Homogenization of spectral problem for elliptic operators with sign-changing density

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Problem setup

The talk will focus on homogenization of spectral problem for a formally self-adjoint elliptic operator of the form

$$\operatorname{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla u^\varepsilon\Big) = \lambda^\varepsilon \rho\big(\frac{x}{\varepsilon}\big)u^\varepsilon, \qquad x \in \Omega,$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega,$$

with a small positive parameter ε; here λ^ε is a spectral parameter. We assume that

- $\Omega \subset \mathbb{R}^n$ is a regular (say, Lipschitz) bounded domain;
- the matrix $a(y) = \{a_{ij}(y)\}$ is symmetric and positive definite;
- $a(y)$ and $\rho(y)$ are $[0,1]^n$-periodic;
- moreover, we assume that $\rho(y)$ changes sign (this is a crucial condition):

$$\text{meas}\{y \in [0,1]^n : \rho(y) > 0\} > 0, \quad \text{meas}\{y \in [0,1]^n : \rho(y) < 0\} > 0.$$

Possible generalizations

For presentation simplicity we will mostly dwell on Dirichlet spectral problem for a divergence form second order scalar operators.

The results can be generalized to the case of

- Higher order self-adjoint elliptic operators;
- Spectral problem for a system of equations of the form

$$\mathcal{L}\Big(\frac{x}{\varepsilon},\nabla_x\Big)U^\varepsilon=\lambda^\varepsilon\rho\big(\frac{x}{\varepsilon}\big)U^\varepsilon$$

  with

$$\mathcal{L}(y,\nabla_y)=\overline{\mathcal{D}(-\nabla_y)}^t\mathcal{A}(y)\mathcal{D}(-\nabla_y),$$

  $\mathcal{D}(-\nabla_y)$ – $K\times k$ matrix of first order homogeneous differential operators with constant coefficients, $\mathcal{A}(y)$ is a periodic Hermitian positive definite $K\times K$ matrix. It is supposed that $\mathcal{D}(\xi)$ is algebraically complete i.e. for some $\sigma_0>0$ and for any row $P(\xi)=(P_1(\xi),\ldots P_k(\xi)))$ of homogeneous polynomials of degree $\sigma>\sigma_0$ there exists a polynomial row $Q(\xi)=(Q_1(\xi),\ldots Q_K(\xi))$ such that $P(\xi)=Q(\xi)\mathcal{D}(\xi)$.
- other boundary conditions on $\partial\Omega$.

Previously, homogenization of spectral problems for elliptic operators with periodic coefficients has been studied by S. Kesavan (1979), the eigenvalue problem in perforated domains – by M. Vanninatan (1981), and then this topic has been considered by many mathematicians.

Among the studied problems are:

- homogenization of spectral problem for elasticity system;
- homogenization of spectral problem for operators with random stationary coefficients;
- the asymptotic behaviour of spectrum in the double porosity model;
- the spectrum of thin periodic structures.

Preliminary results

In what follows for simplicity we assume that $a(y)$ and $\rho(y)$ are smooth functions.
We denote $a^\varepsilon(x) = a(x/\varepsilon)$, $\rho^\varepsilon(x) = \rho(x/\varepsilon)$.
$(\cdot,\cdot)$ – inner product in $L^2(\Omega)$.

Lemma 0.1. *For each $\varepsilon > 0$ the spectrum of problem*

$$\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla u^\varepsilon\Big) = \lambda^\varepsilon \rho\big(\frac{x}{\varepsilon}\big)u^\varepsilon \quad \textit{in } \Omega, \qquad u^\varepsilon = 0 \quad \textit{on } \partial\Omega,$$

consists of two series:

$$0 < \lambda_1^{\varepsilon,+} \le \lambda_2^{\varepsilon,+} \le \dots \le \lambda_j^{\varepsilon,+} \le \dots \to +\infty$$

and

$$0 > \lambda_1^{\varepsilon,-} \ge \lambda_2^{\varepsilon,-} \ge \dots \le \lambda_j^{\varepsilon,-} \ge \dots \to -\infty.$$

Moreover,

$$\lambda_1^{\varepsilon,+} = \min(a^\varepsilon \nabla u, \nabla u)$$

where the minimum is taken over all $u \in H_0^1(\Omega)$ such that $(\rho^\varepsilon u, u) = 1$,

Preliminary results (cont.)

$$\lambda_j^{\varepsilon,+} = \min(a^\varepsilon \nabla u, \nabla u)$$

where the minimum is taken over all $u \in H_0^1(\Omega)$ *such that* $(\rho^\varepsilon u, u) = 1$ *and* $(\rho^\varepsilon u, u_i^{\varepsilon,+}) = 0$ *for* $i = 1, \dots, j-1$. *Similarly,*

$$\lambda_1^{\varepsilon,-} = -\min(a^\varepsilon \nabla u, \nabla u)$$

where the minimum is taken over all $u \in H_0^1(\Omega)$ *such that* $(\rho^\varepsilon u, u) = -1$*,*

$$\lambda_j^{\varepsilon,-} = -\min(a^\varepsilon \nabla u, \nabla u)$$

where the minimum is taken over all $u \in H_0^1(\Omega)$ *such that* $(\rho^\varepsilon u, u) = -1$ *and* $(\rho^\varepsilon u, u_i^{\varepsilon,-}) = 0$ *for* $i = 1, \dots, j-1$.

