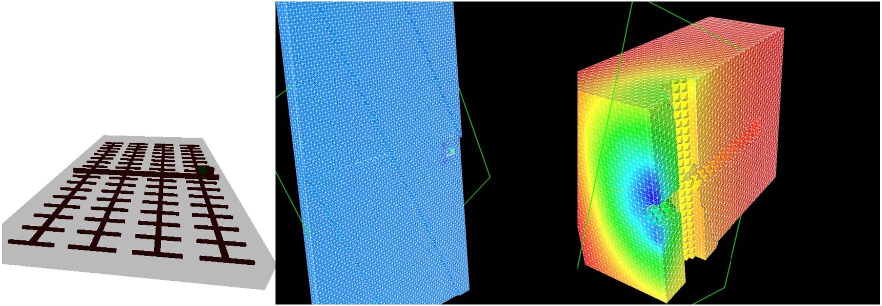


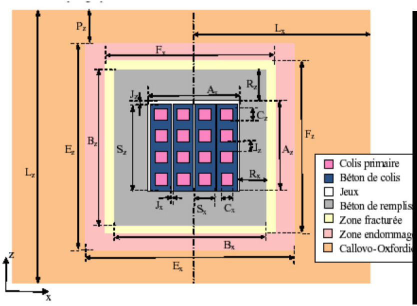
Figure: $U_H$ and $U_H - (xy + 20)$.

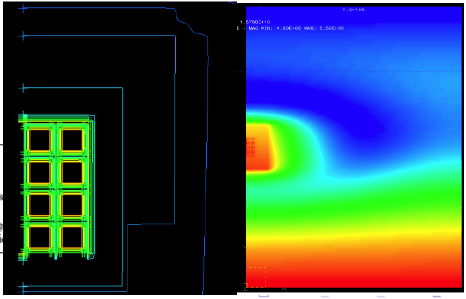
| | | | | | |
|---|---|---|---|---|---|
| Schwarz | 227.383 | 86.0596 | 6.42153 | 0.199725 | 0.0070609 |
| SHE | 507.434 | 0.015881 | 0.0030023 | 0.0013834 | 0.00096568 |

Table: Convergence

# Conclusion

- Numerical zooms are inevitable
- Precision: given by GHLR.
- With embedded meshes:
  - similar to DDM
  - convergence similar to full overlapping Schwarz
- Advice to code developer: since DDM is built in due to computer architecture why not add the zoom facility also!



# Bibliography on www.ann.jussieu.fr/pironneau


- Apoung-Kamga J.B. and J.L., Pironneau : O. Numerical zoom. DDM16 conf. proc, New-York Jan 2005. D. Keyes ed.
- Brezzi,F., Lions, J.L., Pironneau, O. : Analysis of a Chimera Method. C.R.A.S., **332**, 655-660, (2001).
- P. Frey: `medit`, `http://www.ann.jussieu.fr/~frey`
- R. Glowinski, J. He, A. Lozinski, J. Rappaz, and J. Wagner. Finite element approximation of multi-scale elliptic problems using patches of elements. Numer. Math., 101(4):663–687, 2005.
- Hecht F.., O. Pironneau: `http://www.freefem.org`
- Lions, J.L., Pironneau, O. : Domain decomposition methods for CAD. C.R.A.S., **328** 73-80, (1999).
- J. He, A. Lozinski and J. Rappaz: Accelerating the method of finite element patches using harmonic functions. C.R.A.S. 2007.
- Steger J.L. : The Chimera method of flow simulation. Workshop on applied CFD, Univ. of Tennessee Space Institute, (1991).
- Wagner J. : FEM with Patches and Appl. Thesis 3478, EPFL, 2006.