Spectrum asymptotics. Outline.

Our main aim is to study the asymptotic behaviour of $\lambda_j^{\varepsilon,\pm}$, as $\varepsilon \to 0$.

It turns out that this behaviour depend crucially on whether the mean value of $\rho(\cdot)$ over the period is positive, or negative, or equal to zero. We denote this mean value by $\bar{\rho}$.

$\bar{\rho} > 0$ If $\bar{\rho} > 0$, then the limit behaviour of "positive" part of the spectrum $\{\lambda_j^{\varepsilon,+}, u_j^{\varepsilon,+}\}$ is governed by the homogenized equation. In particular, $\lambda_j^{\varepsilon,+}$ converges, as $\varepsilon \to 0$, to a finite limit.

$\bar{\rho} < 0$ If $\bar{\rho} = 0$, then both $\lambda_j^{\varepsilon,+}$ and $\lambda_j^{\varepsilon,-}$ are of order ε^{-1}, and the limit behaviour of $\varepsilon\lambda_j^{\varepsilon,\pm}$ and $u_j^{\varepsilon,+}$ can be described in terms of a simple quadratic operator pencil for the homogenized operator.

$\bar{\rho} < 0$ If $\bar{\rho} < 0$, then $\lambda_j^{\varepsilon,+}$ are of order ε^{-2}, in particular $\lambda_1^{\varepsilon,+} \geq c\varepsilon^{-2}$ with $c > 0$. In this case the corresponding eigenfunctions are rapidly oscillating, and the asymptotics of their amplitude can be described in terms of a homogenized operator which differs from the standard homogenized operator.

Clearly, the negative part of the spectrum in the case $\bar{\rho} > 0$ behaves like its positive part in the case $\bar{\rho} < 0$, and vice versa.

Case $\bar{\rho} > 0$

We begin the detail description by studying the case

$$\bar{\rho} = \int_{[0,1]^n} \rho(y)dy > 0.$$

Denote by $a^{\rm hom}$ the standard homogenized matrix:

$$a^{\rm hom} = \int\limits_{[0,1]^n} a(y)(\mathbf{I} + \nabla\chi(y))dy,$$

where the symbol $\mathbf{I}$ stands for the unit matrix and the vector-function $chi(\cdot)$ is a periodic solution to the following equation

$$\mathrm{div}\big(a(y)\nabla(\chi(y) + y)\big) = 0.$$

Denote $A^{\rm hom} = \mathrm{div}(a^{\rm hom}\nabla)$.

Case $\bar{\rho} > 0$. (cont.)

Theorem 0.1. *Let $\bar{\rho} > 0$. Then for any $j \geq 1$ the sequence $\lambda_j^{\varepsilon,+}$ converges, as $\varepsilon \to 0$, towards the j-th eigenvalue $\bar{\lambda}_j$ of the limit spectral problem*

$$A^{\hom} u = \bar{\rho}\lambda u \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0. \tag{1}$$

Moreover, the sequence of normalized eigenfunctions $u_j^{\varepsilon,+}$ converges (along a subsequence) in $L^2(\Omega)$ towards an eigenfunction of the limit problem that corresponds to $\bar{\lambda}_j$. If the eigenvalue $\bar{\lambda}_j$ is simple, then the whole sequence $u_j^{\varepsilon,+}$ converges to $\bar{u}_j$.

Case $\bar{\rho} = 0$.

We introduce a number of auxiliary quantities.

- $N(y)$ is defined as a periodic solution to the equation

$$\mathrm{div}\big(a(y)\nabla N(y)\big) = \rho(y);$$

- Positive number $\mathbf{m}$ is defined by

$$\mathbf{m} = \int\limits_{[0,1]^n} \rho(y)N(y)dy = \int\limits_{[0,1]^n} a(y)\nabla N(y)\cdot\nabla N(y)dy;$$

- Operator pencil

$$A^{\mathrm{hom}}u - \mathbf{m}\beta^2 u = 0 \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0.$$

Notice that since $\mathbf{m} > 0$, the spectrum of this operator pencil is as follows:

$$\pm\beta_1,\ \pm\beta_2, \ldots, \pm\beta_j, \ldots$$

with

$$0 < \beta_1 < \beta_2 \le \beta_3 \le \cdots \le \beta_j \underset{j\to\infty}{\longrightarrow} +\infty$$

Case $\bar{\rho} = 0$. The result

Theorem 0.2. *Let $\bar{\rho} = 0$. Then*

$$\lim_{\varepsilon \to 0} (\varepsilon \lambda_j^{\varepsilon,+}) = \beta_j, \qquad \lim_{\varepsilon \to 0} (\varepsilon \lambda_j^{\varepsilon,-}) = -\beta_j.$$

Moreover, the normalized sequence $u_j^{\varepsilon,\pm}$ converges (along a subsequence) to an eigenfunction of problem

$$A^{\mathrm{hom}} u - \mathbf{m}\beta^2 u = 0 \quad \textit{in } \Omega, \qquad u\big|_{\partial\Omega} = 0,$$

that corresponds to the eigenvalue β_j.

Remarks

Remark 1. *Under the assumptions of Theorem 0.1 and Theorem 0.2 the asymptotic expansion of* $\lambda_j^{\varepsilon,+}$ *and* $u_j^{\varepsilon,+}$ *can be constructed:*

$$\lambda_j^{\varepsilon,+} = \bar{\lambda}_j + \varepsilon\bar{\lambda}_j^1 + \dots \qquad (\bar{\rho} > 0),$$

$$\lambda_j^{\varepsilon,+} = \beta_j + \varepsilon\beta_j^1 + \dots \qquad (\bar{\rho} = 0),$$

$$u_j^{\varepsilon,+}(x) = \bar{u}_j(x) + \varepsilon\bar{u}_j^1(x) + \dots .$$

Remark 2. *In the general case the operator pencil takes the form*

$$A^{\mathrm{hom}}u(x) - \beta\mathbf{s}\mathcal{D}(\nabla)u(x) + \beta\overline{\mathcal{D}(\nabla)}^t\mathbf{s}^t u - \beta^2\mathbf{m}u.$$

with

$$\mathbf{s} = \int_{[0,1]^n} \rho(y)\chi(y)dy.$$

In this case β_j^+ *need not coincide with* β_j^-. *However, the spectrum remains real and consists of two sequences:*

$$0 < \beta_1^+ \le \beta_2^+ \le \dots \beta_j^+ \dots \to +\infty \ \text{ and } \ 0 > \beta_1^- \ge \beta_2^- \ge \dots \beta_j^- \dots \to -\infty.$$

Case $\bar{\rho} < 0$. Auxiliary quantities

In the space of $[0,1]^n$ periodic functions consider the eigenproblem

$$\mathrm{div}_{\mathrm{y}}\big(a(y)\nabla_y w\big) = \nu\rho(y)w(y).$$

Lemma 0.2. *The spectrum of the above problem consists of two sequences:*

$$0 < \nu_1^+ < \nu_2^+ \le \nu_3^+ \cdots \le \nu_j^+ \to +\infty$$

and

$$0 > \nu_1^- > \nu_2^- \ge \nu_3^- \cdots \ge \nu_j^- \to -\infty.$$

We denote the eigenfunction related to ν_1^+ by $p_1^+(y)$. It is known that $0 < p_1^+(y) < +\infty$.

Proposition 0.3. *The inequality holds*

$$\lambda_1^{\varepsilon,+} \ge \frac{1}{\varepsilon^2}\nu_1^+.$$

Case $\bar{\rho} < 0$. (Cont.)

Denote by $\hat{a}$ the matrix of effective (homogenized) coefficients for the operator

$$\operatorname{div}\Big\{ \Big(p_1^+(\frac{x}{\varepsilon})\Big)^2 a(\frac{x}{\varepsilon})\nabla \Big\},$$

and consider the spectral problem

$$\operatorname{div}\big(\hat{a}\nabla u\big) = \kappa u \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0.$$

Since $\hat{a}$ is positive definite, the spectrum of this problem is discrete and

$$0 < \kappa_1 < \kappa_2 \leq \kappa_3 \leq \cdots \leq \kappa_j \to +\infty.$$

Case $\bar\rho < 0$. The result

Theorem 0.4. *Let $\bar\rho < 0$. Then the eigenvalues $\lambda_j^{\varepsilon,+}$ admits the representation*

$$\lambda_j^{\varepsilon,+} = \frac{1}{\varepsilon^2}\nu_1^+ + \kappa_j + o(1).$$

Also, along a subsequence,

$$u_j^{\varepsilon,+} = p_1^+\big(\frac{x}{\varepsilon}\big)v(x) + r_\varepsilon(x),$$

where $\|r_\varepsilon\|_{L^2(\Omega)} \to 0$, as $\varepsilon \to 0$, and $v(x)$ is an eigenfunction of problem

$$\mathrm{div}\big(\hat a \nabla v\big) = \kappa v \quad \textit{in } \Omega, \qquad v\big|_{\partial\Omega} = 0$$

which corresponds to κ_j.

Bon anniversiare, Alain!